Linear Quotients of Connected Ideals of Graphs

Aryaman Maithani (Joint work with H. Ananthnarayan and Omkar Javadekar)

February 16, 2024

§1. Introduction

Notes I made for my talk at BIKES – the student commutative algebra seminar at the University of Utah. I introduce the notions of linear resolutions and linear quotients, as well as some monomial ideals related to graphs. I mention our results of characterising when the connected ideals of trees have linear quotients, and giving a sufficient condition for general graphs.

§2. Monomial ideals

Throughout the talk, K will denote an arbitrary field, and R will be a polynomial ring over K.

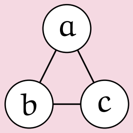
Definition 2.1. A monomial ideal is an ideal of R generated by monomials.

Example 2.2. $I := (ab, bc, ca) \subseteq K[x, y, z]$ is a monomial ideal.

A rich source of monomial ideals are graphs and simplicial complexes. We will focus on the former in this talk, and make passing remarks to the latter. Loosely speaking, a simplicial complex on a set V is a collection of subsets of V that is closed under taking subsets.

Recall that a graph $G = (V, E)$ is a finite set V along with a subset $E \subseteq \binom{V}{2}$, i.e., E is a collection of subsets of V of cardinality two. The elements of V are called the vertices of G and the elements of E the edges.

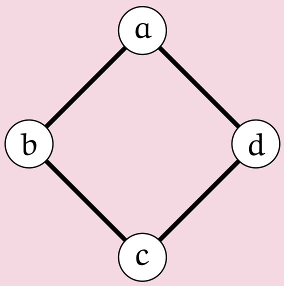
Example 2.3. There is a natural way to depict a graph visually.

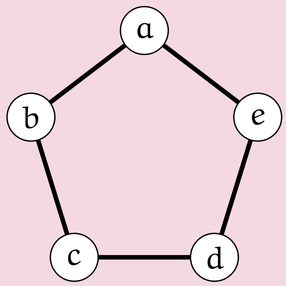


$V = \{a, b, c\}$, $E = \{\{a, b\}, \{b, c\}, \{c, a\}\}$. This is the 3-cycle, denoted C_3.

For ease (and suggestive notation), we may simply write the edges as $\{ab, bc, ca\}$.

Example 2.4. There is the analogous definition of an n-*cycle*, denoted C_n. C_4 and C_5 are drawn below.





Definition 2.5. Given a graph G, we consider the ring $R = K[x_v : v \in V]$, i.e., the polynomial ring with variables indexed by the vertices of G.
The edge ideal of G is the monomial ideal generated by the edges, i.e.,

$$I(G) := \langle x_v x_w : \{v, w\} \in E(G) \rangle.$$

In the case that we label the graph vertices with letters, we will typically use the same letters for the polynomial ring above.

Example 2.6. The edge ideal of C_3 is $\langle ab, bc, ca \rangle$.

§3. Linear resolutions and linear quotients

Let $I \subseteq R$ be a homogeneous ideal generated by elements of d (such as ideal will be referred to as equigenerated in degree d). Consider its minimal graded free resolution:

$$0 \to F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to F_0 \xrightarrow{\partial_0} I \to 0.$$

The following are equivalent:

(a) The entries of ∂_i for $i \geqslant 1$ are linear (or zero).

(b) The height of the Betti table of I is d.

(c) $\operatorname{reg}(I) = d$.

If any of the above conditions hold, we say that I has linear resolution.

Example 3.1. If $R = \mathbb{F}_7[a, b, c]$ and $I := I(C_3) = (ab, bc, ca)$ from earlier, then running `betti res module I` on Macaulay2 gives the following

	0	1
total:	3	2
2 :	3	2

Thus I has linear resolution.

Remark 3.2. In general, given a set of monomials, the property of the ideal having linear resolution can be characteristic-dependent. The Stanley-Reisner ideal of the triangulation of $\mathbb{RP}^2$ has the property of having linear resolutions precisely if the characteristic is two.

A running theme of questions is whether one can characterise ideals (among certain classes) that have linear resolution. After introducing some more terminology, we shall state a celebrated theorem of Fröberg's that characterises squarefree quadratic monomials.

We now introduce a stronger property for a monomial ideal to have: that of *linear quotients*.

Definition 3.3. Let I be a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of I. We say that I has linear quotients, if there exists an order $\sigma = u_1, \ldots, u_m$ on $G(I)$ such that the colon ideal $\langle u_1, \ldots, u_{i-1}\rangle : \langle u_i \rangle$ is generated by a subset of the variables, for $i = 2, \ldots, m$. Any such order is said to be an admissible order.

Remark 3.4. We immediately note that colons of monomial ideals and monomials are straightforward to compute. Indeed, abusing notations, we have

$$\langle u_1, \ldots, u_n \rangle : \langle v \rangle = \langle u_1 : v, \ldots, u_n : v \rangle$$

for monomials u_i, v, where we define $u : v$ to be the monomial $\dfrac{\operatorname{lcm}(u, v)}{v}$.

This also shows that the property of having linear quotients is characteristic-independent.

Theorem 3.5 ([JZ10])**.** Let I be a monomial equigenerated in degree d.

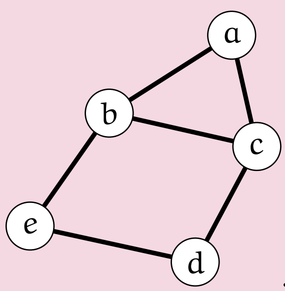
$$I \text{ has linear quotients} \Rightarrow I \text{ has linear resolution.}$$

§4. Some terminology about graphs

Definition 4.1. Let $G = (V, E)$ be a graph. A subgraph of G is a tuple $H = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$.
Further, the subgraph is said to be induced if $E' = E \cap \binom{V'}{2}$.

In words: a subgraph is some subcollection of vertices with some subcollection of edges between those vertices. The subgraph is induced if we pick *all* the edges between the subcolleciton of vertices.

Example 4.2. Consider the *house graph* G



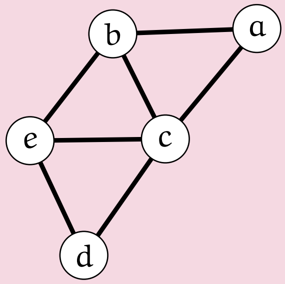
.

G contains C_4 as an induced subgraph. G also contains C_5 as a subgraph, but not as an induced subgraph.

A running theme is to restrict one's attention to graphs that don't contain a forbidden (family of) graph(s) as an induced subgraph and prove results about those. As an example, we have the following definition.

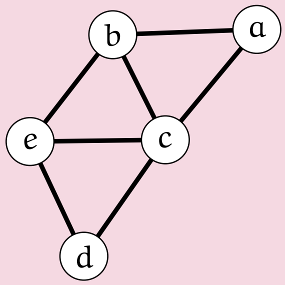
Definition 4.3. G is chordal if G contains no induced C_n for $n \geqslant 4$.

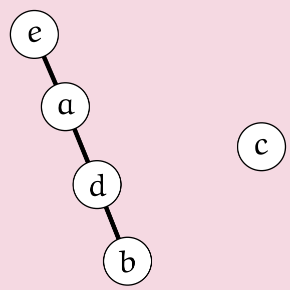
Example 4.4. The house graph (Example 4.2) is not chordal. However, we add an extra edge bd, then it becomes chordal:



We are now almost there at Fröberg's theorem. We recall the notion of the complement of a graph: if $G = (V, E)$ is a graph, the complement is $G^c := \left(V, \binom{V}{2} \setminus E\right)$. In words: we switch the edges and non-edges.

Example 4.5. The following are complements:





Theorem 4.6 ([Frö90]). Let G be a graph. $I(G)$ has linear resolution if and only if G^c is chordal.

Note that since every squarefree monomial ideal is of the form $I(G)$, the above completely characterises linear resolution for such ideals. Combined with the following result, we also have the complete characterisation of such ideals with linear quotients.

Theorem 4.7 ([HHZ04, Theorem 3.2]). Let I be a monomial ideal equigenerated in degree 2. The following are equivalent:

(a) I has a linear resolution.

(b) I has linear quotients.

(c) Each power of I has a linear resolution.

§5. Path ideals

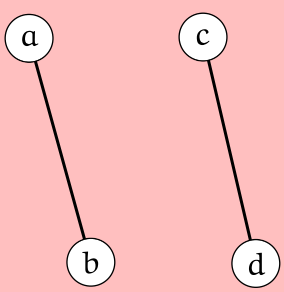
Let $G = (V, E)$ be a graph, and R be the associated polynomial ring. For ease of language, we note that any subset of V corresponds to a monomial. The edge ideal was the ideal generated by the edges of G. One could similarly define, for $t \geqslant 2$, the ideal $I_t(G)$ which is generated by all the t-paths of G.
As an attempt to generalise Fröberg's result, one might when is $I_t(G)$ possessing linear resolution. One result in this direction is the following.

Theorem 5.1 ([Ban17]). If G is a gap-free and claw-free graph, then $I_t(G)$ has linear resolution for all $t \geqslant 3$.

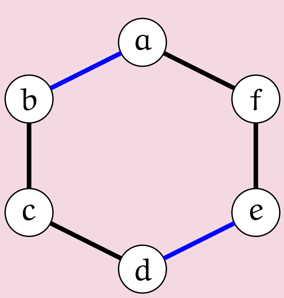
We will define the above terms more generally in a bit. One takeaway is that by prohibiting certain graphs to be induced subgraphs, we get nice properties for $I_t(G)$.

Definition 5.2. We say that a graph (V, E) is t-gap-free if whenever C and C' are two disjoint connected subsets of V, then there is an edge joining a vertex of C to a vertex of C'.
The term gap-free simply stands for 2-gap-free.

Remark 5.3. G being gap-free is the same as saying that G does not contain the following as an induced subgraph:



Example 5.4. Consider the 6-cycle:



C_6 is not gap-free in view of the blue edges. However, C_6 is 3-gap-free.

Recall that $K_{1,t}$ is the graph with $V = \{0, \ldots, t\}$ and $E = \{(0, i) : 1 \leqslant i \leqslant t\}$.

Definition 5.5. A graph is called t-claw-free if it contains no induced subgraph isomorphic to $K_{1,t}$.
The term claw-free simply stands for 3-claw-free.

§6. Connected ideals

Path ideals could be viewed as one generalisation of the edge ideal. A bigger generalisation would be to consider the following class.

Definition 6.1. Given a graph G and $t \geqslant 2$, let $J_t(G)$ be the ideal generated by the connected subsets of G of size t.

Note that $I_t(G) \subseteq J_t(G)$ with equality for $t = 2, 3$. In general, the containment can be strict.

Remark 6.2. $J_t(G)$ can be viewed as an edge ideal of an associated *hypergraph*. Using [HW14, Theorem 1.4], it is relatively straightforward to show that

$$J_t(G) \text{ has a linear resolution} \Rightarrow G \text{ is t-gap-free.}$$

(See [AJM24, Corollary 4.3].)

This project began as an attempt to prove the converse.

Theorem 6.3 ([AJM24])**.** Let T be a tree, i.e., a connected graph with no cycles. For each $t \geqslant 2$, the following are equivalent.

(a) $J_t(T)$ has linear quotients.

(b) $J_t(T)$ has a linear resolution.

(c) T is t-gap-free

Note that for $t = 2$, we recover Fröberg's result for trees.

The above does not hold in general. Indeed, C_5 is (2-)gap-free but $J_2(C_5)$ does not have linear resolution, for it is not co-chordal. In fact, we showed that every cycle (on $\geqslant 5$) is a counterexample to the above for a suitable t.

Sketch for Theorem 6.3. We prove this by induction on $|V(T)|$. If $|V(T)| = t$, this is clear. Assume $|V(T)| > t$.
Let ℓ be a leaf of T. Then, $T \setminus \{\ell\}$ is an induced subgraph and hence, t-gap-free. By hypothesis, there is an admissible order on $G(J_t(T \setminus \{\ell\}))$ (recall Definition 3.3). Furthermore,

$$G(J_t(T)) = G(J_t(T \setminus \{\ell\})) \sqcup \{\text{connected subsets of size } t \text{ containing } \ell\}.$$

We showed in the paper that appending the extra generators in any order gives an admissible order. □

Theorem 6.4 ([AJM24]). Let $t \geqslant 3$ be an integer. Suppose G is a gap-free and t-claw-free graph. Then, $J_t(G)$ has linear quotients. In particular, $J_t(G)$ has a linear resolution.

References

[AJM24] H. Ananthnarayan, Omkar Javadekar, and Aryaman Maithani. "Linear quotients of connected ideals of graphs". In: *arXiv e-prints*, arXiv:2401.01046 (Jan. 2024), arXiv:2401.01046. DOI: 10.48550/arXiv.2401.01046. arXiv: 2401.01046 [math.AC].

[Ban17] Arindam Banerjee. "Regularity of path ideals of gap free graphs". In: *J. Pure Appl. Algebra* 221.10 (2017), pp. 2409–2419. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2016.12.005. URL: https://doi.org/10.1016/j.jpaa.2016.12.005.

[Frö90] Ralf Fröberg. "On Stanley-Reisner rings". In: *Topics in algebra, Part 2 (Warsaw, 1988)*. Vol. 26, Part 2. Banach Center Publ. PWN, Warsaw, 1990, pp. 57–70. ISBN: 83-01-09579-2.

[HW14] Huy Tài Hà and Russ Woodroofe. "Results on the regularity of square-free monomial ideals". In: *Adv. in Appl. Math.* 58 (2014), pp. 21–36. ISSN: 0196-8858,1090-2074. DOI: 10.1016/j.aam.2014.05.002. URL: https://doi.org/10.1016/j.aam.2014.05.002.

[HHZ04] Jürgen Herzog, Takayuki Hibi, and Xinxian Zheng. "Monomial ideals whose powers have a linear resolution". In: *Math. Scand.* 95.1 (2004), pp. 23–32. ISSN: 0025-5521,1903-1807. DOI: 10.7146/math.scand.a-14446. URL: https://doi.org/10.7146/math.scand.a-14446.

[JZ10] Ali Soleyman Jahan and Xinxian Zheng. "Ideals with linear quotients". In: *J. Combin. Theory Ser. A* 117.1 (2010), pp. 104–110. ISSN: 0097-3165,1096-0899. DOI: 10.1016/j.jcta.2009.03.012. URL: https://doi.org/10.1016/j.jcta.2009.03.012.