Linear Quotients of Connected Ideals of Graphs

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§1. Introduction

Notes I made for my talk at the commutative algebra seminar at the University of Utah. I introduce the notions of linear resolutions and linear quotients, as well as some monomial ideals related to graphs. I mention our results of characterising when the connected ideals of trees have linear quotients, and giving a sufficient condition for general graphs.

Throughout the article, V will be a finite set, G a finite simple graph with vertex set V, and E its edge set. K will denote a field.

We let $K[V] := K[x_v : v \in V]$ be the polynomial ring over K with variables indexed by elements of V. Given a subset $C \subseteq V$, we define the monomial

$$x_C := \prod_{c \in C} x_c \in K[V].$$

§2. Simplicial complexes, graphs, and Stanley-Reisner ideals

Let V be a finite set. A simplicial complex Δ on V is a collection of subsets of V such that

- $G \in \Delta$ and $F \subseteq G$ implies $F \in \Delta$,
- $\{v\} \in \Delta$ for all $v \in V$.

The elements of Δ are called faces and maximal faces (with respect to inclusion) are called facets. Subsets of V which are *not* in Δ are called nonfaces (of Δ).

Example 2.1. Our examples of interest will be simplicial complexes associated to (finite, simple) graphs. To begin with, let $G = (V, E)$ be a graph. Recall that a subset $C \subseteq V$ is said to be independent is no two vertices in C share an edge. Singletons are independent and being independence is closed under taking subsets, hence we get a simplicial complex:

$$\mathrm{Ind}(G) := \{C \subseteq V : C \text{ is independent}\}.$$

This is called the independence complex of G.

Definition 2.2. The Stanley-Reisner ideal (with respect to K) of Δ is the ideal $I_\Delta \subseteq K[V]$ defined as

$$\begin{aligned} I_\Delta &:= (x_C : C \text{ is a nonface}) \\ &= (x_C : C \text{ is a minimal nonface}) \\ &= \bigcap_{F \text{ a face}} (x_v : v \notin F) \\ &= \bigcap_{F \text{ a facet}} (x_v : v \notin F) \end{aligned}$$

The last equation is in fact a primary decomposition of I_Δ.

Example 2.3. Turning back to our earlier example, we can examine what the Stanley-Reisner ideal of $\mathrm{Ind}(G)$ is. This is precisely what is known as the edge ideal of the graph:

$$\begin{aligned} I_{\mathrm{Ind}(G)} &= (x_v x_w : \{v, w\} \text{ is an edge}) \\ &= (x_e : e \in E) =: I(G). \end{aligned}$$

This comes down to the observation that a minimal nonface of $\mathrm{Ind}(G)$ is precisely an edge.

There is also the notion of a dual of a simplicial complex (equivalently, dual of a squarefree monomial ideal). For edge ideals, it takes the form

$$I(G)^\vee = I_{\mathrm{Ind}(G)^\vee} = \bigcap_{\{v,w\} \in E} (x_v, x_w).$$

The double-dual gives back the same object.

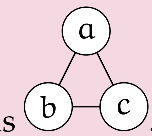
§3. Linear resolutions and linear quotients

Let I be a homogeneous ideal generated by elements of degree d. Let $(\mathbb{F}_\bullet, \partial_\bullet)$ be the minimal graded free resolution of I (as a module). We say that I has linear resolution if any of the following equivalent conditions hold:

(a) The entries of ∂_i for $i \geqslant 1$ are linear (or zero).

(b) The height of the Betti table of I is d.

(c) $\mathrm{reg}(I) = d$.

Example 3.1. Consider the graph G drawn as 

. We have $I(G) = (ab, bc, ca)$.

Running this example on Macaulay2 gives the Betti table (over $K = \mathbb{F}_7$) of $I(G)$ as

	0	1
total:	3	2
2 :	3	2

Note that the resolution of a monomial ideal may depend on the characteristic[1].

Remark 3.2. The Stanley-Reisner ideal of the triangulation of $\mathbb{RP}^2$ has the property of having linear resolutions precisely if the characteristic is two.

Theorem 3.3 ([ER98, Theorem 3])**.** I_Δ has linear resolution if and only if $K[V]/I_{\Delta^\vee}$ is Cohen-Macaulay.

A running theme of questions is whether one can characterise ideals (among certain classes) that have linear resolution. After introducing some more terminology, we shall state a celebrated theorem of Fröberg's that characterises squarefree quadratic monomials.

We now introduce a stronger property for a monomial ideal to have: that of *linear quotients*.

Definition 3.4. Let I be a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of I. We say that I has linear quotients, if there exists an order $\sigma = u_1, \ldots, u_m$ on $G(I)$ such that the colon ideal $\langle u_1, \ldots, u_{i-1} \rangle : \langle u_i \rangle$ is generated by a subset of the variables, for $i = 2, \ldots, m$. Any such order is said to be an admissible order.

Remark 3.5. We immediately note that colons of monomial ideals and monomials are straightforward to compute. Indeed, the colon is generated by the "pairwise colons" which are again monomials.

This also shows that the property of having linear quotients is characteristic-independent.

[1] as well as the version of Macaulay2 that you are using

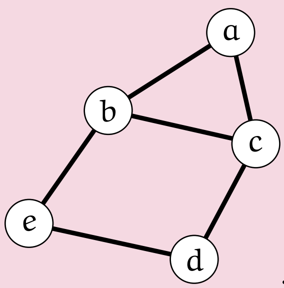
Theorem 3.6 ([JZ10])**.** Let I be a monomial equigenerated in degree d.

$$I \text{ has linear quotients} \Rightarrow I \text{ has linear resolution.}$$

§4. Some terminology about graphs

Recall that a subgraph of a graph is a subcollection of vertices and a subcollection of edges between those vertices. A subgraph is said to be induced if we pick all the edges between those vertices.

Example 4.1. Consider the *house graph* G



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G contains the 4-cycle as an induced subgraph. G also contains the 5-cycle as a subgraph, but not as an induced subgraph.

A running theme is to restrict one's attention to graphs that don't contain a forbidden (family of) graph(s) as an induced subgraph and prove results about those. The celebrated result of Fröberg is one example of this.

Theorem 4.2 ([Frö90])**.** Let G be a graph. The following are equivalent:

- $I(G)$ has linear resolution.
- G^c is chordal, i.e., contains no induced n-cycle for $n \geqslant 4$.

Note that since every squarefree monomial ideal is of the form $I(G)$, the above completely characterises linear resolution for such ideals. Combined with the following result, we also have the complete characterisation of such ideals with linear quotients.

Theorem 4.3 ([HHZ04, Theorem 3.2])**.** Let I be a monomial ideal equigenerated in degree 2. The following are equivalent:

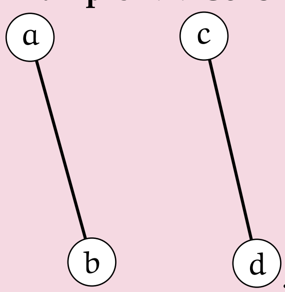
(a) I has a linear resolution.

(b) I has linear quotients.

(c) Each power of I has a linear resolution.

On the other hand, combining this with Theorem 3.3 gives us a way of telling when certain ideals define Cohen-Macaulay rings, perhaps best depicted with an example.

Example 4.4. Consider $I = (a, b) \cap (c, d)$. As noted in Example 2.3, $I^\vee$ is the edge ideal of



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But the complement of the above graph is the 4-cycle, which is not chordal. Thus, $I^\vee$ does not have a linear resolution and hence $K[V]/I$ is not Cohen-Macaulay. This can extended to any ideal I which is given by intersection of ideals generated by two variables.

§5. Higher analogues of the edge ideal

We now look at some generalisations of the edge ideal. Let G be a graph. For $t \geqslant 2$, define the two classes of ideals:

- the t-path ideal $I_t(G)$ is generated by the paths of G of length t;
- the t-connected ideal $J_t(G)$ is generated by the connected subsets of G of size t.

(Recall that we can identify a subset $C \subseteq V(G)$ with the monomial $x_C \in K[V]$. That is what we mean by generated above.)

$J_t(G)$ seems to be a somewhat more natural analogue. For one, it is the Stanley-Reisner ideal of the simplicial complex $\mathrm{Ind}_{t-1}(G)$, where

$$\mathrm{Ind}_r(G) = \{C \subseteq V : \text{every connected component of } G[C] \text{ has size} \leqslant r\}.$$

Note that $\mathrm{Ind}_1(G) = \mathrm{Ind}(G)$ and $I_2(G) = J_2(G) = I(G)$.

Definition 5.1. We say that a graph (V, E) is t-gap-free if whenever C and C' are two disjoint connected subsets of V of size t, then there is an edge joining a vertex of C to a vertex of C'.

Remark 5.2. $J_t(G)$ can be viewed as an edge ideal of an associated *hypergraph*. Using [HW14, Theorem 1.4], it is relatively straightforward to show that

$$J_t(G) \text{ has a linear resolution} \Rightarrow G \text{ is } t\text{-gap-free}.$$

(See [AJM24, Corollary 4.3].)

This project began as an attempt to prove the converse.

Theorem 5.3 ([AJM24])**.** Let T be a tree, i.e., a connected graph with no cycles. For each $t \geqslant 2$, the following are equivalent.

(a) $J_t(T)$ has linear quotients.

(b) $J_t(T)$ has a linear resolution.

(c) T is t-gap-free

Note that for $t = 2$, we recover Fröberg's result for trees.

The above does not hold in general. Indeed, C_5 is (2-)gap-free but $J_2(C_5)$ does not have linear resolution, for it is not co-chordal. In fact, we showed that every ($\geqslant 5$)-cycle is a counterexample to the above for a suitable t ([AJM24, Theorem 5.2]).

Sketch for Theorem 5.3. We prove this by induction on $|V(T)|$. If $|V(T)| = t$, this is clear. Assume $|V(T)| > t$.
Let ℓ be a leaf of T. Then, $T \setminus \{\ell\}$ is an induced subgraph and hence, t-gap-free. By hypothesis, there is an admissible order on $G(J_t(T \setminus \{\ell\}))$ (recall Definition 3.4). Furthermore,

$$G(J_t(T)) = G(J_t(T \setminus \{\ell\})) \sqcup \{\text{connected subsets of size } t \text{ containing } \ell\}.$$

We showed in the paper that appending the extra generators in any order gives an admissible order. □

Continuing the theme of prohibiting certain subgraphs, we introduce some more terminology: Recall that $K_{1,t}$ is the graph with $V = \{0, \dots, t\}$ and $E = \{(0, i) : 1 \leqslant i \leqslant t\}$.

Definition 5.4. A graph is called t-claw-free if it contains no induced subgraph isomorphic to $K_{1,t}$.

For path ideals, Banerjee showed the following.

Theorem 5.5 ([Ban17])**.** If G is a (2-)gap-free and (3-)claw-free graph, then $I_t(G)$ has linear resolution for all $t \geqslant 3$.

Theorem 5.6 ([AJM24])**.** Let $t \geqslant 3$ be an integer. Suppose G is a gap-free and t-claw-free graph. Then, $J_t(G)$ has linear quotients. In particular, $J_t(G)$ has a linear resolution.

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