ABELIAN EXTENSIONS OF EQUICHARACTERISTIC REGULAR RINGS NEED NOT BE COHEN-MACAULAY

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To Professor Paul Roberts, on the occasion of his eightieth birthday

ABSTRACT. By a theorem of Roberts, the integral closure of a regular local ring in a finite abelian extension of its fraction field is Cohen-Macaulay, provided that the degree of the extension is coprime to the characteristic of the residue field. We show that the result need not hold in the absence of this requirement on the characteristic: for each positive prime integer p, we construct polynomial rings over fields of characteristic p, whose integral closure in an elementary abelian extension of order p^2 is not Cohen-Macaulay. Localizing at the homogeneous maximal ideal preserves the essential features of the construction.

1. INTRODUCTION

Paul Roberts [Ro80] proved the following:

Theorem 1.1 (Roberts)**.** *Let R be a regular local ring with fraction field K. Let L be a finite Galois extension of K with an abelian Galois group. Assume moreover that the order of the Galois group is not divisible by the characteristic of the residue field of R. Let S denote the integral closure of R in L. Then S is a Cohen-Macaulay ring.*

If one instead assumes that R is a UFD instead of a regular local ring, with the other hypotheses still in place, Roberts observes that the proof yields that S is a free R-module; if the requirement on the characteristic is dropped, [Ro80, Example 1] provides a UFD R of mixed characteristic 2, and an extension field L of degree 2 over $\operatorname{frac}(R)$, such that if S denotes the integral closure of R in L, then S is not a free R-module; in fact, no nonzero S-module is free over R.

Returning to the case where R is regular, the abelian hypothesis cannot be weakened to solvable or nilpotent in view of [Ro80, Example 2]. Additionally, Koh demonstrated that Theorem 1.1 may fail in the absence of the requirement on the order of the group: [Ko, Example 2.4] is an example of a regular local ring R of mixed characteristic 3, with an extension field L having Galois group $\mathbb{Z}/3\mathbb{Z}$ over $\operatorname{frac}(R)$, such that the integral closure of R in L is not Cohen-Macaulay. Nevertheless, there has been work exploring extensions of Roberts's theorem by two distinct methods: in terms of tracking ramification in codimension one, and in terms of constructing birational maximal Cohen-Macaulay modules, see [Ka, KS, Sr1, Sr2]. The purpose of this brief note is to record equal characteristic examples where the integral closure of a regular ring in an elementary abelian extension of order p^2 is not Cohen-Macaulay—indeed, the Cohen-Macaulay defect can be arbitrarily large—and to demonstrate that such examples arise quite naturally in the context of modular invariant theory. While the examples are recorded in the framework of $\mathbb{N}$-graded rings, the relevant properties are preserved upon localization at the respective homogeneous maximal ideals.

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2. MODULAR INVARIANT RINGS

Let V be a finite rank vector space over a field; an element of $\mathrm{GL}(V)$ is a *bireflection* if it fixes a subspace of V of codimension at most 2. The following is [Ke, Corollary 3.7]:

Theorem 2.1 (Kemper). *Let K be a field of characteristic $p > 0$, and H a finite subgroup of $\mathrm{GL}_n(K)$, with its natural action on the polynomial ring $T := K[x_1, \dots, x_n]$.*

If H is a p-group, and T^H is Cohen-Macaulay, then H is generated by bireflections.

Example 2.2. For each positive prime integer p, we construct a polynomial ring R over the finite field $\mathbb{F}_p$, and an elementary abelian extension L of $\mathrm{frac}(R)$ with order p^2, such that the integral closure of R in L is not Cohen-Macaulay. The ring R will be obtained as the invariant ring T^G for the action of a group G on a polynomial ring T, while the extension ring S that is not Cohen-Macaulay equals T^H, for a subgroup H of G. Schematically,

$$\begin{array}{c} T := \mathbb{F}_p[x_1, y_1, x_2, y_2, x_3, y_3] \\ | \\ S := T^H \\ | \\ R := T^G, \end{array}$$

with R being a polynomial ring.

Let G be the subgroup of $\mathrm{GL}_6(\mathbb{F}_p)$ consisting of the matrices

$$\left[\begin{array}{cc|cc|cc} 1 & a & & & & \\ & 1 & & & & \\ \hline & & 1 & b & & \\ & & & 1 & & \\ \hline & & & & 1 & c \\ & & & & & 1 \end{array}\right] \qquad \text{for } a, b, c \in \mathbb{F}_p.$$

The group G is abelian, isomorphic to the direct product of three copies of $\mathbb{Z}/p\mathbb{Z}$. Consider the linear action of G on the polynomial ring T, where $M \in G$ acts via

$$M\colon X \longmapsto MX,$$

with $X := (x_1, y_1, x_2, y_2, x_3, y_3)^{\mathrm{tr}}$ denoting the column vector of indeterminates. We claim that T^G, i.e., the invariant ring for the action of G on T, is the polynomial ring

$$R := \mathbb{F}_p[y_1\,,\ x_1^p - x_1 y_1^{p-1}\,,\ y_2\,,\ x_2^p - x_2 y_2^{p-1}\,,\ y_3\,,\ x_3^p - x_3 y_3^{p-1}].$$

It is readily seen that $R \subseteq T^G$. For each i, the element $x_i \in T$ is a root of the polynomial

$$Z^p - Z y_i^{p-1} - \left(x_i^p - x_i\, y_i^{p-1}\right) \ \in\ R[Z],$$

so T is integral over R and $[\mathrm{frac}(T) : \mathrm{frac}(R)] \leqslant p^3$. Since G has order p^3, it follows that

$$\mathrm{frac}(R) \ =\ \mathrm{frac}(T)^G \ =\ \mathrm{frac}(T^G),$$

where the second equality holds since G is finite. Since R has dimension 6 and is generated by the 6 elements displayed, R is a polynomial ring, hence normal. Each element of T^G lies in $\mathrm{frac}(R)$ and is integral over R, so the normality of R yields that $R = T^G$, as claimed.

We note that while one direction of the Shephard-Todd-Chevalley-Serre theorem may fail in the modular case, [Se, page 3], one may instead use [Na, Theorem 1.4] or Remark 2.3

below to conclude that the invariant ring T^G is a polynomial ring; this is in lieu of the direct proof that we have chosen to include above.

Next, consider the cyclic subgroup H of G generated by the matrix

$$\left[\begin{array}{cc|cc|cc} 1 & 1 & & & & \\ & 1 & & & & \\ \hline & & 1 & 1 & & \\ & & & 1 & & \\ \hline & & & & 1 & 1 \\ & & & & & 1 \end{array}\right].$$

Since H is a p-group that is *not* generated by bireflections—the only bireflection in H is the identity—Theorem 2.1 implies that the invariant ring $S := T^H$ is not Cohen-Macaulay.

To summarize, we have constructed a polynomial ring R with a finite normal extension ring S that is not Cohen-Macaulay; the Galois group of $\mathrm{frac}(S)$ over $\mathrm{frac}(R)$ is

$$G/H \;\cong\; \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z},$$

the elementary abelian group of order p^2. Note that S is the integral closure of R in $\mathrm{frac}(S)$.

We mention that the ring S is a UFD since there is no nontrivial character $H \longrightarrow \mathbb{F}_p^\times$, see, for example, [Be, Corollary 3.9.3]. While S is not Cohen-Macaulay, it evidently admits a small Cohen-Macaulay algebra, namely the polynomial ring T.

Remark 2.3. Extending the example above, let P be an abelian p-group acting on a finite rank $\mathbb{F}_p$-vector space V, such that the invariant ring $\mathbb{F}_p[V]^P$ is polynomial; this holds, for example, if the fixed subspace V^P has codimension one, see, for example, [CW, Theorem 3.9.2]. For an integer $d \geqslant 3$, consider the d-fold product $G := P \times \cdots \times P$ acting on the polynomial ring $T := \mathbb{F}_p[V^{\oplus d}]$. Then the invariant ring T^G is the tensor product of d copies of $\mathbb{F}_p[V]^P$, hence a polynomial ring. Set H to be the diagonal copy of P in G, which contains no bireflections other than the identity, since $d \geqslant 3$. One obtains a tower of rings

$$\begin{array}{c} \mathbb{F}_p[V^{\oplus d}] \\ | \\ S := \mathbb{F}_p[V^{\oplus d}]^H \\ | \\ R := \mathbb{F}_p[V^{\oplus d}]^G, \end{array}$$

where S is not Cohen-Macaulay in view of Kemper's result, Theorem 2.1. The Galois group of $\mathrm{frac}(S)$ over $\mathrm{frac}(R)$ is the abelian group G/H.

Specifically, suppose P is the cyclic group generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

with its usual action on the rank two vector space $\mathbb{F}_p^2$. Let G be the product of d copies of P for an integer $d \geqslant 3$, and let T and H be as above; note that $\dim T = 2d$. The fact that the invariant ring $S := T^H$ is not Cohen-Macaulay also follows from the work of Ellingsrud and Skjelbred: [ES, Corollaire 2.4] implies that S does not satisfy the Serre condition S_3, while [ES, Corollaire 3.2] implies that S has depth $d+2$. In particular, the Cohen-Macaulay defect of the ring S is $2d - (d+2) = d - 2$.

REFERENCES

[Be] D. J. Benson, *Polynomial invariants of finite groups*, London Math. Soc. Lecture Note Ser. **190**, Cambridge University Press, Cambridge, 1993. 3

[CW] H. E. A. Campbell and D. L. Wehlau, *Modular invariant theory*, Encyclopaedia Math. Sci. **139**, Springer-Verlag, Berlin, 2011. 3

[ES] G. Ellingsrud and T. Skjelbred, *Profondeur d'anneaux d'invariants en caractéristique p*, Compositio Math. **41** (1980), 233–244. 3

[Ka] D. Katz, *On the existence of maximal Cohen-Macaulay modules over p-th root extensions*, Proc. Amer. Math. Soc. **127** (1999), 2601–2609. 1

[KS] D. Katz and P. Sridhar, *On abelian extensions in mixed characteristic and ramification in codimension one*, Int. Math. Res. Not. **11** (2025), rnaf153. 1

[Ke] G. Kemper, *On the Cohen-Macaulay property of modular invariant rings*, J. Algebra **215** (1999), 330–351. 2

[Ko] J. Koh, *Degree p extensions of an unramified regular local ring of mixed characteristic p*, J. Algebra **99** (1986), 310–323. 1

[Na] H. Nakajima, *Regular rings of invariants of unipotent groups*, J. Algebra **85** (1983), 253–286. 2

[Ro80] P. Roberts, *Abelian extensions of regular local rings*, Proc. Amer. Math. Soc. **78** (1980), 307–310. 1

[Se] J. P. Serre, *Groupes finis d'automorphismes d'anneaux locaux réguliers*, in: Colloque d'Algèbre (Paris, 1967), École normale supérieure de jeunes filles **8**, Secrétariat mathématique, Paris, 1968. 2

[Sr1] P. Sridhar, *On the existence of birational maximal Cohen-Macaulay modules over biradical extensions in mixed characteristic*, J. Pure Appl. Algebra **225** (2021), paper no. 106790. 1

[Sr2] P. Sridhar, *Existence of birational small Cohen-Macaulay modules over biquadratic extensions in mixed characteristic*, J. Algebra **582** (2021), 100–116. 1

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