Evil Rings

Not All Rings Are Round: A Journey Through Misfit Math

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Depending on your taste, you may think that any nonnoetherian ring is evil. We will look at some examples of strange behaviour of nonnoetherian rings, but then stick to noetherian rings for the majority of the talk.

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1. The Ring of Integers Modulo n ($\mathbb{Z}/n\mathbb{Z}$)

- **Weird Behavior**: In the ring of integers modulo a non-prime n, certain elements may not have multiplicative inverses. This contrasts with fields (e.g., $\mathbb{Z}/p\mathbb{Z}$ where p is prime), where every non-zero element has an inverse.
- **Example**: In $\mathbb{Z}/6\mathbb{Z}$, 2 does not have a multiplicative inverse because $2 \times 3 = 6 \equiv 0 \mod 6$.

2. The Ring of Gaussian Integers ($\mathbb{Z}[i]$)

- **Weird Behavior**: This is a ring consisting of complex numbers of the form $a + bi$ where a and b are integers. It has unique factorization properties that can fail in other complex integer rings.
- **Example**: Unlike $\mathbb{Z}$, some other rings of integers don't have unique factorization, showing that even slightly changing the number system can lead to strange and different properties.

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Such valuation domains do exist!

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$R = k[x_1, x_2, \ldots]$ and $\mathfrak{m} = (x_1, x_2, \ldots)$ serve as a uniform (counter)example. [Stacks, Tag 05JA]

Turbulence over...

Rings in the forthcoming slides will all be noetherian!

Revisiting Krull's height theorem

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Question. Does there exist a noetherian ring with infinite Krull dimension?

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Punchline (Nagata)

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This was Nagata's original example given in [Nag62, Appendix A1].

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There exists a Gorenstein ring with infinite injective dimension.

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Moreover, if R is a domain, then $\mathfrak{a} \neq (0)$.

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Note that $S_{\mathfrak{p}_n S}$ is not normal and hence not regular. Every nonzero element avoids some $\mathfrak{p}_n$. Thus, S_f is not regular for any $f \in S \setminus \{0\}$. Since S is a domain, this shows that $\mathrm{Sing}(S)$ is not closed. □

Projective dimensions

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Try running *this* on M2...

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Theorem (Osofsky [Oso68])

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- *The continuum hypothesis holds.*

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Punchline (Heitmann–Jorgensen)

There exists a three-dimensional complete intersection domain which is not a homomorphic image of a regular local ring.

Are complete intersections complete intersections?

Recall that a local ring R is defined to be a *complete intersection* if

$$\widehat{R} \cong \frac{\text{regular local ring}}{(\text{regular sequence})}.$$

Question. Is every c.i. R itself a quotient of the above form?

Punchline (Heitmann–Jorgensen)

There exists a three-dimensional complete intersection domain which is not a homomorphic image of a regular local ring.

This is from the paper *"Are complete intersections complete intersections?"* [HJ12], that has an example where the completion is $\mathbb{R}[\![x, y, z, w]\!]/(x^2 + y^2)$.

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For a Cohen–Macaulay ring, being the image of a Gorenstein ring is equivalent to possessing a canonical module. Thus, the above shows that there exist CM rings without canonical modules.

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Such a ring is thus not the image of a Gorenstein ring.

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See [Lip75, §4] for a discussion.

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Jacking up the previous example, there exists a noetherian local UFD R such that $\widehat{R} \cong \mathbb{C}[\![x, y, z]\!]/(x^2)$; this is again not reduced.

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(Heitmann, 1979) The difference in lengths of maximal chains of primes between (0) *and* $\mathfrak{m}$ *can be arbitrarily large in a local noetherian domain.*

(Ogoma, 1980) There exists a noncatenary <u>*normal*</u> *noetherian domain.*

See [Nag56; Hei79; Ogo80].

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This was constructed by Heitmann in his 1993 paper. The ring satisfies $\widehat{R} \cong \mathbb{C}[\![x, y, z, w, v]\!]/(wx, wy)$.

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There exists a local UFD R such that $R[\![x]\!]$ is not a UFD.

Construction: $R = k[x, y, z]/(x^2 + y^3 + z^7)$ localised at (x, y, z).

See [fer] and [kar] for more (Mathematics Stack Exchange).

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