Evil Rings

Not All Rings Are Round: A Journey Through Misfit Math

Aryaman Maithani

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University of Utah

Conventions

All rings mentioned will be commutative and unital.

(Nonunital rings might just be too evil...)

Depending on your taste, you may think that any nonnoetherian ring is evil. We will look at some examples of strange behaviour of nonnoetherian rings, but then stick to noetherian rings for the majority of the talk.

One mathematician's pathology is another's normalcy

Such a talk if of course subjective. For example, ChatGPT [Ope24] suggested the following (*surely* benign) rings as evil.

1. The Ring of Integers Modulo n ($\mathbb{Z}/n\mathbb{Z}$)

- **Weird Behavior**: In the ring of integers modulo a non-prime n, certain elements may not have multiplicative inverses. This contrasts with fields (e.g., $\mathbb{Z}/p\mathbb{Z}$ where p is prime), where every non-zero element has an inverse.
- **Example**: In $\mathbb{Z}/6\mathbb{Z}$, 2 does not have a multiplicative inverse because $2 \times 3 = 6 \equiv 0 \mod 6$.

2. The Ring of Gaussian Integers ($\mathbb{Z}[i]$)

- **Weird Behavior**: This is a ring consisting of complex numbers of the form $a + bi$ where a and b are integers. It has unique factorization properties that can fail in other complex integer rings.
- **Example**: Unlike $\mathbb{Z}$, some other rings of integers don't have unique factorization, showing that even slightly changing the number system can lead to strange and different properties.

Tensoring noetherian algebras

Have you thought about $k[\![x]\!] \otimes_k k[\![y]\!]$? This is *not* $k[\![x, y]\!]$. In fact:

Punchline

Each of the following rings is not noetherian: $k[\![x]\!] \otimes_k k[\![y]\!]$, $k[\![x]\!] \otimes_k k(\!(y)\!)$, *and* $k(\!(x)\!) \otimes_k k(\!(y)\!)$.

Krull's height theorem

Krull's principal ideal theorem states: in a noetherian ring, every minimal prime over a proper principal ideal has height at most one.

This fails in every valuation domain $(R, \mathfrak{m})$ with dimension at least two: pick a prime $\mathfrak{p}$ strictly between 0 and $\mathfrak{m}$. Pick $x \in \mathfrak{m} \setminus \mathfrak{p}$. Since ideals in a valuation domain are linearly ordered, we get

$$0 \subsetneq \mathfrak{p} \subsetneq (x) \subseteq \mathfrak{m}.$$

Any minimal prime over (x) has height at least two.

Such valuation domains do exist!

Incomplete completions

Let R be a ring and $\mathfrak{m}$ a maximal ideal. The *completion* of R (with respect to $\mathfrak{m}$) is defined as

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

The following is true: $\widehat{R}$ is a local ring with unique maximal ideal $\mathfrak{M} := \ker(\widehat{R} \to R/\mathfrak{m})$.

Punchlines

All of the following have negative answers:
Is $\mathfrak{m}\widehat{R} = \mathfrak{M}$?
Is $\widehat{R}$ viewed as an R-module $\mathfrak{m}$-adically completely?
Is $\widehat{R}$ viewed as an $\widehat{R}$-module $\mathfrak{M}$-adically completely, i.e., is $\widehat{R}$ a complete local ring?

$R = k[x_1, x_2, \ldots]$ and $\mathfrak{m} = (x_1, x_2, \ldots)$ serve as a uniform (counter)example. [Stacks, Tag 05JA]

Turbulence over...

Rings in the forthcoming slides will all be noetherian!

Revisiting Krull's height theorem

Krull was the first to show that a great deal of the geometric theory of the polynomial ring could be carried over to the noetherian case, indicating that is a good class of rings to work with.

The Krull dimension of a ring can be defined as the supremum of heights of all of its prime ideals. By Krull's height theorem, we see that every prime ideal in a noetherian ring has finite height. In particular, noetherian local rings have finite dimension.

Question. Does there exist a noetherian ring with infinite Krull dimension?

Nagata's example

Punchline (Nagata)

There exists a noetherian domain with infinite Krull dimension.

Construction: Consider the polynomial ring in ω-many variables:

$$R := k[x_{11},$$
$$x_{21}, x_{22},$$
$$x_{31}, x_{32}, x_{33}, \ldots].$$

For $n \geq 1$, define the prime ideal $\mathfrak{p}_n := (x_{n1}, \ldots, x_{nn})$ and set $W := R \setminus \bigcup_{n \geq 1} \mathfrak{p}_n$.

Then, $S := W^{-1}R$ is a noetherian domain whose maximal ideals are $W^{-1}\mathfrak{p}_n$. As $\text{height}(W^{-1}\mathfrak{p}_n) = n$, we see that $\dim(S) = \infty$. $\square$

This was Nagata's original example given in [Nag62, Appendix A1].

Nagata's example continued

Moreover, observe that the localisation of S at each maximal ideal is regular and thus, S is itself a regular domain. In particular, S is also Gorenstein but has infinite injective dimension. This gives us:

Punchline

There exists a Gorenstein ring with infinite injective dimension.

Excellent rings

The definition of an *excellent* ring is available at https://en.wikipedia.org/wiki/Excellent_ring. This is considered a "well-behaved" class of rings to work with. Most naturally occurring commutative rings in number theory or algebraic geometry are excellent. Such rings are noetherian, universally catenary, have finite Krull dimension, have a closed singular locus, ...

The *singular locus* of a ring R is

$$\mathrm{Sing}(R) := \{\mathfrak{p} \in \mathrm{Spec}(R) : R_{\mathfrak{p}} \text{ is not regular}\}.$$

Note: If $\mathrm{Sing}(R) = V(\mathfrak{a})$, then any $f \in \mathfrak{a} \setminus \{0\}$ has the property that R_f is regular.
Moreover, if R is a domain, then $\mathfrak{a} \neq (0)$.

Non-excellent noetherian rings

Our previous example was infinite-dimensional and hence, not excellent. Thus, there exist nonexcellent noetherian rings. Another example is furnished with the following.

Punchline

There exists a noetherian domain whose singular locus is not closed.

Construction: Consider $R := k[x_1^2, x_1^3, x_2^2, x_2^3, \ldots]$, $\mathfrak{p}_n := (x_n^2, x_n^3)$, $W := R \setminus \bigcup_{n \geq 1} \mathfrak{p}_n$, and $S := W^{-1}R$. S is noetherian for similar reasons as before.

Note that $S_{\mathfrak{p}_n S}$ is not normal and hence not regular. Every nonzero element avoids some $\mathfrak{p}_n$. Thus, S_f is not regular for any $f \in S \setminus \{0\}$. Since S is a domain, this shows that $\mathrm{Sing}(S)$ is not closed. $\square$

Projective dimensions

Consider $R := \mathbb{R}[x, y, z]$. This ring has global dimension 3, this means that that every R-module has a projective resolution of length (at most) 3. Consider $Q := \mathrm{Frac}(R)$ as a module over R.

What is $\mathrm{pdim}_R(Q)$?

Theorem (Osofsky [Oso68])

The following are equivalent:

- $\mathrm{pdim}_R(Q) = 2$.
- *The continuum hypothesis holds.*

Try running *this* on M2...

Are complete intersections complete intersections?

Recall that a local ring R is defined to be a *complete intersection* if

$$\widehat{R} \cong \frac{\text{regular local ring}}{(\text{regular sequence})}.$$

Question. Is every c.i. R itself a quotient of the above form?

Punchline (Heitmann–Jorgensen)

There exists a three-dimensional complete intersection domain which is not a homomorphic image of a regular local ring.

This is from the paper *"Are complete intersections complete intersections?"* [HJ12], that has an example where the completion is $\mathbb{R}[\![x, y, z, w]\!]/(x^2 + y^2)$.

Lack of imagery

The previous example was from 2011. In 1978, one knew:

Punchlines (Marinari)

There exists a one-dimensional local Gorenstein domain which is not a homomorphic image of a regular local ring.

There exists a one-dimensional local Cohen–Macaulay domain which is not a homomorphic image of a Gorenstein ring.

These examples were constructed in the paper *"Examples of bad Noetherian local rings"* [Mar78] using a technique attributed to Larfeldt–Lech [LL81].

For a Cohen–Macaulay ring, being the image of a Gorenstein ring is equivalent to possessing a canonical module. Thus, the above shows that there exist CM rings without canonical modules.

UFD + CM $\Rightarrow$ G?

Murthy [Mur64] showed that a Cohen–Macaulay UFD possessing a canonical module is Gorenstein.

Punchline

There exists a two-dimensional UFD ($\Rightarrow$ CM) which is not a Gorenstein ring.

Such a ring is thus not the image of a Gorenstein ring.

UFD $\Rightarrow$ CM?

For a while, the question "UFD $\Rightarrow$ CM?" was open under various contexts.

Punchlines

There exists a local UFD which is not Cohen–Macaulay.

There exists a complete local UFD which is not Cohen–Macaulay.

Examples for both can be obtained via invariant subrings: Consider the action of $G := \mathbb{Z}/4$ on $S := \mathbb{F}_2[w, x, y, z]$ by cyclically permuting the variables. The fixed subring $R := S^G$ is a (graded) UFD which is not CM. Localising and completing at the homogeneous maximal yields the examples.

See [Lip75, §4] for a discussion.

Characterising completions: Lech

Lech's *"A Method for Constructing Bad Noetherian Local Rings"* [Lec86] characterises what rings can be obtained as the completion of a noetherian local domain.

A corollary: if S is a complete noetherian local ring containing a field and $\operatorname{depth}(S) \geq 1$, then S can be obtained so.

Thus, there exists a noetherian local domain R such that $\widehat{R} \cong \mathbb{C}[\![x, y]\!]/(x^2)$; this is not reduced.

Characterising completions: Heitmann

Similarly, Heitmann [Hei93] characterised in 1993 which rings can be obtained as the completion of a local UFD.

A corollary: if S is a complete noetherian local ring containing a field and $\mathrm{depth}(S) \geq 2$, then S can be obtained so.

Jacking up the previous example, there exists a noetherian local UFD R such that $\widehat{R} \cong \mathbb{C}[\![x, y, z]\!]/(x^2)$; this is again not reduced.

Catenary

A ring is *catenary* if for any pair of prime ideals $\mathfrak{p}$, $\mathfrak{q}$, any two maximal chains of primes from $\mathfrak{p}$ to $\mathfrak{q}$ have the same length.

Examples: Cohen–Macaulay rings are catenary. Thus, so are regular rings. In turn, quotients of regular rings are catenary. In particular, completions of noetherian local rings are catenary.

Non-examples?

For some time it was thought that all noetherian rings are catenary.

Punchlines

(Nagata, 1956) There exists a noncatenary noetherian ring.

(Heitmann, 1979) The difference in lengths of maximal chains of primes between (0) *and* $\mathfrak{m}$ *can be arbitrarily large in a local noetherian domain.*

(Ogoma, 1980) There exists a noncatenary <u>*normal*</u> *noetherian domain.*

See [Nag56; Hei79; Ogo80].

UFDs. Catenary?

Fact: A three-dimensional noetherian local UFD is catenary.

See [Mur] for a discussion.

[Fuj77]: "noetherian" cannot be dropped.

Punchline

There exists a four-dimensional noncatenary noetherian local UFD.

This was constructed by Heitmann in his 1993 paper. The ring satisfies $\widehat{R} \cong \mathbb{C}[[x, y, z, w, v]]/(wx, wy)$.

UFD$[\![x]\!]$

Have you thought about why $k[\![x]\!]$ is a UFD? $k[\![x_1, \ldots, x_n]\!]$? Is $R[\![x]\!]$ a UFD whenever R is so?

Punchline

There exists a local UFD R such that $R[\![x]\!]$ is not a UFD.

Construction: $R = k[x, y, z]/(x^2 + y^3 + z^7)$ localised at (x, y, z).

See [fer] and [kar] for more (Mathematics Stack Exchange).

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