Gorenstein Rings

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§1. Dualising complexes

Let R be a commutative noetherian ring. A complex ω_R is a dualising complex if

(1) $\omega_R \in D^b(\mathbf{mod}\text{-}R)$,

(2) $\operatorname{injdim}(\omega_R) < \infty$, i.e., ω_R is quasi-isomorphic to a bounded complex of injectives,

(3) $R \xrightarrow{\simeq} \operatorname{RHom}_R(\omega_R, \omega_R)$.

ω_R is called semi-dualising if (1) and (3) hold.

Remark 1.1. R is always semi-dualising. Thus, R is dualising iff $\operatorname{injdim} R < \infty$.
Recall that R is said to be regular if $\operatorname{gldim}(R_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R$.
In particular, if R is regular local, then the global dimension of R is finite and thus, $\operatorname{injdim}(R) < \infty$. Thus, R is even a dualising complex.

Remark 1.2 (Existence)**.** Dualising complexes need not exist. Indeed, if $\omega_R \simeq I^\bullet$ is a minimal injective resolution, then up to shifts, one has that

$$I^{-n} \cong \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec}(R) \\ \dim(R/\mathfrak{p}) = n}} E_R(R/\mathfrak{p}),$$

where $E_R(-)$ denotes the injective hull.

The key point here is that every prime shows up exactly once, at location prescribed by its "co-height". In particular, if $\mathfrak{p} \subsetneq \mathfrak{q}$ are primes, then any two saturated chains of prime ideals joining $\mathfrak{p}$ and $\mathfrak{q}$ must have the same length. That is, R must be catenary.
However, as Nagata showed, there exists a noetherian local domain $(R, \mathfrak{m})$ of dimension three that is not catenary.

Remark 1.3 (Uniqueness)**.** Even if ω_R exists, it need not be unique. For one, $\Sigma^n \omega_R$ is another dualising complex, for all $n \in \mathbb{Z}$.
Slightly less trivial, if P is a projective R-module of rank 1 (i.e., $P_\mathfrak{p} \cong R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$), then $\omega_R \otimes_R P$ is again dualising.

But this is all: Given any two dualising complexes, one can obtain the other by applying the above operations successively. In particular, if R is local, then ω_R is unique up to shifts. (In fact, one can set $P := \Sigma^n \operatorname{RHom}_R(\omega_R, \omega_R')$ for a suitable shift n.)

Theorem 1.4 (Local Duality Theorem)**.** Let R be a commutative noetherian ring with a dualising complex ω_R. The functor

$$(-)^\dagger := \operatorname{RHom}_R(-, \omega_R) : D(R)^{op} \to D(R)$$

restricts to auto-equivalences

$$\begin{array}{ccc} D^b(\mathbf{mod}\text{-}R)^{op} & \overset{\cong}{\longleftrightarrow} & D^b(\mathbf{mod}\text{-}R) \\ \cup\! \mid & & \cup\! \mid \\ \operatorname{Perf}(R)^{op} & \overset{\cong}{\longleftrightarrow} & \operatorname{InjPerf}(R), \end{array}$$

where $\operatorname{Perf}(R)$ (resp. $\operatorname{InjPerf}(R)$) is the subcategory of objects with finite projective (resp. injective) dimension.

In fact, for $M \in D^b(\mathbf{mod}\text{-}R)$, one has that the natural map $M \to M^{\dagger\dagger}$ is a quasi-isomorphism. This is seen by checking that the map factors as

$$M \xrightarrow{\simeq} M \otimes_R^{\ell} \operatorname{RHom}_R(\omega_R, \omega_R) \xrightarrow{\simeq} \operatorname{RHom}_R(\operatorname{RHom}_R(M, \omega_R), \omega_R).$$

Connection to Matlis duality. If we further assume that $(R, \mathfrak{m}, k)$ is local and $\ell(M) < \infty$, then we see that M is $\mathfrak{m}$-torsion. Then, letting $I^\bullet$ be the minimal injective resolution (as described below), we see that

$$M^\dagger = \operatorname{Hom}_R(M, I^\bullet) = \operatorname{Hom}_R(M, E_R(k))$$

since every other $\operatorname{Hom}_R(M, E_R(R/\mathfrak{p}))$ must vanish. In particular, $M^\dagger = M^\vee$. (Strictly speaking, we first "normalise" the complex appropriately.)
Thus, Matlis duality is a special case of local duality.

§2. Gorenstein rings

Definition 2.1. A noetherian local ring $(R, \mathfrak{m})$ is Gorenstein if $\operatorname{injdim} R < \infty$.
More generally, a noetherian ring R is Gorenstein if $R_\mathfrak{m}$ is Gorenstein for all $\mathfrak{m} \in \operatorname{Max} R$.

In the case that $\dim(R) < \infty$, R is Gorenstein iff $\operatorname{injdim}(R) < \infty$. In this case, being Gorenstein is the same as R being self-dualising.
Side note: Recall Nagata's example of a noetherian domain with infinite Krull dimension. This is a regular ring and hence, Gorenstein. However, this ring has infinite injective dimension.

Remark 2.2. One can show that R being Gorenstein (in our sense) is equivalent to $\operatorname{RHom}_R(-, R)$ being an auto-equivalence on $D^b(\mathbf{mod}\text{-}R)$.

For the occasional noncommutative ring A, we shall use Gorenstein to mean that $\operatorname{injdim}({}_A A) < \infty$ and $\operatorname{injdim}(A_A) < \infty$. This is what was defined as (Iwanaga-)Gorenstein in an earlier talk.

Example 2.3. Let k be an arbitrary field, and G a finite group. The group ring kG is Gorenstein since it is self-injective. In the case that $\operatorname{char}(k) \nmid |G|$, this follows since kG is semisimple and thus, every kG-module is injective.

As we remarked earlier, we have

$$\text{regular} \Rightarrow \text{Gorenstein}.$$

In fact, one can check the following.

Lemma 2.4. Let $(R, \mathfrak{m})$ be local, and $x \in \mathfrak{m}$ be a nonzerodivisor. Then,

$$R \text{ is Gorenstein} \Leftrightarrow R/xR \text{ is Gorenstein}.$$

Corollary 2.5. For local rings, we have: Regular $\Rightarrow$ complete intersection $\Rightarrow$ Gorenstein.

Theorem 2.6. For a noetherian local ring $(R, \mathfrak{m}, k)$ of Krull dimension d, the following are equivalent:

1. R is Gorenstein.
2. $\operatorname{Ext}^n_R(k, R) = 0$ for $n \neq d$ and $\operatorname{Ext}^d_R(k, R) \cong k$.

For a local ring $(R, \mathfrak{m}, k)$ with $d := \operatorname{depth}(R)$, we define the type of R to be the d-th *Bass number*:

$$\operatorname{type}(R) := \operatorname{rank}_k \operatorname{Ext}_R^{\operatorname{depth}(R)}(k, R).$$

The above is simply the number of copies of $E_R(k)$ that appear in the d-th spot of the minimal injective resolution of R.

Then, has the following.

Theorem 2.7. Let $(R, \mathfrak{m}, k)$ be local. Then,

$$R \text{ is Gorenstein} \Leftrightarrow R \text{ is CM and } \operatorname{type}(R) = 1.$$

In particular, if R is artinian, then R is Cohen-Macaulay with $d = 0$. This gives us the following.

Theorem 2.8. Let $(R, \mathfrak{m}, k)$ be local artinian. TFAE:

1. R is Gorenstein.
2. $\operatorname{type}(R) = 1$.
3. $\operatorname{Hom}_R(k, R) \cong k$.
4. $\operatorname{soc}(R)$ is one-dimensional.

Note that $\operatorname{Hom}_R(k, R) = (0 :_R \mathfrak{m}) =: \operatorname{soc}(R)$ is the socle of R. This is the largest submodule of R which has a k-module structure.
Note that in the zero-dimensional case, the type is the dimension of the socle.

Example 2.9. The ring $R = k[X, Y]/(X^2, Y^2)$ is an artinian ring. The socle is one-dimensional, being generated by xy. This was the example of the group ring $\mathbb{F}_2 V_4$ which we saw yesterday.

On the other hand, if we quotient further to get $R = k[X, Y]/(X^2, XY, Y^2)$, then the socle is two dimensional: generated by x and y.

Both the rings *are* Cohen-Macaulay, being zero-dimensional.

Theorem 2.10 (Watanabe)**.** Let K be a field of characteristic zero. Let G be a finite subgroup of $GL_n(K)$ acting on $S := K[x_1, \ldots, x_n]$.
If $G \leqslant SL_n(K)$, then S^G is Gorenstein.
If G contains no pseudoreflections, then the converse holds too.

Recall that an element $g \in GL_n(k)$ is called a pseudoreflection if $\operatorname{rank}(g - I) = 1$.

Example 2.11. $\mathbb{C}[x^2, xy, y^2]$ is Gorenstein, but $\mathbb{C}[x^3, x^2y, xy^2, y^3]$ is not. These appear as invariant rings of $\left\langle \begin{bmatrix} \zeta & \\ & \zeta \end{bmatrix} \right\rangle$ for $\zeta = -1$ and $\exp(2\pi\iota/3)$. This is in $SL_2(\mathbb{C})$ precisely in the former case.
More generally, the d-th Veronese of $\mathbb{C}[x_1, \ldots, x_n]$ is Gorenstein iff $d \mid n$.

$\mathbb{C}[x^n, xy, y^n]$ is also Gorenstein for all $n \geqslant 1$. This is the invariant ring corresponding $\left\langle \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} \right\rangle$ for $\zeta = \exp(2\pi\iota/n)$.

§3. Symmetry

We now look at some more examples of Gorenstein rings.

Example 3.1 (Poincaré Algebras)**.** Suppose that R is a graded k-algebra of the form

$$R = k \oplus R_1 \oplus \cdots \oplus R_d$$

with $\operatorname{rank}_k R < \infty$ and $R_d \neq 0$. Assume R is either commutative or graded-commutative. R being graded gives us bilinear maps

$$\langle -, - \rangle_i : R_i \times R_{d-i} \to R_d$$

for all $i \in [d]$. Equivalently, we have maps

$$\rho_i : R_i \to \operatorname{Hom}_k(R_{d-i}, R_d).$$

Then, the following are equivalent:

1. R is Gorenstein.
2. $R_d \cong k$ and $\langle -, - \rangle_i$ is nondegenerate for all i.
3. ρ_i is a bijection for all i.
4. $R \cong \operatorname{Hom}_k(R, k)[d]$ as R-modules. (Note that $A = k$ is a Noether normalisation since R is finite-dimensional over k.)

Note that in this case, we must necessarily have $\operatorname{rank}_k(R_i) = \operatorname{rank}_k(R_{d-i})$.

Example 3.2. Poincaré Duality tells us that for a compact manifold M, the ring $H^*(M; \mathbb{F}_2)$ is Gorenstein.
For example, $k[x]/(x^{n+1})$ is Gorenstein, as can be seen using the criteria above. (For $k =$

$\mathbb{F}_2$, this ring is the cohomology ring of $\mathbb{R}P^n$.)

Example 3.3. As a concrete example of the above, one can create many zero-dimensional Gorenstein rings as follows: Let V be any finite-dimensional k-vector space, and let $\langle -, - \rangle$ be any (anti-)symmetric nondegenerate bilinear form on V. Then, give

$$R := k \oplus V \oplus k$$

a graded ring structure in the obvious way.

In fact, our earlier example $R = k[X, Y]/(X^2, Y^2)$ fits in this form. We have

$$R = k \oplus k \cdot \{x, y\} \oplus k \cdot \{xy\}$$

with the pairing being given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Similarly, the example $k[X, Y]/(X^2, XY, Y^2)$ can be seen as non-Gorenstein using the above criteria since the top graded piece has dimension > 1.

On the other hand, $k[X, Y]/(X^3, XY, Y^2) \cong k \oplus k^2 \oplus k$ is not Gorenstein because the pairing is degenerate. (Note that the dimensions are still palindromic!)

Let us consider $R = k \oplus k \cdot \{x, y, z\} \oplus k \cdot T$ with pairing given by

$$z^2 = xy = yx = T,$$

and all other pairs in $\{x, y, z\} \times \{x, y, z\}$ get mapped to zero. This is nondegenerate as this is given by the invertible matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the ring R is Gorenstein. In fact, it is not too difficult to check that we have

$$R \cong k[X, Y, Z]/(X^2, Y^2, XZ, YZ, Z^2 - XY).$$

One can also show that the ideal above does not have fewer than 5 generators. Thus, R gives an example of a Gorenstein ring which is not a complete intersection.

Example 3.4. Suppose R is a noetherian $\mathbb{N}$-graded ring with $K := R_0$ a field. Let $A \subseteq R$ be a Noether normalisation. Then, R is Gorenstein iff R is Cohen-Macaulay and there exists

$q \in \mathbb{Z}$ such that $\operatorname{Hom}_A(R, A) \cong R[q]$ as graded R-modules.

Using this, one gets that "the" numerator of the Hilbert series is a palindrome. More precisely: consider $A \subseteq R$, where A is a homogeneous Noether normalisation and R is Cohen-Macaulay. Say $A = K[f_1, \ldots, f_n]$, $\deg f_i = k_i$. Then,

$$\operatorname{Hilb}(A, t) = \frac{1}{\prod(1 - t^{k_i})} \quad \text{and} \quad \operatorname{Hilb}(R, t) = \frac{c_0 + \cdots + c_m t^m}{\prod(1 - t^{k_i})},$$

where c_j is the number of basis elements of degree j in a homogeneous A-basis for R. Then, $\operatorname{Hom}_A(R, A) \cong R[q]$ gives us that

$$(c_0, \ldots, c_m) = (c_m, \ldots, c_0),$$

since $\operatorname{Hom}_A(-, A)$ negates the degrees of the basis elements.

(We assume $c_m \neq 0$.)

A more inherent symmetry is given as:

$$\operatorname{Hilb}(R, t^{-1}) = (-1)^{\dim R} t^{\ell} \operatorname{Hilb}(R, t) \tag{3.1}$$

for some $\ell \in \mathbb{Z}$.

In fact, Stanley proved a converse as well: If R is a Cohen-Macaulay graded domain satisfying (3.1), then R is Gorenstein.
Without the additional hypothesis, (3.1) does not suffice: we already have an example from earlier.

Example 3.5 (Numerical monoids)**.** A numerical monoid Σ is a subset $\Sigma \subseteq \mathbb{N}_0$ such that

(1) $0 \in \Sigma$,

(2) Σ is closed under addition,

(3) $\mathbb{N}_0 \setminus \Sigma$ is finite.

Let k be a field. Corresponding to Σ above, we get a ring

$$k[\Sigma] := k[x^i : i \in \Sigma] \subseteq k[x].$$

That is, $k[\Sigma]$ is the k-subalgebra of $k[x]$ generated by $\{x^i : i \in \Sigma\}$.

By (3), there exists a smallest $c \geqslant 0$ such that $x^{\geqslant c} \in k[\Sigma]$. Then, R is Gorenstein iff exactly half the numbers in $[0, c-1]$ are not in Σ. Equivalently, the holes are anti-palindromic, i.e., for all $0 \leqslant i \leqslant c-1$, $i \in \Sigma \Leftrightarrow c - 1 - i \notin \Sigma$.

Example 3.6. As concrete examples, one can check $R = K[x^3, x^5, x^7]$ is not Gorenstein. (Check with $c = 5$.)

On the other hand, $R = K[x^4, x^5, x^6]$ is Gorenstein. (Check with $c = 8$.)