Classical Invariant Theory

Aryaman Maithani

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University of Utah

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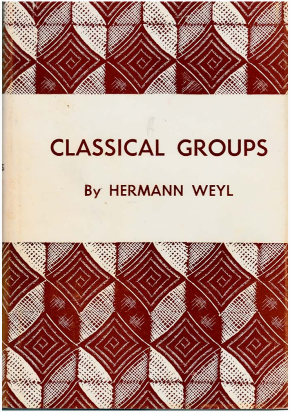

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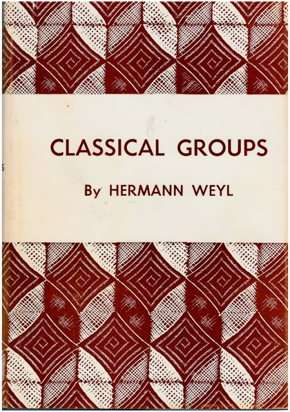
These groups have their natural actions on $V = \mathbb{C}^n$.



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If one wishes to be coordinate-free, these notions can be defined in terms of symmetric algebras, duals, and tensor products.

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Note: This would work for any infinite field.

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Theorem

$\mathbb{C}[X_{n\times m}]^{\mathrm{O}_n} = \mathbb{C}[X^\top X]$ *and* $\mathbb{C}[X_{n\times m}]^{\mathrm{Sp}_n} = \mathbb{C}[X^\top \Omega X]$.

A Brief Word about the proofs

Define the following operators on $R := \mathbb{C}[X_{n\times m}]$:

$$D_{ij} = \sum_{k=1}^{n} x_{ki} \frac{\partial}{\partial x_{kj}} \quad \text{and} \quad \Omega = \det\left[\frac{\partial}{\partial x_{ij}}\right]_{1\leq i,j\leq m}.$$

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The *Capelli identity* says that

$$\det \begin{bmatrix} D_{11}+m-1 & D_{12} & \cdots & D_{1m} \\ D_{21} & D_{22}+m-2 & \cdots & D_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mm} \end{bmatrix} = \det(X_{n\times m}) \cdot \Omega$$

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If F is an invariant of G, so is each $D_{ij}(F)$. Note that D_{ij} lowers the j-th degree by 1 while increasing the i-th degree by 1. Induct...

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In particular, in characteristic zero, the inclusion

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splits (as R^G-modules) when G is a classical group with the natural action.

This is similar to the finite group case when we have a splitting given by $r \mapsto \frac{1}{|G|} \sum_{g \in G} g(r)$.

When do they split in positive characteristic?

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Theorem 1.1. *Let K be a field of characteristic $p > 0$. Fix positive integers d, m, n, and t, and let $R \subseteq S$ denote one of the following inclusions:*

(a) $K[YZ] \subseteq K[Y, Z]$, *where Y and Z are $m \times t$ and $t \times n$ matrices of indeterminates;*
(b) $K[Y^{\text{tr}}\Omega Y] \subseteq K[Y]$, *where Y is a $2t \times n$ matrix of indeterminates;*
(c) $K[Y^{\text{tr}}Y] \subseteq K[Y]$, *where Y is a $d \times n$ matrix of indeterminates;*
(d) $K[\{\Delta\}] \subseteq K[Y]$, *where Y is a $d \times n$ matrix of indeterminates with $d \leqslant n$.*

Then $R \subseteq S$ is pure if and only if, in the respective cases,

(a) $t = 1$ *or* $\min\{m, n\} \leqslant t$;
(b) $n \leqslant t + 1$;
(c) $d = 1$; $d = 2$ *and p is odd;* $p = 2$ *and* $n \leqslant (d+1)/2$; *or p is odd and* $n \leqslant (d+2)/2$;
(d) $d = 1$ *or* $d = n$.

These cases are essentially the "obvious" ones: in these cases, either the subring R is regular, or the corresponding group was linearly reductive.

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These cases are essentially the "obvious" ones: in these cases, either the subring R is regular, or the corresponding group was linearly reductive.

Note that the above theorem applies even if K is finite; however, the subring does not arise from the corresponding group action in those cases.

What does this mean?

Concretely: let us consider the inclusion $R \subset S$ where $S = \mathbb{Q}[X_{2\times 3}]$ and $R = \mathbb{Q}[\Delta_1, \Delta_2, \Delta_3]$ is the subring generated by the three 2×2 minors of X.

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What this means is that for every prime p, there is some monomial $m_p \in S$ such that the expression for $f(m_p)$ in terms of the Δ_i has a p in the denominator.

Finite fields?

Natural question

What are the invariant subrings when $K = \mathbb{F}_p$? Do they split?

Even the first action of $\mathrm{GL}_n(K)$ on $K[X_{n\times m}]$ is not trivial now.

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Even the first action of $\mathrm{GL}_n(K)$ on $K[X_{n\times m}]$ is not trivial now. In the case that $m = 1$, the fixed subring is generated by the n algebraically independent *Dickson invariants*.

Fin

Thank you for your attention!

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[2] Melvin Hochster, Jack Jeffries, Vaibhav Pandey, and Anurag K. Singh. **"When are the natural embeddings of classical invariant rings pure?"** In: *Forum Math. Sigma* 11 (2023), Paper No. e67, 43.