Lecture 1 (09-01-2023)

Monday, January 9, 2023 1:23 PM

PLAN.

\# Bruns & Herzog → Cohen-Macaulay rings
- 1st Part

\# Affine algebra

X

Derived Category

- Gelfand & Manin
- E. Riehl's : Cat theory in context
- Dwyer & Spalinski

$R \to$ ring (possibly non comm.)

R-complexes :

$$\cdots \to M_{i+1} \xrightarrow{\partial} M_i \xrightarrow{\partial} M_{i-1} \to \cdots \qquad \partial^2 = 0$$

$$\mathrm{im}(\partial_{i+1}) \subset \ker(\partial_i)$$

$$H_i(M) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

$$H(M) = (H_i(M))_{i \in \mathbb{Z}}$$

$\mathcal{C}(R)$ = category of R-complexes (morphisms as usual)

If $f : M \to N$, we get an induced map $H(f) : H(M) \to H(N)$.

Def^n f is a quasi isomorphism (or weak equivalence) if $H(f)$ is bijective. (Automatically iso.)

$W :=$ collection of weak equivalences in $\mathcal{C}(R)$

$D(R) := \mathcal{C}(R)[W^{-1}]$. (or $W^{-1}\mathcal{C}(R)$)

- Key property : W has the 2-out-of-6 property:

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot,$

- Key property: W has the 2-out-of-6 property:

Given composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$,

if gf and $hg \in W$, then f, g, h, hgf are in W.

Ex: $\Rightarrow$ 2-out-of-3 property

If f, g, fg are defined and 2 are in W, then so is the third.

Concretely: $\mathcal{C}(R) \rightsquigarrow K(R) \rightsquigarrow D(R)$.

$\uparrow$ homotopy category

$M, N \rightarrow$ R-complexes

$\mathrm{Hom}_R(M, N) :=$ Hom-complex of abelian groups (when R is comm. this is an R-complex)

$\mathrm{Hom}_R(M, N)_n :=$ Maps of degree n from M to N (no compatibility!)

$$\begin{tikzcd} \cdots \arrow[r] & M_i \arrow[r] \arrow[dl] & M_{i-1} \arrow[r] \arrow[dl] & \cdots \\ \cdots \arrow[r] & N_{i+n} \arrow[r] & N_{i+n-1} \arrow[r] & \cdots \end{tikzcd}$$

$$= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$$

$$\partial : \mathrm{Hom}_R(M, N)_{n+1} \longrightarrow \mathrm{Hom}_R(M, N)$$

$$\partial(f) = \partial^N f - (-1)^{n+1} f \partial^M.$$

Check: $\partial^2 = 0$.

Observe: $Z_0(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_{\mathcal{C}}(M,N)$.

Def. $f, g \in \mathrm{Hom}_{\mathcal{C}}(M,N)$ are homotopic if

$f - g \in B_0(\mathrm{Hom}_R(M,N))$, i.e.,

$f - g = \partial h$ for some $h \in \mathrm{Hom}_R(M,N)_1$

$K(R) := \mathcal{C}(R) / \text{homotopy relation.}$

Object = R-complexes

$\mathrm{Hom}_K(M,N) = H_0(\mathrm{Hom}_R(M,N))$.

- $f \sim g$ in $\mathcal{C}$ $\Rightarrow$ $f = g$ in $K(R)$.

  $\Rightarrow H(f) = H(g)$

Def^n. M an R-complex.
ΣM (or $M[1]$) is the R-complex

$$(\Sigma M)_i = M_{i-1}$$

with $\partial^{\Sigma M} = -\partial^M$.

$\mathrm{Proj}\, R :=$ Projective R-modules

$$\begin{tikzcd} K(R) \arrow[dr, "q"] & \\ \cup\!\!\mid & \\ K(\mathrm{Proj}\, R) \arrow[r, shift right, "q"'] & D(R) \arrow[l, dashed, shift right, "p"'] \end{tikzcd}$$

q = localisation

$\exists p : D(R) \to K(\mathrm{Proj}\ R)$, a full and faithful embedding

"projective resolutions"

left adjoint to q.

$$\mathrm{Hom}_K(pM, N) = \mathrm{Hom}_D(M, qN)$$

$f : M \to N$ morphism

$\mathrm{cone}(f) := N \oplus \Sigma M$ with differential

$$\begin{matrix} N_{i+1} \\ \oplus \\ M_i \end{matrix} \longrightarrow \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \qquad \partial = \begin{bmatrix} \partial^N & f \\ 0 & -\partial^M \end{bmatrix}$$

$$0 \to N \hookrightarrow \mathrm{cone}(f) \to \Sigma M \to 0.$$

f is w.e. $\Leftrightarrow$ $H(\mathrm{cone}(f)) = 0$.

Image of p?

K-projectives.

P an R-complex is K-projective if given any solid diagram

$$\begin{tikzcd} & M \arrow[d, "\pi"', "\wr"] & \text{w.e.} \\ P \arrow[r, "\alpha"'] \arrow[ur, dashed, "\tilde{\alpha}"] & N & \end{tikzcd}$$

, $\exists$ lift $\tilde{\alpha}$.

FACT: $p : D(R) \xrightarrow{\sim} K\text{-}\mathrm{Proj}(R) \subseteq K(\mathrm{Proj}\, R)$.

morphisms here are up to homotopy

• $\mathrm{Hom}_{\mathcal{C}}(R, M) = Z_0(M)$

$$\begin{tikzcd} 0 \arrow[d] & \\ R \arrow[r] \arrow[d] & M_0 \arrow[d, "\partial"] \\ 0 \arrow[r] & M_{-1} \end{tikzcd}$$

Using this,

check: R is K-projective.

Use: surjective + w.e.

$\Downarrow$

$Z(M) \to Z(N)$ onto

• $(P_\lambda)_\lambda$ family of K-projectives

Then, $\oplus P_\lambda$ is also K-projective.

Conversely closed under direct summands.

• K-projectives are closed under suspensions.

$$\cdots \xrightarrow{\partial} P_{i+1} \xrightarrow{\partial} P_i \xrightarrow{\partial} P_{i-1} \xrightarrow{\partial} \cdots$$

is K-proj, if P_i projective $\forall i$.

Ex: $0 \to P' \xrightarrow{f} P \xrightarrow{g} P'' \to 0$ exact seq. of complexes.

Then, if P' and P'' are K-projective, so is P.

If P', P, P'' are complexes of projectives, then any two being K-projective $\Rightarrow$ third is K-proj.

any two being K-projective $\Rightarrow$ third is K-proj.

<u>Corollary</u>. Any bounded complex of projectives is K-projective.

Proof. $P_\bullet$: $0 \to P_b \dashrightarrow P_a \to 0$, each P_i proj.

Induce on $b-a$.

$b-a = 0$ done earlier.

$$0 \to \underset{\substack{\parallel \\ 0 \to P_{b-1} \to \cdots \to P_a \to 0}}{P_{\leq b-1}} \to P \to \Sigma^b P_b \to 0. \quad \blacksquare$$

Next: Any complex of projectives with $P_i = 0$ $\forall i \ll 0$ is K-projective.

$$P_\bullet : \quad \cdots \to P_{a+1} \to P_a \to 0$$

$P = \underset{\substack{\longrightarrow \\ n \geq a}}{\mathrm{colim}}\, P_{\leq n}$, each $P_{\leq n}$ is projective since bounded.

$$0 \to \bigoplus_n P_{\leq n} \xrightarrow{1-\delta} \bigoplus_n P_{\leq n} \to P \to 0.$$

K-proj

projectives

Use 2-out-of-3.

Do directly...

- $\begin{array}{c} M \\ \pi \downarrow \text{w.e.} \\ N \end{array}$ $\quad$ P is K-projective $\Leftrightarrow$ $\mathrm{Hom}_K(P, M) \cong \mathrm{Hom}_K(P, N)$.

Lecture 2 (11-01-2023)

11 January 2023 13:24

R ring.

$D(R) \simeq k\mathrm{Proj}(R)$.

Recall: $P \in \mathcal{C}(\mathrm{Proj}\ R)$ is K-projective if

$$\begin{tikzcd} & X \arrow[d, "\simeq"] \\ P \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Example $P \in \mathcal{C}(\mathrm{Proj}\ R)$ with $P_i = 0$ for all $i \ll 0$.

$$\cdots \to P_{a-1} \to P_a \to 0 \to 0 \to \cdots$$

Sketch. Construct lifting one step at a time.

Suppose $\tilde{\alpha} : P_{\leq n} \to X$ is a lifting.

Want $\tilde{\alpha} : P_{\leq n+1} \to X$ compatibly.

Let $s \in P_{n+1}$

We must have

$$\begin{tikzcd} & & X \arrow[d, "\varepsilon", "\simeq"'] \\ \cdots \arrow[r] & P_{n+1} \arrow[r] & (P_n \to P_{n-1} \to \cdots \to P_a \to 0) \arrow[r, "\alpha"] \arrow[u, "\tilde{\alpha}"] & Y \end{tikzcd}$$

- $\varepsilon(\tilde{\alpha}(s)) = \alpha(s)$
- $\partial \tilde{\alpha}(s) = \tilde{\alpha}(\partial s)$

Check that the above can be solved.

This uses three things:
- ε surjective $\Rightarrow$ ε surjective on boundaries
- $H(\varepsilon)$ iso $\Rightarrow$ ε surjective on cycles
- $\ker(\varepsilon)$ is acyclic.

$\Rightarrow$ Every module has a K-projective resolution.

$$\begin{tikzcd} F: \cdots & F_2 \arrow[r, "\partial_2"] \arrow[dr] & F_1 \arrow[r, "\partial_1"] \arrow[dr] & F_0 \arrow[dr, "\varepsilon"] & \\ & & \ker \partial_1 \arrow[u] & \ker \varepsilon \arrow[u] & M \end{tikzcd}$$

(Can avoid choices by taking generating set to be M, $\ker \varepsilon$, $\ker \partial_1, \ldots$)

$$F \xrightarrow{\varepsilon} M.$$

Def^n: A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism

$$\varepsilon : P_\cdot \longrightarrow M \quad \text{s.t.}$$

① ε is a quasi iso,

② $P_\cdot$ is K-projective.

(Not insisting surjective.)

② $P_\cdot$ is K-projective.

This makes it functorial!

Thm $\forall\, M_\cdot \in \mathcal{C}(R)$, $\exists$ surjective K-projective resolution:

$$\underset{\text{K-proj.}}{P_\cdot} \xrightarrow{\sim} M_\cdot$$

Defn. An R-complex F is semi-free if F admits a filtration:

$$(0) = F(0) \subseteq F(1) \subseteq \cdots \subseteq \bigcup_{n \geq 0} F(n) = F$$

s.t. ① $F(n) \subseteq F$ is a subcomplex,

② $F(n+1)/F(n)$ graded free module with $\partial = 0$,

i.e. $\partial(F(n+1)) \subseteq F(n)$

Fact/Ex. Semifree $\Rightarrow$ K-proj.

Example. $\cdots \to F_{a+1} \to F_a \to 0$

$$F(n) = F_{\leq n} = 0 \to F_n \to \cdots \to F_a \to 0$$

$$\frac{F(n+1)}{F(n)} = \cdots \to 0 \to F(n+1) \to 0 \to \cdots$$

$$= \Sigma^{n+1} F_{n+1}.$$

Thm. Each $M_\cdot \in \mathcal{C}(R)$ has a surjective semi-free resolution

$$\underset{\text{K-projective}}{F_\cdot} \xrightarrow{\sim} M_\cdot.$$

Corollary. Every K-projective is a retract of a semifree.

Proof. P is K-projective:

$$\begin{tikzcd} & F \arrow[d, "\wr"] \\ P \arrow[ur, dashed] \arrow[r, equal, "\mathrm{id}"'] & P \end{tikzcd}$$

$\blacksquare$

→ **Sketch** (Baby ver of Quillen's "small object argument"):

Construct $0 = F(0) \hookrightarrow F(1) \xhookrightarrow{i(1)} F(2) \xhookrightarrow{i(2)} \cdots \hookrightarrow F(n) \xhookrightarrow{i(n)} \cdots \hookrightarrow \bigcup F(n) = F$

Construct

$$\begin{tikzcd} 0 = F(0) \arrow[r, hook] \arrow[dr, "\varepsilon(0)"'] & F(1) \arrow[r, hook, "i(1)"] \arrow[d, "\varepsilon(1)"] & F(2) \arrow[r, hook, "i(2)"] \arrow[dl, "\varepsilon(2)"] & \cdots \arrow[r, hook] & F(n) \arrow[r, hook, "i(n)"] \arrow[dll, bend left, "\varepsilon(n)"] & \cdots \arrow[r, hook] & \bigcup_{n \geq 0} F(n) = F \\ & M & & & & & \end{tikzcd}$$

s.t. ① $F(n+1)/F(n)$ is graded free with $\partial = 0$,

② $\varepsilon(1)$ is surjective on homology.

(In turn, each $\varepsilon(n)$ is surjective on homology.)

③ $\ker(H(\varepsilon(n))) \subseteq H(F(n))$ maps to 0 under $H(i(n))$.

This does the job. ["Something 0 in colim is 0 at finite stage."]

Why is ε surjective?

Remark. $\varepsilon: X \to Y$ s.t. $Z(\varepsilon)$ surjective + $H(\varepsilon)$ bijective. Then, ε is surjective.

Indeed, we have s.e.s.es:

$$\begin{tikzcd} 0 \arrow[r] & B(X) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & H(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B(Y) \arrow[r] & Z(Y) \arrow[r] & H(Y) \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \Sigma B(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z(Y) \arrow[r] & Y \arrow[r] & \Sigma B(Y) \arrow[r] & 0 \end{tikzcd}$$

Snake lemma $\Downarrow$ $B(X) \to B(Y)$ is epi $\Downarrow$ $X \to Y$ is surj

Construction of $F(n), \varepsilon(n)$:

$\varepsilon(1): F(1) \to M$ free cover of cycles. ($\varepsilon(1)$ morphism since mapping on cycles)

0 diffs

Say we have constructed $\varepsilon(n): F(n) \longrightarrow M$.

Choose cycles $(z_\lambda)_\lambda \subseteq F(n)$ that map to a generating set of $\ker(H(\varepsilon(n)))$

Pick w_λ s.t. $\partial(w_\lambda) = \varepsilon(n)(z_\lambda)$.

Set $F(n+1) = F(n) \oplus R e_\lambda$ $\quad \deg(e_\lambda) = \deg(z_\lambda) + 1$.

with $\partial|_{F(n)} = \partial^{F(n)}$

$$\partial(e_x) = z_x.$$

Define $\varepsilon(n+1) : F(n+1) \to M$
$$\varepsilon(n+1)|_{F(n)} = \varepsilon(n),$$
$$\varepsilon(n+1)(e_x) = \omega_x.$$
$\boxtimes$

Remarks. ① As before the above construction can be made functorial by avoiding choices (consider all choices!).

② Depending on what we wish to do with the resolution, there are other constructions.

Given a module, we have the graded homology module $H(M) = \langle H_i(M) \rangle_{i \in \mathbb{Z}}$.

(Recall: for us, a graded module is a collection of modules.)

If $P. \xrightarrow{\sim} H(M)$ is a projective res^n, one can "perturb" the differentials of $P.$ to construct a K-projective resolution of M.

(Adam's resolution, Cartan/Eilenberg resolution)

Exercise. $P.$ K-proj $\Rightarrow$ P_i projective $\forall i$.

Converse of above NOT true.

Example (Dold's): Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the complex

$$P. : \quad \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} \cdots$$

One way of seeing that the above is not K-projective is to do the following exercise and note that $P.$ is acyclic but not contractible.

is to do the following exercise ...

$P_\cdot$ is acyclic but not contractible.

Exercise. If P is K-proj and $H(P) = 0$, then P is contractible, i.e., $\mathrm{id}_P \sim 0$.
(or: P is the mapping cone of some id.)

We saw we have an inclusion
$$\mathrm{KProj}(R) \hookrightarrow K(\mathrm{Proj}\ R).$$

FACT. Let R be comm. Noetherian.
The above inclusion is an equality iff R is regular.

Examples of reg. rings: $\mathbb{Z}$, $k[x_1, \dots, x_n]$.

Derived functors

Let $M_\cdot$ be an R-complex.

FACT. If $P_\cdot \xrightarrow{\sim} M_\cdot$ and $Q_\cdot \xrightarrow{\sim} M_\cdot$ are K-projective resolutions, then $P_\cdot \simeq Q_\cdot$ in $K(\mathrm{Proj}\ R)$.

Given any $N_\cdot \in \mathcal{C}(R)$, set
$$\mathrm{RHom}_R(M_\cdot, N_\cdot) := \mathrm{Hom}_R(P_\cdot, N_\cdot),$$
where $P_\cdot$ is a K-proj. res^n of $M_\cdot$

The object on the right is defined in the homotopy category of abelian groups, i.e.,
$\mathrm{RHom}_R(-, N_\cdot)$ is a functor
$$\mathcal{C}(R) \to K(\mathbb{Z}).$$

$R\mathrm{Hom}_R(-, N.)$ is a functor
$$\mathcal{C}(R) \longrightarrow K(\mathbb{Z}).$$
(If R is comm, then $\longrightarrow K(R)$.)

Define $\mathrm{Ext}^i_R(M., N.) := H^i(R\mathrm{Hom}_R(M., N.))$
$$= H_{-i}(\mathrm{Hom}_R(P., N.))$$

$\mathrm{Ext}^0_R(M, N) = H_0(\mathrm{Hom}_R(P., N.))$
= morphisms $P. \to N.$, up to homotopy

$\mathrm{Ext}^i_R(M, N) =$ — " — $P. \to \Sigma^i N.$, — " —

If $Q \xrightarrow{\sim} N$ is a K-proj resol^n, then
$$\mathrm{Hom}_R(P., Q.) \cong \mathrm{Hom}_R(P., N.).$$
quasi - iso

Tensors Let $M.$ be a chain complex of right R-modules.

Let $N. \in \mathcal{C}(R)$.

$M. \otimes N.$ is a complex of $\mathbb{Z}$-modules defined by
$$(M. \otimes N.)_i = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_i$$

$$\partial(m \otimes n) = \partial m \otimes n + (-1)^{|m|} m \otimes \partial n$$

FACT. If $X. \xrightarrow{\sim} Y.$ is a quasi iso, then $P. \otimes_R X. \xrightarrow{\sim} P. \otimes Y.$ for any K-proj $P.$

Defn. $M_\cdot \otimes_R^{L} N_\cdot := P_\cdot \otimes_R N_\cdot$ where $P_\cdot \xrightarrow{\sim} M_\cdot$ is a K-proj. resolⁿ.

$$\mathrm{Tor}_i^R(M_\cdot, N_\cdot) = H_i(P_\cdot \otimes_R N_\cdot).$$

$X \xrightarrow{\sim} Y$ quasi iso $\Rightarrow \mathrm{Tor}_i^R(M_\cdot, X_\cdot) = \mathrm{Tor}_i^R(M_\cdot, Y_\cdot)$.

Lecture 3 (18-01-2023)

Wednesday, January 18, 2023 1:26 PM

$R \to$ comm. Noetherian ring

$M \to$ R-module

$r \in R$ is a zerodivisor on M if $r \cdot m = 0$ for some $m \neq 0$.

nzd = "not a zero divisor"

$$Z_R(M) = \{ r \in R : r \text{ is a z.d. on } M \}$$

$$= \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

(M need not be finite. Union need not be)

Fix R, M. Let $\underline{x} = x_1, \dots, x_n$ be a sequence in R.

$\underline{x}$ is weakly M-regular or a weakly regular sequence on M

if

$$x_{i+1} \text{ is nzd on } \frac{M}{(x_1, \dots, x_i)M} \quad \text{for } 0 \leq i \leq n-1.$$

$\underline{x}$ is M-regular (or ...) if further $M/(\underline{x})M \neq 0$.

Ex. $R = k[x_1, \dots, x_n]$.
$\underline{x} := x_1, \dots, x_n$ is a regular sequence on R.

Koszul Complexes. Given $r \in R$,

$$K(r; R) = 0 \to \underset{\deg 1}{R} \xrightarrow{r} \underset{\deg 0}{R} \to 0.$$

$H_1(K(r;R)) = 0 \iff r$ is nzd on R.

Given $\underline{x} = x_1, \dots, x_n$, we define

$$K(\underline{x}; R) = \bigotimes_{i=1}^{n} K(x_i; R).$$

$$K(\underline{x}; R) = 0 \to R \xrightarrow{\pm \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{\cdot (x_1 \cdots x_n)} R \to 0$$

$$K(\underline{x};R) = 0 \to R \xrightarrow{\pm\begin{bmatrix} x_1 \\ -x_2 \\ \vdots \end{bmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \xrightarrow{\cdot(x_1 \cdots x_n)} R^n \to R \to 0$$

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \mapsto \sum r_i x_i$$

Now, given $M \in \mathcal{C}(R)$,

$$K(\underline{x};M) := K(\underline{x},R) \otimes M.$$

↳ Koszul complex on $\underline{x}$ with coefficients in M.

$$H_i(\underline{x};M) := H_i(K(\underline{x};M)).$$ → Koszul homology

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If M is simply an R-module (viewed in degree 0), then

$K(\underline{x};M)$:

$$0 \to M \to M^n \to \cdots \to M^n \to M \to 0$$

"same" differentials

$$H_0(\underline{x};M) = M/\underline{x}M,$$

$$H_n(\underline{x};M) = \{m \in M : x_i m = 0 \ \forall i\}$$
$$= (0 :_M (\underline{x})).$$

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① $K(\underline{x};M) = K(x_1;R) \otimes_R K(x_2;R) \otimes_R \cdots \otimes K(x_n;R) \otimes M$

$$= K(x_1;R) \otimes K(x_{\geq 2}, M)$$

$$= K(x_1; K(x_{\geq 2}, M))$$

② $X, Y \in \mathcal{C}(R) \rightsquigarrow X \otimes_R Y \xrightarrow{\sim} Y \otimes_R X$ as R-complexes.

$$x \otimes y \longmapsto (-1)^{|x||y|} \, y \otimes x$$

$\therefore\ K(\underset{\sim}{x}; R) \cong K(\underset{\sim}{x}^\sigma; R)$ for any $\sigma \in S_n$.

$\Rightarrow K(\underset{\sim}{x}; M) \cong K(\underset{\sim}{x}^\sigma; M)$ —"—.

(Can apply this to Obs ①.)

2nd Perspective: "Koszul complexes are iterative mapping cones."

$f: X \to Y$ morphism of complexes

$$\mathrm{Cone}(f) = \left(Y \oplus \Sigma X,\ \begin{pmatrix} \partial^Y & f \\ & -\partial^X \end{pmatrix}\right).$$

s.e.s. : $0 \to Y \to \mathrm{Cone}(f) \to \Sigma X \to 0.$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$

$\begin{pmatrix} y \\ x \end{pmatrix} \mapsto x$

Homology l.e.s. reads

$$H_i(X) \xrightarrow[H_i(f)]{} H_i(Y) \to H_i(\mathrm{Cone}(f)) \to H_i(\Sigma X) \to \cdots$$

$H_i(\Sigma X) \cong H_{i-1}(X)$

$H_i(f)$ connecting map

Consider: $x \in R$

$f: R \xrightarrow{x} R$

$1 \mapsto x$

$$\mathrm{Cone}(R \xrightarrow{x} R) = \left(\underset{\deg 0}{R} \oplus \underset{\deg 1}{R},\ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = K(x; R).$$

Ditto: If $x \in R$, and $M \in \mathcal{C}(R)$ → complex, not necessarily in deg 0

$$\mathrm{Cone}(M \xrightarrow{x} M) = K(x; M)$$

$\underline{x} = x_1, x_2, \dots, x_n$

$$K(\underline{x};M) = K(x_1 ; K(x_{\geq 2};M))$$

$$= \mathrm{Cone}\left(K(x_{\geq 2};M) \xrightarrow{x_1} K(x_{\geq 2};M)\right)$$

on homology

iterate

$\vdots$

$$H_i(x_{\geq 2};M) \xrightarrow{\pm x_1} H_i(x_{\geq 2};M) \to H_i(\underline{x};M) \to H_{i-1}(x_{\geq 2};M) \xrightarrow{\pm x_1} \vdots$$

s.e.s.

$$0 \to \frac{H_i(x_{\geq 2};M)}{x_1 H_i(x_{\geq 2};M)} \to H_i(\underline{x};M) \to (0 :_{H_{i-1}(x_{\geq 2};M)} x_1) \to 0$$

$M \to$ R-module

$$K(\underline{x};M) \rightsquigarrow H_0(x;M) = \frac{M}{\underline{x}M}$$

So, $K(\underline{x},M) \to \frac{M}{\underline{x}M}$ is a w.e. (quasi iso)

$\Leftrightarrow H_i(\underline{x};M) = 0 \quad \forall i > 0.$

Defn. $\underline{x}$ is Koszi-regular on M (or...) if

$$H_i(\underline{x};M) = 0 \quad \forall i \geq 1.$$

Lemma. When $\underline{x}$ is weakly M-reg, $(M \to R\text{-mod})$

(weakly-reg $\Rightarrow$ Koszi-reg)

$$K(\underline{x},M) \twoheadrightarrow M/(\underline{x})M \text{ is a w.eq.}$$

Proof. $n=1$: $\quad 0 \to M \xrightarrow{x} M \to 0.$

$H_1(x;M) = 0 \Leftrightarrow x$ nzd on M

$n \geq 2$: $\quad K(\underline{x};M) = K(x_n ; K(x_{<n};M)).$

By induction,

$$K(x_{<n}; M) \xrightarrow{\sim} \frac{M}{(x_{<n})M}.$$

Now, $0 \to R \xrightarrow{x_n} R \to 0$ is K-proj. (Even semifree.)

$$\Rightarrow K(\underline{x}; M) = K(x_n; R) \otimes_R K(x_{<n}; M)$$

$$\simeq K(x_n; R) \otimes_R \frac{M}{(x_{<n})M}$$

now note that x_n is a nzd on $\cancel{\frac{M}{(x_{<n})M}}$ and we are done. ∎

Instead of semifree, can use l.e.s. of homology and induction.

(Semifree business $\Rightarrow$ ($M \simeq N$ quasi $\Downarrow$ $K(\underline{x}; M) \simeq K(\underline{x}; N)$ quasi))

Note: ① $\underline{x}$ Koszul-reg $\Rightarrow$ $\underline{x}^\sigma$ is Koszul-reg $\forall \sigma \in S_n$.

② Not true for weakly regular.

Theorem. Say $\underline{x} \subseteq J(R)$ and $M \neq 0$ f.g. R-module.

TFAE:

1) $\underline{x}$ is M-regular. ($\equiv$ weakly M-reg, by NAK.)

2) $\underline{x}$ is Koszul M-regular, i.e., $H_i(\underline{x}; M) = 0$ $\forall i \geq 1$.

3) $H_1(\underline{x}; M) = 0$.

In particular, take R local and x_i nonunits.

Proof. ① $\Rightarrow$ ② $\Rightarrow$ ③ is clear.

Only need to prove ③ $\Rightarrow$ ①.

Already saw for $n=1$.

Induction:

$$K(\underline{x}; M) = K(x_n; K(x_{<n}; M)).$$

l.e.s.

$$0 \to \frac{H_i(x_{<n};M)}{x_n H_i(x_{<n};M)} \to H_i(\underline{x};M) \to (0 :_{H_{i-1}(x_{<n};M)} x_n) \to 0. \quad (*)$$

Put $i=1$ to get $\dfrac{H_1(x_{<n};M)}{x_n H_1(x_{<n};M)} = 0$

$$\xRightarrow{\text{NAK}} H_1(x_{<n};M) = 0.$$

(Note: Koszul homology modules are f.g. when M is f.g.)

$\xRightarrow{\text{induction}}$ $x_1, \ldots, x_{n-1}$ is M-reg. — (1)

Moreover, $(*)$ now tells us

$$(0 :_{H_0(x_{<n};M)} x_n) = 0.$$

$$H_0(x_{<n};M) = \ker\left(\frac{M}{x_{<n}M} \xrightarrow{x_n} \frac{M}{x_{<n}M}\right).$$

$\therefore$ x_n is nzd on $M/(x_{<n})M$. — (2)

(1) & (2) finish. ∎

Corollary. $\mathfrak{I} \subseteq J(R)$, M f.g., the property of $\underline{x}$ being regular is not dependent on the order of x_i.

(Permutation of regular is regular.)

$R = k[x, y, z]$

$x, y(1-x), z(1-x)$ reg

$y(1-x), z(1-x), x$ NOT

Lemma. If $\underline{x} = x_1, \ldots, x_n \subseteq R$, M an R-module.

TFAE:

① $\underline{x}$ is M-Koszul-regular.

② $\underline{x}^{\underline{a}}$ is M-Koszul-regular for some $\underline{a} \geq \underline{1}$.

Proof. Suffices to prove:

$x_1, \boxed{x_2, \ldots, x_n}$ is M-KR

$\Leftrightarrow x_1^a, \boxed{x_2, \ldots, x_n}$ is M-KR for some (/all) $a \geq 1$.

$x_1, x_2, \ldots, x_n$ KR

$$\Rightarrow K(x_1 ; K(x_{\geq 2} ; M)) \xrightarrow{\sim} K\left(x_1 ; \frac{M}{(x_{\geq 2})M}\right).$$

Replacing M by $M/(x_{\geq 2})M$ we are reduced to $n = 1$.

But

x is reg on M

$\Leftrightarrow x$ is nzd on M

$\Leftrightarrow x^a$ is nzd on M for some $a \geq 1$

$\Leftrightarrow x^a$ is reg on M —— " ——.

Theorem. (Rigidity of Koszul homology)

$\underline{x} \subset J(R)$ and M f.g. R-module.

Let $i \geq 0$ be s.t. $H_i(\underline{x} ; M) = 0$.

Then, $H_j(\underline{x} ; M) = 0 \quad \forall\, j \geq i$.

HW.

Lecture 4 (23-01-2023)

Monday, January 23, 2023 1:19 PM

$R \to$ commutative Noetherian

Given complexes, $M, N \in \mathcal{C}(R)$.

$$\mathrm{RHom}_R(M,N) := \mathrm{Hom}_R(pM, N)$$

$pM \xrightarrow{\sim} M$ is a K-proj res^n

$$\mathrm{Ext}^*_R(M,N) := H^i(\mathrm{RHom}_R(pM,N)).$$

Given $M, N, P \in \mathcal{C}(R)$, we have a morphism

$$\theta : \mathrm{RHom}_R(M,N) \otimes^L_R P \longrightarrow \mathrm{RHom}_R(M, N \otimes^L_R P).$$

Lemma. θ is a w.e. when P is perfect, i.e.,

$P \simeq (0 \to P_b \to \cdots \to P_a \to 0)$,

each P_i f.g. R-module.

$$\left.\begin{aligned} \mathrm{Hom}_R(pM, N) \otimes_R pP &\longrightarrow \mathrm{Hom}_R(pM, N \otimes_R pP) \\ f \otimes x &\longmapsto \left(m \mapsto (-1)^{|m||x|} f(m) \otimes x\right) \end{aligned}\right\} \text{ in } \mathcal{C}(R)$$

Check this is a morphism.

Proof. Replace P with $0 \to P_b \to \cdots \to P_a \to 0$.

"Things true for $R \rightsquigarrow$ true for perfect."

Then it reduces to checking for one f.g. projective P.

But that can be reduced to f.g. free.

That reduces to R. Obvious. ∎

Rees' Lemma

Setup. $\underline{x} = x_1, \dots, x_c \subset R$ finite subset.

$N \rightsquigarrow R$-module s.t. $\underline{x} N = 0$.

$M \rightsquigarrow R$-module s.t. $\underline{x}$ is Koszul-regular on M.

i.e. $K(\underline{x}; M) \xrightarrow{\sim} M/\underline{x}M$

Lemma. Then,
$$\mathrm{RHom}_R(N, M/\underline{x}M) \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \Lambda^*(\Sigma R^c).$$

In particular,
$$\mathrm{Ext}^*_R\left(N, \frac{M}{\underline{x}M}\right) \cong \mathrm{Ext}^*_R(N, M) \otimes_R \Lambda^*(\Sigma R^c).$$

$\Lambda^*(\Sigma R^c) : 0 \to \underset{\text{deg } c}{R} \xrightarrow{0} R^c \xrightarrow{0} \cdots \xrightarrow{0} R^{\binom{c}{2}} \xrightarrow{0} R^c \to \underset{\text{deg } 0}{R} \to 0$

Proof.
$$\begin{aligned}
\mathrm{RHom}_R(N, M/\underline{x}M) &\simeq \mathrm{RHom}_R(N, K(\underline{x}; M)) \\
&\simeq \mathrm{RHom}_R(N, M \otimes^L_R K(\underline{x}; R)) \\
&\simeq \mathrm{RHom}_R(N, M) \otimes^L_R K(\underline{x}; R)
\end{aligned}$$
perfect

Since $\underline{x} N = 0$, $\underline{x}\, \mathrm{Ext}^*_R(N, M) = 0$.

Alter: $\mathrm{RHom}_R(N, M) \simeq \mathrm{Hom}_R(N, I)$ where $M \xrightarrow{\sim} I$ is an injective res^n.

Now, $\underline{x} \cdot \mathrm{Hom}_R(N, I) = 0$.

$\therefore$

$$\begin{aligned}
&\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{x}; R) \\
&\simeq K(\underline{x}; \mathrm{Hom}_R(N, I)) \\
&\simeq K(\underset{\text{length } c}{\underline{0}}; \mathrm{Hom}_R(N, I)) \\
&\simeq \mathrm{Hom}_R(N, I) \otimes_R K(\underline{0}; R)
\end{aligned}$$

$$\cong \operatorname{Hom}_R(N, I) \otimes_R K(Q; R)$$

$$= \operatorname{Hom}_R(N, I) \otimes_R \Lambda^*(\Sigma R^c)$$

Last statement means:

$$\operatorname{Ext}^n_R\left(N, \frac{M}{\underline{x}M}\right) \cong \left(\operatorname{Ext}_R(N, M) \otimes_R \Lambda(\Sigma R^c)\right)^n$$

$$= \bigoplus_i \operatorname{Ext}^i_R(N, M) \otimes_R (\Lambda(\Sigma R^c))^{n-i} \quad \text{think}$$

$$= \bigoplus_i \operatorname{Ext}^i_R(N, M) \otimes_R R^{\binom{c}{i-n}}.$$

$V \to \mathbb{Z}$-graded object

View V as having both an upper grading and lower grading via $V^i = V_{-i}$.

Notation:

$$\sup V^* = \sup\{i : V^i \neq 0\}$$

$$\inf V^* = \inf\{i : V^i \neq 0\}.$$

$$\sup V_* = \sup\{i : V_i \neq 0\} \quad \ldots$$

$$= -\inf V^*$$

Corollary. $\rho := \inf \operatorname{Ext}^*_R(N, M) = \inf \operatorname{Ext}^*_R(N, M/\underline{x}M) + c.$

$$\operatorname{Ext}^\rho_R(N, M) = \operatorname{Ext}^{\rho - c}_R(N, M/\underline{x}M)$$

Defn. Fix $I \subseteq R$ ideal. $M \in \mathcal{C}(R)$.

$$\operatorname{depth}_R(I, M) := \inf \operatorname{Ext}^*(R/I, M).$$

I-depth of M

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences / of complexes on . .

Properties. ① $0 \to L \to M \to N \to 0$ exact sequences (of complexes or modules).

Then, $\operatorname{depth}_R(I, M) \geq \min\{\operatorname{depth}_R(I,L), \operatorname{depth}_R(I,N)\}$.

Similarly relations with other twos using l.e.s.

$$\to \operatorname{Ext}^i_R(R/I, L) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, N)$$
$$\to \operatorname{Ext}^{i+1}_R(R/I, L) \to \cdots$$

Thm (depth).

② Let $\underline{x} = x_1, \dots, x_c$ be a gen. set for the ideal I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M) \qquad \forall M \in \mathcal{C}(R).$$

Will prove when M is an R-module.

(recall Koszul homology)

Koszul complexes revisited

Giving $x_1, \dots, x_c$ in R

$\leftrightarrow$ Giving a map $f: F \to C$, where F is a free mod of rank c and chosen basis

$K(f) := (\Lambda^*(\Sigma F), \partial)$, where ∂ is

$$e_{i_1} \wedge \dots \wedge e_{i_n} \longmapsto \sum_{j=1}^{n} (-1)^{j-1} f(e_{i_j})\, x_{i_1} \wedge \dots \wedge \hat{x}_{i_j} \wedge \dots \wedge x_{i_n}.$$

Lemma. Fix $\underline{x} = x_1, \dots, x_c \subseteq R$. For any $y \in (\underline{x})$, we have

$$K(\underbrace{\underline{x}, y}_{c+1 \text{ seq.}}; M) \cong K(\underline{x}, 0; M)$$
$$\cong K(\underline{x}; M) \otimes (0 \to R \to \underset{\text{deg } 0}{R} \to 0).$$

Proof.

$$\begin{array}{ccc} R^{c+1} & \xrightarrow{[x_1 \ \cdots \ x_c \ y]} & R \\ \uparrow \cong & \text{?} & \| \end{array}$$

$\left[\begin{array}{ccc|c} \ddots & & & -r_1 \\ & \ddots & & \vdots \\ & & & -r_c \\ \hline & & & 1 \end{array}\right]$

$y = \sum r_i x_i$

$$\begin{bmatrix} \ddots & & -x_1 \\ & & \vdots \\ & & -x_c \\ & & 1 \end{bmatrix} \Big\uparrow \cong \quad R^{c+1} \xrightarrow[{[x_1 \cdots x_c \;\; 0]}]{\quad \supseteq \quad} R \qquad \cdots \qquad y = \sum r_i x_i$$

In particular,

$$\sup H_*(\underline{x}, y; M) = 1 + \sup H_*(\underline{x}; M)$$

Thus,

$$c+1 - \sup H_*(\underline{x}, y; M) = c - \sup H_*(\underline{x}; M).$$

Corollary. $c - \sup_* H(\underline{x}, y; M)$ is independent of a generating set of I. (If $\underline{x}$, $\underline{y}$ generate, look at $\underline{x}, \underline{y}$ & $\underline{y}, \underline{x}$.)

Proof of Thm (Depth) When M is a module

$$\begin{aligned}
\operatorname{depth}(I, M) = 0 &\iff \operatorname{Hom}(R/I, M) \neq 0 \\
&\iff I \subseteq \operatorname{Zdv}_R(M) \\
&\iff H_c(\underline{x}; M) \neq 0 \\
&\iff \sup H_*(\underline{x}; M) = c.
\end{aligned}$$

Actually, $H_c(\underline{x}; M) \cong \operatorname{Hom}(R/I, M)$

Can assume $\operatorname{depth}_R(I, M) \geq 1$, i.e., $\exists y \in I$ nzd on M

Then, in particular, y is Koszul-reg on M.

By Rees, $\because y(R/I) = 0$

$$\inf \operatorname{Ext}^*_R(R/I, M/yM) = \inf \operatorname{Ext}^*_R(R/I, M) - 1.$$

$$\therefore \operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R\left(I, \frac{M}{yM}\right)$$

induction now applies

$$= 1 + \left(c - \sup H_*\left(\underline{x}; \frac{M}{yM}\right)\right)$$

$H_*(\underline{x}; M/yM)$

$($ $yM))$

$$\begin{aligned} & H_*(\underline{x};\ M/yM) \\ \cong\ & H_*(\underline{x};\ K(y;M)) \\ \cong\ & H_*(K(\underline{x},y;M)) \\ \cong\ & H_*(K(\underline{x},0;M)) \end{aligned}$$

$$= c + 1 - \sup H_*(\underline{x},0;\ M)$$

$$= c - \sup H_*(\underline{x};M).$$

Ex. Show $\operatorname{depth}_R(I, M)$ is the length of longest __ M-regular in I.

Lecture 5 (25-01-2023)

Wednesday, January 25, 2023 1:19 PM

$R \to$ comm. noetherian ring

$I \subsetneq R$ ideal

$M \to$ R-complex

$$\operatorname{depth}(I, M) = \inf \operatorname{Ext}^*_R(R/I, M)$$

$I = (x_1, \ldots, x_c)$

$$= c - \sup H_*(\underline{x}; M)$$

- When M is a module, $\operatorname{depth}(I, M) =$ length of any maximal M-Koszul-regular sequence in I

(Maybe $IM \neq M$ needed?)

If M f.g. and $I \subseteq \operatorname{Jac}(R)$

$=$ length of any maximal M-reg. seq in I

Observation. $\underline{x} = x_1, \ldots, x_c$

$\underline{y} = y_1, \ldots, y_d$

$$\sup H_*(\underline{x}, \underline{y}; M) \leq \sup H_*(\underline{x}; M) + d.$$

(Can use l.e.s. to see this.)

$$K(\underline{x}, \underline{y}; M) \cong K(\underline{x}; K(\underline{y}; M))$$

This implies

$$I \subseteq J \Rightarrow \operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(J, M).$$

Also, $\sqrt{I} = \sqrt{J} \Rightarrow \operatorname{depth}_R(I, M) = \operatorname{depth}_R(J, M)$.

Today. $(R, \mathfrak{m}, k)$ is a local ring:

- R is commutative Noetherian,
- $\mathfrak{m}$ is the unique maximal ideal of R,

- $k = R/\mathfrak{m}$.

In this case, $\operatorname{depth}(M) = \mathfrak{m}\text{-depth of } M$

$$= \inf \operatorname{Ext}^*_R(k, M).$$

It suffices to compute the above using $\underline{x} = x_1, \ldots, x_c$

s.t. $\sqrt{(\underline{x})} = \mathfrak{m}$.

Thus we can take c minimal as $c = \dim(R)$.

(Then, $\underline{x}$ is a system of parameters.)

$\therefore$ Can compute using $\dim(R)$ elements.

Ausland-Buchsbaum Equality

- $F \to$ an R-complex

F has finite flat dimension if

$$F \simeq (0 \to F_b \to \cdots \to F_a \to 0)$$

with F_i flat.

We write $\operatorname{flat\,dim}_R F < \infty$.

Examples
- Flat modules.
- Perfect complexes.
- Koszul complexes.

If $\operatorname{flat\,dim}_R F < \infty$, then $\operatorname{Tor}_i(-, F) = 0 \quad \forall\, |i| \gg 0$

on Mod R.

$$\mathrm{Tor}_i^R(M, F) = H_i\big(M \otimes_R (0 \to F_b \to \cdots \to F_a \to 0)\big)$$

$$= 0 \quad \text{for } i \notin [a, b].$$

In fact, the above characterises flat $\dim_R F < \infty$.

Theorem (AB equality) (R, m, k) local.

$F \to$ finite flat dimension. Then,

$$\mathrm{depth}_R(M \otimes^L_R F) = \mathrm{depth}_R M - \underbrace{\sup H_*(k \otimes^L_R F)}_{\mathrm{Tor}_*^R(k, F)}$$

for **ANY** R-complex M.

Specialise: ① $M = R$.

$$\mathrm{depth}_R(F) = \mathrm{depth}_R(R) - \sup H_*(k \otimes^L_R F).$$

Now, let N be a f.g. R-module.

Such an N has a minimal free resolution.

$$\begin{tikzcd}
\cdots R^{b_2} \arrow[rr, "\partial_2"] \arrow[dr, two heads] & & R^{b_1} \arrow[rr, "\partial_1"] \arrow[dr, two heads, "\min"] & & R^{b_0} \arrow[r, two heads, "\varepsilon"] & N \text{ minimally} \\
& \ker \partial_1 \subseteq m R^{b_1} \arrow[ur, hook] & & \ker \varepsilon \arrow[ur, hook] &
\end{tikzcd}$$

$\ker \varepsilon \subseteq m R^{b_0}$, by minimality

This gives a complex

$$\begin{array}{c} G : (\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to 0) \\ \qquad\qquad\qquad\qquad \downarrow \simeq \\ \qquad\qquad\qquad\qquad N \end{array}$$

$\partial G \subseteq m G$. G turns out to be unique up to isomorphism of complexes.

up to isomorphism of complexes.

"The" minimal free resolution of N.

$$\mathrm{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes G)_i.$$

H_i = everything since $\partial \otimes k = 0$

$$\therefore \quad \mathrm{Tor}_i^R(k, N) = 0 \iff G_i = 0.$$

$\therefore$ flat $\dim_R N < \infty \iff N$ has a finite free resolution, i.e., N is perfect.

$$\sup \mathrm{Tor}_*^R(k, N) = \text{length of } G$$
$$=: \operatorname{proj} \dim_R N.$$

② If N is a f.g. R-module with $\operatorname{proj}\dim_R N < \infty$, then

$$\operatorname{proj}\dim_R(N) + \operatorname{depth}_R(N) = \operatorname{depth}_R(R).$$

(Classical AB Equality.)

<u>Corollary</u>. $\operatorname{proj}\dim_R N < \infty \Rightarrow \operatorname{depth}(N) \leq \operatorname{depth}(R)$.

Furthermore, equality iff N is projective.

False without assumption of proj dim $< \infty$

Note we had:

$$\operatorname{depth}_R(F) = \operatorname{depth}_R(R) - \sup H_*(k \otimes_R^L F)$$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F)$$

Subtract:

$$\operatorname{depth}_R(F) - \operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}(R) - \operatorname{depth}(M).$$

Note: Some terms above may be ∞.
when $H_i(\underline{x}; M) = 0 \ \forall i$, then
$\sup H_*(\underline{x}; M) = -\infty$.

Proof of AB Equality

$$\operatorname{RHom}_R(k, M \otimes_R^L F) \xleftarrow{\simeq} \operatorname{RHom}_R(k, M) \otimes_R^L F$$

quasi iso because k is a f.g. module
$\therefore \exists\, G \xrightarrow{\simeq} k$ where each G_i is finite free
AND flat dim $F < \infty$.

[More generally:
$\operatorname{Hom}_R(N, M) \otimes R \xrightarrow{\simeq} \operatorname{Hom}_R(N, M \otimes_R F)$
N f.g. R-mod, F flat]

Key observation:

$\operatorname{RHom}_R(k, M) \simeq$ complexes of k-vector spaces
(take injective resolution of M)

$$\operatorname{RHom}_R(k, M) \otimes_R^L F \simeq \operatorname{RHom}_R(k, M) \otimes_k^L (k \otimes_R^L F)$$

$$\therefore \operatorname{Ext}_R^*(k, M \otimes_R^L F) = \operatorname{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^L F). \quad \blacksquare$$

Observation: Let M be an R-complex.
Suppose $s := \sup H_*(M)$ is finite.
Then, $\operatorname{depth}_R M \geq -s$

Then, $\operatorname{depth}_R M \geqslant -s$

with equality iff $\operatorname{depth}_R H_s(M)$.

Note. For an R-module M, $\operatorname{depth} M = 0 \Leftrightarrow \inf \operatorname{Ext}^{i}(k,M) = 0$

$\Leftrightarrow \operatorname{Hom}(k, M) \neq 0$

$\Leftrightarrow k \hookrightarrow M$

$\Leftrightarrow \mathfrak{m} \in \operatorname{Ass}_R M$.

One proof. M as above.

$$\operatorname{Ext}_R^{-s}(N, M) = \operatorname{Hom}_R(N, H_s(M))$$

N any R-module.

- key. $M \simeq M'$ with $M_i' = 0 \ \forall\ i > s$.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "0"] & M_{s+1} \arrow[r, "\partial"] \arrow[d, "0"'] & M_s \arrow[r] \arrow[d] & M_{s-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[d, equal] & = M \arrow[d, "\simeq"] \\
\cdots & 0 \arrow[r] & \dfrac{M_s}{\partial(M_{s+1})} \arrow[r] & M_{s-1} \arrow[r] & \cdots & = M'
\end{tikzcd}$$

$\therefore$ Can assume $M_i = 0$ for $i \geqslant s+1$.

In particular,

$$0 \to \Sigma^s H_s(M) \hookrightarrow M \to M'' \to 0.$$

iso on homology in degrees $\leq s-1$

$H_i(M'') = 0 \quad i \geqslant s$.

Let $\underline{x} = x_1, \ldots, x_n$ gen set for $\mathfrak{m}$.

Then,

Then,

$$H_{i+1}(\underline{x} ; M'') \to H_i(\underline{x}, \Sigma^* H_s(M)) \to H_i(\underline{x} ; M) \to H_i(\underline{x} ; M'') \quad \text{— } (*)$$

$H_j(M'') = 0 \ \forall j \geq s.$

So, $M'' \simeq M'''$ with $M'''_j = 0$ for $j \geq s.$

$K(\underline{x} ; M''')_j = 0$ for $j \geq s+n+1.$

Thus, $H_j(\underline{x} ; M''') = 0 \quad \forall j \geq s+n+1.$

$$\Rightarrow H_i(\underline{x} ; \Sigma^s H_s(M)) = 0 \quad \forall j \geq s+n+1.$$

Put $i \geq n+s$ in $(*)$:

$$H_j(\underline{x} ; M) = 0 \quad \forall j \geq n+s+1$$

$$\Rightarrow \sup H_*(\underline{x} ; M) \leq n+s$$

$$\Rightarrow -s \leq n - \sup H_*(\underline{x} ; M) = \operatorname{depth} M.$$

Moreover,

$$H_{n+s}(\underline{x} ; M) \cong H_{n+s}(\underline{x} ; \Sigma^s H_s(M))$$
$$\cong H_n(\underline{x}, H_s(M))$$

The above is nonzero iff $\operatorname{depth} H_s(M) = 0.$ $\boxtimes$

---

① $\operatorname{flat\,dim}_R F < \infty.$ Then $\forall M$

$$\operatorname{depth}_R(M \otimes_R^L F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^L F).$$

② $s = \sup H_*(M)$ is finite.

Then, $\operatorname{depth}_R(M) \geq -s$.

Equality $\Leftrightarrow$ $\operatorname{depth}(H_s(M)) = 0$.

Application. Say

$$F = \quad 0 \to F_d \to \cdots \to F_0 \to 0$$

minimal, $\partial F \subseteq \mathfrak{m} F$

is a finite free complex s.t.

$$0 < \operatorname{length}_R H_*(F) < \infty.$$

(All $H_i(F)$ are finite length and at least one is nonzero.)

Then, for any M,

$$\operatorname{depth}_R M = d - \sup H_*(F \otimes_R M).$$

Thus, such an F is depth sensitive.

When $\mathfrak{m}M \neq M$, one can check some $H_*(F \otimes_R M) \neq 0$.

Then, one gets

$$d \geq \operatorname{depth}_R(M).$$

Note: Over any local ring, $\exists M$ s.t. $\mathfrak{m}M \neq M$ and $\operatorname{depth}_R M = \dim R$.

M need not be f.g.

In particular, $d \geq \dim(R)$. $\longrightarrow$ New Intersection Theorem

Hochster ('70)
André (2016)
Bhatt (2021)

<u>Proof.</u> Assume $s = \sup H_*(F \otimes_R M)$.

Take any prime $\mathfrak{p} \neq \mathfrak{m}$.

$$H_i(F \otimes_R M)_{\mathfrak{p}} \cong H_i(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$$

$$H_i(F_{\mathfrak{p}}) = 0 \quad (\because \text{length } H_i(F) < \infty)$$

i.e. $F_{\mathfrak{p}} \simeq 0$ in $D(R_{\mathfrak{p}})$.

$$\therefore H_*(F_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$$

Thus, $H_i(F \otimes_R M)$ is $\mathfrak{m}$-power torsion.

(I.e., each $a \in H_i(F \otimes_R M)$ is killed by some $\mathfrak{m}^n$.)

$\therefore$ depth $H_s(F \otimes_R M) = 0$.

Thus, by previous result,

$$\text{depth}_R(F \otimes_R M) = -s$$

$$\| \text{ AB}$$

$$\text{depth}(M) - \sup H(k \otimes_R^L F)$$

$$\Rightarrow \text{depth}(M) = \underbrace{\sup H_*(k \otimes_R^L F)}_{d} - s. \qquad \blacksquare$$

Lecture 6 (30-01-2023)

Monday, January 30, 2023 1:26 PM

Recap. $I \subseteq R$ Comm. Noe
M an R-complex.

$$\operatorname{depth}_R(I, M) = \inf \operatorname{Ext}_R^*(R/I, M).$$

. Choose any fin gen set $\underline{x} = x_1, \dots, x_c$ of I.

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\underline{x}; M).$$

(Focus on the case $H_*(M)$ bounded.)

$(R, m_R, k) \to$ local

$$\operatorname{depth}_R M := \operatorname{depth}_R(m_R, M).$$

$$\operatorname{depth}_R(M) \geq -\sup H_*(M). \quad \text{--- (1)}$$

Equality $\iff m \in \operatorname{Ass}(H_s(M))$

$s = \sup H_*(M)$.

Exercise 1.

Suppose $M = 0 \to M_b \to \cdots \to M_a \to 0$

$$\operatorname{depth}_R M \geq \inf\{\operatorname{depth}(M_i) - i : a \leq i \leq b\}.$$

$$\operatorname{depth}_R M \geq \inf\{\operatorname{depth} H_i(M) - i : \inf H_*(M) \leq i \leq \sup H_*M\}. \quad \text{--- (2)}$$

(Note (2) $\Rightarrow$ (1).)

Setup. $H_*(M)$ bounded. $\underline{x} = x_1, \dots, x_c$. (R not necessarily local.)

$$\sup H_*(M) \underset{(i)}{\leq} \sup H_*(\underline{x}; M) \underset{(ii)}{\leq} \sup H_*(M) + c$$

Lemma 2. (a) Inequality (ii) always holds.
Equality iff $\operatorname{depth}_R(\underline{x}; H_s(M)) = 0$ $\quad s = \sup H_*(M)$

(b) (i) holds if $\underline{x} \subseteq J(R)$, each $H_i(M)$ is f.g.

equality iff ... $= \sup H_*(M)$

(b) (i) holds if $\underline{x} \subseteq J(R)$, each $H_i(M)$ is f.g.

Equality holds iff $\underline{x}$ is $H_s(M)$-regular.

Proof. (a) $H(\underline{x}; M) = H(x_1; K(x_2, \dots, x_c; M))$.

Reduce to $c = 1$. $x := x_1$.

In this case, we have

$$H_{i+1}(M) \xrightarrow{x} H_{i+1}(M) \longrightarrow H_{i+1}(x;M) \longrightarrow H_i(M) \xrightarrow{x} H_i(M)$$

For $i \geq s$, we get $H_{i+1}(x;M) = 0$.

$\therefore$ (ii) follows.

Moreover, $H_{s+1}(x;M) \neq 0 \iff x$ is a zd on $H_s(M)$.

(b) $H_i(M) \neq 0 \overset{NAK}{\Longrightarrow} H_i(x;M) \neq 0$

$\overset{?}{=} \dfrac{H_i(M)}{xH_i(M)}$

$\therefore \sup H_*(x;M) \geq \sup H_*(M)$.

Moreover,

$$0 \to H_{s+1}(x;M) \to H_s(M) \xrightarrow{x} H_s(M).$$

Corollary 3. $\operatorname{depth}_R(\underline{x}; M) \geq -\sup H_*(M)$.

Equality $\iff \operatorname{depth}(\underline{x}, H_s(M)) = 0$

$\overset{?}{\iff} (\underline{x}) \subseteq \mathfrak{p} \in \operatorname{Ass} H_s(M)$.

Prop^n 4. $(R, \mathfrak{m}, k)$ local.

Let M be any bounded complex.

Then, for any $I \subset R$,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

In particular, if M is a f.g. module

$$\operatorname{depth}_R(M) \leq \inf\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M\}$$

$$\leq \dim_R(M). \quad \rightsquigarrow \sup\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass} M\}$$

Proof. Say $I = (y_1, \dots, y_c)$ and $d := \dim(R/I)$.

... s.t. they form a SoP in R/I

Proof. Say $I = (y_1, \dots, y_c)$ and $d := \dim(R/I)$.

Let $x_1, \dots, x_d \in R$ s.t. they form a SOP in R/I.

Then, $\sqrt{(\underline{y}, \underline{x})} = \sqrt{\mathfrak{m}_R}$.

Apply Lemma (2)(b) to $K(\underline{y}; M)$ to get

$$\sup H_*(\underline{y}; M) \leq \sup H_*(\underline{x}; K(\underline{y}; M))$$
$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \; \| $$
$$\sup H_*(\underline{x}, \underline{y}; M).$$

$$\therefore \; d + \underbrace{c - \sup H_*(\underline{y}; M)} \geq \underbrace{d + c - \sup H_*(\underline{x}, \underline{y}; M)}$$

$$\Rightarrow d + \operatorname{depth}(I, M) \geq \operatorname{depth} M. \qquad \blacksquare$$

Local case:

AB Equality F an R-complex with $\operatorname{flatdim}_R F < \infty$.

Then, for any R-complex M,

$$\operatorname{depth}(M \overset{L}{\underset{R}{\otimes}} F) = \operatorname{depth}_R(M) - \sup H_*(k \overset{L}{\underset{R}{\otimes}} F)$$

$$\operatorname{depth}_R M \geq -\sup H_*(M) = s$$

with equality iff $\operatorname{depth}(H_s(M)) = 0$.

Prop^n. $R \to$ comm. Noe.

$I \subseteq R$ ideal.

$$\operatorname{depth}(I, M) = \inf \{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in V(I)\}.$$

Proof. $I \subset \mathfrak{p} \Rightarrow \operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(\mathfrak{p}, M) \leq \operatorname{depth}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$

check using Koszul

This gives $\leq$.

We now construct $\mathfrak{p}$ achieving $\operatorname{depth}_R(I, M)$.

Let $I = (x_1, \dots, x_c)$. Let $s := \sup H_*(\underline{x}; M)$.

We now construct $\mathfrak{p}$ achieving $\operatorname{depth}_R(I, M)$.

Let $I = (x_1, \dots, x_c)$. Let $s := \sup H_*(\underline{x}; M)$.

Pick $\mathfrak{p} \in \operatorname{Ass} H_s(\underline{x}; M)$. (Maybe even minimal.)

Then, $\operatorname{depth}_{R_\mathfrak{p}} H_s(\underline{x}; M)_\mathfrak{p} = 0$.

Consider $K(\underline{x}; M)_\mathfrak{p} = K(\underline{x}; M_\mathfrak{p})$.

$\sup H_*(K(\underline{x}; M_\mathfrak{p})) = \sup H_*(\underline{x}; M_\mathfrak{p}) = s$

$$\operatorname{depth}_{R_\mathfrak{p}} K(\underline{x}; M_\mathfrak{p}) = -s$$

$$\operatorname{depth}_{R_\mathfrak{p}} \left(K(\underline{x}; R_\mathfrak{p}) \otimes M_\mathfrak{p} \right)$$

$\| \ AB$

$$\operatorname{depth}(M_\mathfrak{p}) - \sup H_*\left(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} K(\underline{x}; R_\mathfrak{p})\right)$$

$$\geq \operatorname{depth}(M_\mathfrak{p}) - c$$

$\Rightarrow -s \geq \operatorname{depth}(M_\mathfrak{p}) - c$

$\Leftrightarrow c - s \geq \operatorname{depth}(M_\mathfrak{p})$

$\operatorname{depth}_R(I, M)$

Remark. The proof says

$$\operatorname{depth}_R(I, M) = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \quad \forall \mathfrak{p} \in \operatorname{Ann} H_s(\underline{x}; M)$$

Thm. $(R, \mathfrak{m}_R) \longrightarrow (S, \mathfrak{m}_S)$ local map.

(i.e., $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$.)

Let M be an R-complex, N an S-module s.t. N is flat as an R-module.

Then,

$$\operatorname{depth}_S(N \otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N).$$

Corollary When $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ is flat, then $\left(\begin{smallmatrix} M = R \\ N = S \end{smallmatrix} \text{ gives} \right)$

$$\operatorname{depth}_S(S) = \operatorname{depth}_R(R) + \underbrace{\operatorname{depth}_{S/\mathfrak{m}_R S}\left(S/\mathfrak{m}_R S\right)}_{\text{"fiber"}}.$$

$$R \to S \to \frac{S}{\mathfrak{m}_R S}$$

$$\begin{tikzcd} \operatorname{Spec}\left(R/\mathfrak{m}_R S\right) \arrow[r, hook] \arrow[d] & \operatorname{Spec} S \arrow[d] \\ \{\mathfrak{m}_R\} \arrow[r, hook] & \operatorname{Spec}(R) \end{tikzcd}$$

Under the same hypothesis, we have

$$\dim(S) = \dim(R) + \dim\left(S/\mathfrak{m}_R S\right).$$

Proof. $\underline{x} = x_1, \dots, x_c \in R$ s.t. $\mathfrak{m}_R = (\underline{x})$.

Pick $\underline{y} = y_1, \dots, y_d \in S$ s.t. $\underline{y}\left(\frac{S}{\mathfrak{m}_R S}\right)$ is the maximal ideal of $S/\mathfrak{m}_R S$ (i.e., $\mathfrak{m}_S/\mathfrak{m}_R S$).

Then, $(\underline{x} S, \underline{y}) = \mathfrak{m}_S$.

$$K(\underline{x} S, \underline{y} ; N \otimes_R M) \underset{\hookrightarrow \text{ assoc. of } \otimes}{\cong} K(\underline{y} ; N) \otimes_R K(\underline{x} ; M)$$

N flat over $R \Rightarrow K(\underline{y}, N)$ has fin flat $\dim_R$

$$\operatorname{depth}_R K(\underline{x}, \underline{y} ; N \otimes_R M) = \operatorname{depth}_R K(\underline{x} ; M) - \sup H_*\left(k \otimes K(\underline{y}, N)\right)$$

$(k = R/\mathfrak{m}_R)$

Note. $(\underline{x}, \underline{y}) \cdot H_*(\underline{x}, \underline{y} ; N \otimes_R M) = 0$

$\mathfrak{m}_S \, H_*(\underline{x}, \underline{y} ; N \otimes_R M) = 0$

$\Rightarrow \mathfrak{m}_R \, H_*(\underline{x}, \underline{y} ; N \otimes_R M) = 0$

Similarly $\mathfrak{m}_R \, H_*(\underline{x} ; M) = 0.$

$$-\sup H_*(\underline{x}, \underline{y}; N \otimes_R M) = -\sup H_*(\underline{x}, M) - \sup H_*\left(\underline{y}; \frac{N}{\underline{m}_R N}\right)$$

Add $c+d$

$\hookrightarrow$ depths $(N \otimes_R M)$

Exercise. R local. M f.g. R-module

$$\operatorname{depth} R - \dim_R M \leq \operatorname{grade}_R M \leq \operatorname{codim}_R M \leq \dim R - \dim M \leq \operatorname{pdim}_R M.$$

$\operatorname{grade}_R M = \operatorname{depth}_R(\operatorname{ann}_R M, R)$

$\operatorname{codim}_R M = \operatorname{height}(\operatorname{ann} M) = \inf\{\dim R_p : p \supseteq \operatorname{ann} M\}$

use intersection thm

Recall. New Intersection Theorem (P. Roberts '86)

$0 \to F_d \to \cdots \to F_0 \to 0$ finite free

$0 < \operatorname{length} H_*(F) < \infty.$

Then, $d \geq \dim \dim(R)$.

"Simple consequence" of AB + Existence of big CM modules

Corollary. (Intersection theorem)

R local.

$M \neq 0$ f.g. R-module s.t. $\operatorname{pdim}_R M < \infty$.

Then, for any f.g. R-module N,

$$\dim_R(N) - \dim_R(M \otimes_R N) \leq \operatorname{pdim}_R(M).$$

Inspired by (Serre): R regular local (e.g. $k[[x_1, \dots, x_n]]$)

M, N any finite R-module.

(Serre) M, N any finite R-module.

$$\dim_R N - \dim_R (M \otimes_R N) \leq \dim R - \dim M.$$

Lecture 7 (01-02-2023)

Wednesday, February 1, 2023 1:24 PM

$R \to$ Commutative noetherian.

$I \subseteq R$ ideal

Invariance of domain

Let $R \to S$ finite map. (S is a finite R-module.)

Let M be an S-complex.

Then,
$$\operatorname{depth}_R(I, M) = \operatorname{depth}_S(IS, M).$$

[Only need S noetherian for this.]

Obvious using Koszul.

$I = (\underline{x})$

$$K(\underline{x}; M) = K(\underline{x}; R) \otimes_R R$$
$$= \left(K(\underline{x}; R) \otimes_R S\right) \otimes_S M$$
$$= K(\underline{x}; S) \otimes_S M.$$

$\dim_R M = \dim_S M.$ [Finiteness needed here.]

Say $(R, m_R) \to (S, m_S)$ is finite and local.

$$\begin{aligned}\operatorname{depth}_R(M) &= \operatorname{depth}(m_R, M)\\ &= \operatorname{depth}(m_R S, M)\\ &= \operatorname{depth}(m_S, M)\\ &= \operatorname{depth}_S(M).\end{aligned}$$

finiteness $\Rightarrow \sqrt{m_R S} = m_S$

Special case. R, M an R-module.

Take $S = R/\operatorname{ann}_R(M)$.

Recall. (R, m_R) local, $M \neq 0$ a f.g R-module.

$$\operatorname{depth}(R) - \dim(M) \overset{①}{\leq} \operatorname{grade}(M) \overset{②}{\leq} \operatorname{codim}(M) \overset{③}{\leq} \dim R - \dim M$$

$$\operatorname{depth}(R) - \dim(M) \overset{①}{\leq} \operatorname{grade}(M) \overset{②}{\leq} \operatorname{codim}(M) \overset{③}{\leq} \dim R - \dim M$$
$$\overset{④}{\leq} \operatorname{pdim}_R M.$$

$$\operatorname{depth}(M) \leq \operatorname{depth}_R(I, M) + \dim(R/I).$$

Using invariance of domain, can tighten the above to

$$\operatorname{depth}_R M \leq \operatorname{depth}_R(I, M) + \underbrace{\dim(M/IM)}_{= \dim\left(R/_{I + \operatorname{ann} M}\right)}.$$

Examples. ① $R = \dfrac{k[[x, y]]}{(x^2, xy)}$

$\mathfrak{m}_R = (x, y)$.

$\dim(R) = 1$: $\sqrt{(x^2, xy)} = (x)$

$\therefore \dim R = \dim \dfrac{k[[x, y]]}{(x)} = \dim k[[y]] = 1.$

$\operatorname{depth}(R) = 0$, i.e., $\operatorname{Hom}_R(k, R) \neq 0$

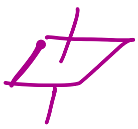
i.e, $\underbrace{(0 : \mathfrak{m}_R)}_{\text{socle}} \neq 0$.

Taking $M = R$ shows
$\operatorname{depth} R - \dim M < \operatorname{grade}(M)$ is strict.

② Take same R.

$M = R/\mathfrak{m}$. Then, $\operatorname{grade} M < \operatorname{codim} M$.

③ $k[[x, y, z]]/_{(x) \cap (y, z)} = R$.



Spec R $= V(x) \cup V(y,z)$

Pick $M = R/\mathfrak{p}$ where $\mathfrak{p}$ is a min'l prime
but $\dim(R/\mathfrak{p}) < \dim(R)$.

$\mathfrak{p} = (y, z)$.

⑤ Say $\operatorname{pdim}_R M < \infty$.

$\dim R - \dim M \leq \operatorname{pdim}_R M$

?, AB

$$\dim R - \dim M \le \operatorname{pdim}_R M$$
$$\Updownarrow \qquad\qquad \}\text{AB}$$
$$\dim R - \dim M \le \operatorname{depth} R - \operatorname{depth} M$$
$$\Updownarrow$$
$$\dim R - \operatorname{depth} R \le \dim M - \operatorname{depth} M \quad — \quad (5)$$

Now, take $R = k[[x,y]]$ and $M = R/(x^2, xy)$.

Then, LHS of (5) is 0 but RHS is not.

Def^n: $(R, \mathfrak{m})$ local, $M \overset{\neq 0}{}$ f.g. over R.

$$\operatorname{Cmd}_R(M) := \dim_R M - \operatorname{depth}_R M.$$

Cohen-Macaulay defect

If $\operatorname{Cmd}_R(M) = 0$, i.e., $\dim_R(M) = \operatorname{depth}_R(M)$, then

M is Cohen-Macaulay (CM).

M is maximal Cohen-Macaulay (MCM) if $\operatorname{depth}(M) = \dim(R)$.

0 is also considered CM and MCM.

$\}$ $\underset{\infty}{\operatorname{depth} M} \ge \underset{-\infty}{\overset{\shortparallel}{\dim M}}$ for $M = 0$.

One can define dim etc. for complexes as well.

$(R, \mathfrak{m}_R)$ local. $M \overset{\neq 0}{}$ f.g. module.

$$\operatorname{depth}_R(M) \le \inf\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R M\} \le \dim_R M.$$

Corollary. M MCM $\Rightarrow$ $\operatorname{Ann}_R M = \operatorname{Min}_R M$ and $\dim(R/\mathfrak{p}) = \dim(M)$ $\forall\, \mathfrak{p} \in \operatorname{Min}(M)$.

Geometrically: $\operatorname{Supp}(M)$ has no "embedded components" and all components have the same dimension.

Prop^n $(R, \mathfrak{m})$ local, M f.g. and CM.

① $\operatorname{depth}_R(I, M) = \dim M - \dim(M/IM) \quad \forall I \subseteq R$

② Given $\underline{x} = x_1, \ldots, x_c \in \mathfrak{m}$, then

$\underline{x}$ is M-regular $\iff \dim(M) - \dim(M/\underline{x}M) = c.$

Note ($\Rightarrow$) is always true.

($\Leftarrow$) Apply ① to $I = (\underline{x})$.

The hypothesis gives

$\operatorname{depth}_R(\underline{x}, M) = c$

$\iff H_i(\underline{x}; M) = 0 \quad \forall i \geq 1$

$\iff \underline{x}$ is regular. ∎

③: $\underline{x}$ is part of an SoP for M ⇝ SoP for $R/\operatorname{ann}(M)$

$\iff \underline{x}$ is M-regular.

Thm. $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ local map.

M f.g. R-module, N f.g. S-module, flat over R.

$\neq 0$ $\neq 0$

Then,

$$\operatorname{Cmd}_S(N \otimes_R M) = \operatorname{Cmd}_R(M) + \operatorname{Cmd}_{(S/\mathfrak{m}_R S)}(N/\mathfrak{m}_R N).$$

Proof: We saw the above for depth instead of Cmd.

Same holds for dim. ∎

Corollary. Under same hypothesis,

$N \otimes_R M$ is CM$/S \iff M$ CM$/R$ + $N/\mathfrak{m}_R N$ CM$/S/\mathfrak{m}_R S$

Special case: $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_R)$ flat local.

S CM $\iff R$ and $S/\mathfrak{m}_R S$ are CM.

Def^n A local ring R is CM if it is so as a module over itself.

Key Consequence: R CM: for any f.g. R-module M,

$$\mathrm{grade}_R(M) = \mathrm{codim}_R(M) = \dim(R) - \dim(M).$$

Exercise $(R,\mathfrak{m})$ M f.g.

M CM $\Rightarrow$ $M_{\mathfrak{p}}$ CM over $R_{\mathfrak{p}}$ $\forall\, \mathfrak{p} \in \mathrm{Supp}\, M$.

Def^n $R \to$ comm. noe.
$M \to$ f.g. R-module

Then, M is CM if $M_{\mathfrak{p}}$ CM$/R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Supp}_R M$.

Equivalently, for all $\mathfrak{m} \in \mathrm{Max}(\mathrm{Supp}_R M)$.

R is CM if . . .

Examples: • $k[x_1, \ldots, x_c]$ is CM. (How?)

• $k[[x_1, \ldots, x_c]]$ is CM. $\because$ $\dim = c$
depth $= c$ since $x_1, \ldots, x_c$.

• R CM $\Leftrightarrow$ $\hat{R}^I$ is CM for some ($\equiv$ all) I.
$\hookrightarrow$ completion wrt I

If R local,
$(R,\mathfrak{m}) \longrightarrow \overset{S}{\hat{R}^I}$ is flat and local.

Check $S/\mathfrak{m}_R S = R/\mathfrak{m}_R \to$ field (CM)

$\therefore$ $\hat{R}^I$ is CM $\Leftrightarrow$ R is CM.

Thm: $(R, \mathfrak{m})$ CM local.
M f.g. R module s.t. $\mathrm{pdim}_R M < \infty$

Then,

M is CM $\Leftrightarrow$ $\mathrm{pdim}_R M = \mathrm{grade}_R M$.

Proof. $\operatorname{pdim}_R M = \operatorname{grade}_R M$

$$\Updownarrow$$

$\operatorname{depth} R - \operatorname{depth} M = \dim R - \dim M$

$$\Updownarrow$$

$\operatorname{depth} M = \dim M.$ ⊠

Lecture 8 (06-02-2023)

Monday, February 6, 2023 1:25 PM

Regular Rings

For now, $(R, \mathfrak{m}, k)$ is a comm. noetherian local ring.

Recall:
$$\operatorname{depth}(R) \overset{①}{\leq} \dim(R) \overset{②}{\leq} \operatorname{edim}(R).$$
$$\operatorname{edim}(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$$

When equality holds in ①, then R is Cohen-Macaulay. (By defn.)

② holds by Krull's height theorem.

Notation: (embedding) codepth of R is
$$\operatorname{codepth}(R) = \operatorname{edim}(R) - \operatorname{depth}(R).$$

Defn. R is regular if $\operatorname{codepth}(R) = 0$.

Exercise. R is regular $\iff$ $\mathfrak{m}$ is generated by a regular sequence

$\iff \operatorname{edim}(R) = \dim(R)$

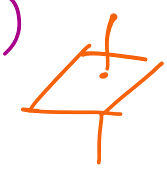
Example

① $k[[x_1, \dots, x_n]]$ with k a field is regular since $\mathfrak{m}$ is gen by a reg. seq.

② $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ —— " ——.

③ $\mathbb{Z}_{(p)}$ similar.

④ Regular ⇒ CM. (Non CM ⇒ Non reg.)



$$\frac{k[[x, y, z]]}{(xz, yz)} \quad \text{not reg.}$$

⑤ $R = \dfrac{k[[x\, y]]}{(xy)}$. codepth $= 1 > 0$.

NOT regular.

Flat Maps

Let $\varphi : (R, m) \to (S, \eta)$ be a local flat extension.

Recall

S is CM $\iff$ R is CM and S/mS are CM.

Analog fails for regularity. ($\Leftarrow$) does hold.

Consider $R = k[[x^2]] \hookrightarrow k[[x]] = S$. (flat ext^n since S is free)

regular

$S/mS = k[[x]]/(x^2)$

↳ not regular, codepth $= 1$

Also, if R is CM, then R_p is CM $\forall p$.

Question ['30s, Krull, Zariski, ...]

Does R regular imply R_p regular $\forall p \in \operatorname{Spec} R$?

Homological characterisation of regularity. (Used to prove ↑)

Recall: for a f.g. R-module M, a minimal free resolution over (R, m, k) is a free resolution

$$F \xrightarrow{\sim} M$$

with $\partial(F) \subseteq \mathfrak{m} F$.

Recall/Prove. Minimal resolutions exist and are unique up to isomorphism of complexes.

The ith Betti number of M is

$$\beta_i^R(M) = \operatorname{rank}_R(F_i)$$
$$= \operatorname{rank}_k(\operatorname{Tor}_i^R(M, k))$$
$$= \operatorname{rank}_k(\operatorname{Ext}^i_R(M, k)).$$

Example. ① $R = k[\![x_1, \dots, x_n]\!]$. $f \in \mathfrak{m} \setminus \{0\}$.

$M := R/(f)$.

Then the min'l free res^n of M is

$$0 \to R \xrightarrow{\cdot f} R \to 0.$$

↪ Koszul

$$\beta_i(M) = \begin{cases} 1 & ; \quad i = 0, 1 \\ 0 & ; \quad \text{else} \end{cases}$$

The min'l res^n of k is $K(\underline{x})$.

$$\beta_i(k) = \binom{n}{i} \quad \text{for } i \geq 0.$$

② $R = \dfrac{k[\![x, y]\!]}{(xy)}$.

$M = R/(x)$ has min'l free res^n:

$$\cdots \xrightarrow{\cdot y} \overset{3}{R} \xrightarrow{\cdot x} \overset{2}{R} \xrightarrow{\cdot y} \overset{1}{R} \xrightarrow{\cdot x} \overset{0}{R} \to 0$$

$$\therefore \beta_i(M) = 1 \quad \forall i.$$

Res'^{n} for k: $\quad k = R/(x,y)$

repeat
$$\cdots \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x\ y)} R \longrightarrow 0$$

$$\beta_i(M) = \begin{cases} 1 & ; i = 0 \\ 2 & ; i \geq 1 \end{cases}$$

FACT. $0 \neq M$ f.g.:
(/Defn)

$$\operatorname{projdim}_R(M) = \sup\{i \geq 0 : \beta_i^R(M) \neq 0\}.$$

$\quad\quad\quad\quad \|$

length of min'l free res'^{n}

$\quad\quad \|$

$\inf(\text{length}(\text{free res}'^n))$

Theorem. [Auslander-Buchsbaum-Serre, '50s]
(ABS)

$(R, \mathfrak{m}, k)$ local. TFAE:

① R is regular.

② $\operatorname{projdim}_R(M) < \infty$ for all f.g. M.

③ $\operatorname{projdim}_R(k) < \infty$.

Proof. ① ⇒ ②

Let $\underline{x}$ ($x_1, \ldots, x_d$) be a min'l gen set for $\mathfrak{m}$. (Hence reg. seq.)

Then, $K(\underline{x})$ is a min'l free res'^{n} for k.

$$\beta_i^R(M) = \operatorname{rank}_k(\operatorname{Tor}_i^R(M,k))$$
$$= \operatorname{rank}_k H_i(M \otimes_R K(\underline{x}))$$
$$= 0 \quad \text{for } i > d.$$

$\therefore \operatorname{projdim}_R(M) < \infty.$

② ⇒ ③. —

② ⇒ ③. ✓

③ ⇒ ①. Serre's proof of ABS Theorem relies on:

Lemma. $(R, \mathfrak{m}, k)$ local.

Then, $\beta_i^R(k) \geq \binom{\operatorname{edim}(R)}{i}$ for $i \geq 0$.

Proof of Lemma. Let $F \xrightarrow{\simeq} k$ be a min'l resol^n. $(\because \beta_i^R(k) = \operatorname{rk}_R F_i.)$

Let $\underline{x} = x_1, \ldots, x_e$ be a min gen set for $\mathfrak{m}$.

$$\begin{tikzcd} K := K(\underline{x}) \arrow[r] \arrow[dr, dashed] & k \\ & F \arrow[u, "\simeq"] \end{tikzcd}$$

$\exists \varphi$ a map of complexes $(\because$ perfect$)$

Claim. φ_i is a split injection $\forall i$. (Then it is clear.)

$\varphi_i : K_i \to F_i$ is split injective

$\overset{\text{NAK}}{\Leftrightarrow}$ $k \otimes_R K_i \to k \otimes_R F_i$ is an injection.

By induction, we show φ_i is split inj.

$i = 0$ ✓ $R \xrightarrow{\simeq} R$

$i > 0$: Let $a \in K_i$ with $\varphi_i(a) \in \mathfrak{m} F_i$. (we are using the NAK result.)

WTS : $a \in \mathfrak{m} K_i$.

$\therefore \partial\varphi(a) \in \mathfrak{m}^2 F_{i-1}$

$\quad \Vert$

$\varphi(\partial a)$

By induction,

$\partial^K a \in \mathfrak{m}^2 K_{i-1}$. (*)

$$\begin{tikzcd} a \in K_i \arrow[r] \arrow[d] & K_{i-1} \arrow[d] \\ F_i \arrow[r] & F_{i-1} \\ \varphi(a) \in \mathfrak{m} F_i \arrow[u, phantom, "\cup"] \arrow[r, bend right, "\partial(F) \subseteq \mathfrak{m}F"'] & \partial(\varphi(a)) \subseteq \mathfrak{m}^2 F_{i-1} \end{tikzcd}$$

Notice that by def^n of ∂^K and since x is a min gen set for $\mathfrak{m}$, Notice that by any of [illegible] and

since x is a min gen set for $\mathfrak{m}$,

$(*) \Rightarrow a \in \mathfrak{m} K_i$. This does it. ∎

Back to : $\text{proj dim}_R (k) < \infty \Rightarrow R$ is regular.

From Serre's inequality:

$$\text{proj dim}_R\, k \geq \text{edim } R.$$

$\| \rightsquigarrow$ By AB Equality, since $\text{proj dim } k < \infty$.

$\text{depth}(R)$

But $\text{depth}(R) \leq \text{edim}(R)$ always true. ∎

The above now solves the localisation problem.

<u>Corollary.</u> R regular $\Rightarrow R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \text{Spec } R$.

<u>Proof.</u> R is reg.

$\Rightarrow \text{proj dim}_R\, R/\mathfrak{p} < \infty$

$\Rightarrow \text{proj dim}_{R_{\mathfrak{p}}} (k(\mathfrak{p})) < \infty \Rightarrow R_{\mathfrak{p}}$ is reg. ∎

<u>Prop^n.</u> $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{m}_S)$ local flat.

① R and $S/\mathfrak{m}S$ reg $\Rightarrow S$ is reg.

② S is regular $\Rightarrow R$ is regular.

<u>Proof.</u> ① $\bar{S} = S/\mathfrak{m}S$

$\text{depth } S \overset{\text{flat}}{=} \text{depth}(R) + \text{depth}(\bar{S})$

$= \text{edim}(R) + \text{edim}(\bar{S})$ regular hypothesis

There is always a right exact sequence

[illegible]

$$\frac{m_R}{m_R^2} \otimes_k \ell \to \frac{m_S}{m_S^2} \longrightarrow \frac{m_{\bar S}}{m_{\bar S}^2} \longrightarrow 0$$

$\ell = S/m_S$

$\therefore$ $\operatorname{edim} S \leq \operatorname{edim} R + \operatorname{edim}(\bar S) = \operatorname{depth}(S)$.

② Let $F \xrightarrow[R]{\sim} k$ min'l, since φ flat AND local,

$$F \otimes_R S \longrightarrow \bar S \quad \text{min'l.}$$

$$\beta_i^R(k) = \beta_i^S(\bar S) = 0 \quad \text{for} \quad i \gg 0.$$

$\therefore$ $\operatorname{projdim}_R k < \infty$.

$\Rightarrow R$ is reg. ∎

Lecture 9 (08-02-2023)

Wednesday, February 8, 2023 1:26 PM

Recall: R local is regular if $\operatorname{codepth}(R) = 0$,

i.e., $\mathfrak{m}$ generated by a regular sequence.

We proved Auslander-Buchsbaum Theorem:

$(R, \mathfrak{m}, k)$ TFAE:

① R is regular,

② $\operatorname{projdim}_R M < \infty$ for all f.g. M

③ $\operatorname{projdim}_R(k) < \infty$.

---

① $\Rightarrow$ ② $\Rightarrow$ ③ was ok.

③ $\Rightarrow$ ①: showed $\operatorname{projdim}_R(k) \geq \operatorname{edim}(R)$ and then AB Equality.

Sketch of second proof.

Theorem [Nagata]. Let $(R, \mathfrak{m}, k)$ be local (not necessarily regular) and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be nzd.

Set $\overline{R} = R/x$.

Then, for any f.g. R/x-module M, we have

$$\operatorname{Tor}^R_*(M, k) \cong \operatorname{Tor}^{\overline{R}}_*(M, k) \otimes_k \Lambda(\Sigma k).$$

$= \quad 0 \to k \to k \to 0$

In particular, $\beta^R_i(M) = \beta^{\overline{R}}_i(M) + \beta^{\overline{R}}_{i-1}(M)$.

In fact, one can show the min'l R-free res'n of M

has the form

$$\cdots \longrightarrow \begin{matrix} G_3 \\ \oplus \\ G_2 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_3 & x \\ \beta_3 & -\alpha_2 \end{pmatrix}} \begin{matrix} G_2 \\ \oplus \\ G_1 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_2 & x \\ \beta_2 & -\alpha_1 \end{pmatrix}} \begin{matrix} G_1 \\ \oplus \\ G_0 \end{matrix} \xrightarrow{(\alpha_1 \;\; x)} G_0$$

multiplication by x

$G_i \to$ free

and $\cdots \to \bar{G}_3 \xrightarrow{\alpha_3} \bar{G}_2 \xrightarrow{\alpha_2} \cdots$ is the min'l $\bar{R}$-resol^n of M.

(Stronger result but not proving this.)

<u>Example.</u> $R = k[[x,y]]/(xy)$ (non regular)

Minimal R-free resol^n of k:

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x \;\; y)} R \to 0$$

Consider $y - x^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$ (nzd on R).

By row/col operations, the min'l resol^n is iso to,

$$\cdots \xrightarrow{\begin{pmatrix} x & y-x^2 \\ & -y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & y-x^2 \\ & -x \end{pmatrix}} R^2 \xrightarrow{(x \;\; y-x^2)} R \to 0$$

Now, going mod $y-x^2$, [using the result] we get

$$\cdots \to \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{y} \bar{R} \xrightarrow{x} \bar{R} \to 0$$

$\bar{R} = k[[x,y]]$

$$\cdots \to \bar{R} \to \bar{R}$$

$$= \quad \cdots \to \bar{R} \xrightarrow{x} \bar{R} \xrightarrow{x^2} \bar{R} \xrightarrow{x} \bar{R} \to 0$$

$$\bar{R} = \frac{k[[x,y]]}{(xy,\, y-x^2)} \cong \frac{k[[x]]}{(x^3)}$$

Proof of Nagata's Thm.

If $a \in R$ is nzd, and set $\bar{R} = R/a$.

$M \rightsquigarrow \bar{R}$-module.

There is a long exact sequence

$$\cdots \xrightarrow{\chi} \mathrm{Tor}^{\bar{R}}_{n-1}(M,k) \to \mathrm{Tor}^{R}_{n}(M,k) \to \mathrm{Tor}^{\bar{R}}_{n}(M,k)$$

$$\xrightarrow{\chi} \mathrm{Tor}^{\bar{R}}_{n-2}(M,k) \to \mathrm{Tor}^{R}_{n-1}(M,k) \to \mathrm{Tor}^{\bar{R}}_{n-1}(M,k)$$

Specialising to our setting, we wish to show that each χ is 0.

One can compute χ in the following way:

$\bar{F} \xrightarrow{\sim} M$ min'l $\bar{R}$-res'n.

Lift this to a sequence of free R-modules:
(not saying complex!)

$$\cdots \to F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \to \cdots$$

∂^2 may not be 0 but

$\partial^2 = \ldots$ where

∂^i may not be 0 on

$\partial^2 = x\,\Theta$ where

$\Theta = \{\Theta_i : F_i \longrightarrow F_{i-2}\}$ is a chain map

and the following diagram commutes: (up to sign)

$$\begin{tikzcd}
F_i \otimes_R k \arrow[r, "\Theta_i \otimes_R k"] \arrow[d, phantom, "\cong" rotate=90] & F_{i-2} \otimes_R k \arrow[dd, "\cong"] \\
\bar{F}_i \otimes_{\bar{R}} k \arrow[d, phantom, "="] & \\
\mathrm{Tor}_i^{\bar{R}}(M,k) \arrow[r, "x"'] & \mathrm{Tor}_{i-2}^{\bar{R}}(M,k)
\end{tikzcd} \qquad (*)$$

Now, since x is linear and $\partial^2 = x\,\Theta$, we have that $\Theta(F) \subseteq \mathfrak{m} F$.

$\therefore$ the top map in $(*)$ is zero. $\therefore x = 0$. $\blacksquare$

Second proof of ③ $\Rightarrow$ ① in ABT.

Given $\mathrm{projdim}_R(k) < \infty$. WTS: R is regular.

Induct on $d := \mathrm{depth}(R)$

$d=0$: AB Equality gives $\mathrm{projdim}_R(k) = 0$.

$\therefore R = k$ is a field. R is reg.

$d>0$: By prime avoidance, $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$ nzd.

$\therefore \mathrm{codepth}(R) = \mathrm{codepth}(R/x)$.

By Nagata's thm,

$\mathrm{projdim}_{R/x} k = \mathrm{projdim}_R k - 1 < \infty$.

By Nagata's Thm,

$$\operatorname{projdim}_{R/x} k = \operatorname{projdim}_R k - 1 < \infty.$$

But R/x has smaller depth. ☒

$(R, \mathfrak{m}, k)$ local

Prop^n. $x \in \mathfrak{m}$ nzd.

R is CM $\iff$ R/x is CM.

($\mathrm{CMD}(R) = \mathrm{CMD}(R/x)$.)

($\mathrm{CMD} = \dim - \operatorname{depth}$.)

Proposition. $x \in \mathfrak{m}$ nzd.

① Suppose R is regular.

R/x is regular $\iff$ $x \notin \mathfrak{m}^2$.

② If R/x is regular, then R is regular.

(Hence, $x \notin \mathfrak{m}^2$.)

Proof.

$$\operatorname{codepth}(R/x) = \begin{cases} \operatorname{codepth}(R) & x \notin \mathfrak{m}^2, \\ \operatorname{codepth}(R) + 1 & x \in \mathfrak{m}^2. \end{cases}$$

☒

Global Setting

$R \to$ comm. noetherian, not necessarily local or of finite Krull dim.

Def^n. R is *regular* if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$

($\iff \forall\, \mathfrak{m} \in \operatorname{mSpec}(R)$).

($\Leftrightarrow$ $\forall\, \mathfrak{m} \in \mathrm{mSpec}(R)$).

(Since regularity localises for local rings, the above makes sense.)

Exercise. R is regular $\Leftrightarrow$ $R[x]$ is regular
$\Leftrightarrow$ $R[[x]]$ is regular

Example ① $k[x_1, \dots, x_n]$ is regular. (k a field.)

② $\mathbb{Z}[x_1, \dots, x_n]$ —"— . ($\mathbb{Z}$ is regular since every localisation is $\mathbb{Q}$ or a DVR.)

③ Nagata's example. (Infinite Krull dimension but regular)

Thm. [Bass–Murthy '60s].

$R \to$ comm noetherian
$M \to$ f.g.

$\mathrm{projdim}_R M < \infty \Leftrightarrow \mathrm{projdim}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

Corollary. $R \to$ comm noe.

R is regular $\Leftrightarrow$ $\mathrm{projdim}_R M < \infty$ for all M f.g.

Proof. ($\Rightarrow$) ✓

($\Leftarrow$) Let $F_\bullet \xrightarrow{\sim} M$ be a free res^n of M, F_i f.g. for all $i \geq 0$.

For $n \geq 0$, define

For $n \geq 0$, define

$$D_n := \{ \mathfrak{p} \in \operatorname{Spec} R : \operatorname{proj\,dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leq n \}$$

$$= \{ \mathfrak{p} \in \operatorname{Spec} R : \operatorname{im}(\partial_n^F)_\mathfrak{p} \text{ is free over } R_\mathfrak{p} \}.$$

Since $\operatorname{im}(\partial_n^F)$ is f.g., the above set is open in $\operatorname{Spec}(R)$. "free locus is open"

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$$

and $\operatorname{Spec} R = \bigcup_{n \geq 0} D_n$.

Since R is noetherian, $\operatorname{Spec} R$ is a noe. top space.

$\therefore \operatorname{Spec} R = D_n$ for some n.

$\Rightarrow \operatorname{im}(\partial_n^F)$ is locally free.

$\Rightarrow \operatorname{im}(\partial_n^F)$ is projective.

Now, $0 \to \operatorname{im}(\partial_n^F) \hookrightarrow F_{n-1} \to \cdots \to F_0 \to 0$ is a proj. res^n. $\boxtimes$

Lecture 10 (13-02-2023)

13 February 2023 13:26

$R \to$ comm noetherian

$$\varphi : F_1 \longrightarrow F_0 \qquad \text{free modules of finite rank}$$
$$\begin{array}{ccc} F_1 & \longrightarrow & F_0 \\ \cong & & \cong \\ R^s & \xrightarrow{(a_{ij})} & R^r \end{array}$$

$I_c(\varphi) =$ ideal generated by $c \times c$ minors of (a_{ij}).

(well defined)

$$I_0(R) = R \supseteq I_1(R) = (a_{11}, \dots, a_{rs}) \supseteq \cdots \supseteq I_i(R) \supseteq I_{i+1}(R) = 0$$
$$i = \min(r, s)$$

M f.g. R-module

$$F_1 \xrightarrow{\varphi} \underbrace{F_0}_{\text{rank} = r} \longrightarrow M \longrightarrow 0$$

finite free presentations

$$G_1 \xrightarrow{\psi} \underbrace{G_0}_{\text{rank} = s} \longrightarrow M \longrightarrow 0$$

Then, $\quad I_{r-c}(\varphi) = I_{s-c}(\psi)$ for all c.

(Exercise.)

Defn. $\mathrm{Fitt}_c^R(M) := I_{r-c}(\varphi)$, $\quad c^{\text{th}}$ Fitting ideal of M.

We get $\quad \mathrm{Fitt}_0(M) \subseteq \mathrm{Fitt}_1(M) \subseteq \cdots$ eventually R.

$\mathrm{Fitt}_0(M) \neq 0$ only if $r \leq s$

$\mathrm{Fitt}_c(M) = R$ for $c \geq r$.

Examples.

① $R = k$ a field, V some vector space (f.g.) of rank n.

$0 \to k^n$ is a presentation

$\mathrm{Fitt}_c(V) = k \iff c \geq n.$

② R a PID.

$$0 \to R^s \xrightarrow{\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \\ \hline & 0 & \end{pmatrix}} R^r \to M \to 0$$

(can assume $s \leq r$)

can put in normal form

$\mathrm{Fitt}_0(M) \neq 0 \iff r = s \iff M$ is torsion.

In this case, $\mathrm{Fitt}_0(M) = (d_1 \cdots d_r)$.

Properties. $R^s \xrightarrow{\varphi} R^r \to M \to 0.$

① If $R \to S$ is any map of rings.

$$\mathrm{Fitt}_c^S(S \otimes_R M) = S \cdot \mathrm{Fitt}_c^R(M).$$

② $(\mathrm{ann}_R(M))^r \subseteq \mathrm{Fitt}_0(M) \subseteq \mathrm{ann}_R(M).$

Proof. Pick an $(r\times r)$-minor 'a' in φ.

WTP: $a \cdot R^r \subseteq \mathrm{im}(\varphi)$

Can assume

$$\varphi = \underbrace{\Bigg(\underbrace{\diagdown\diagdown\diagdown}_{\substack{r\times r \\ \det = a}} \Bigg| \quad \Bigg)}_{r \times s} = A$$

Take the $s\times r$ matrix $\begin{pmatrix} \text{signed cofactor matrix}_{r\times r} \\ \hline 0 \end{pmatrix}_{s\times r} = B$

Then, $AB = \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}.$

$\therefore\ a R^n \subseteq \operatorname{im} \varphi.$ Proves $\operatorname{Fitt}_0 \subseteq \operatorname{ann}.$

Next: $\operatorname{ann}(M)^r \subseteq \operatorname{Fitt}_0(M).$

Fix $a_1, \ldots, a_r \in \operatorname{ann}(M).$

$$\begin{tikzcd} & R^r \arrow[dl, dashed] \arrow[d, "\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix}"] \\ R^s \arrow[r, "\varphi"'] & R^r \end{tikzcd}$$

Apply $\Lambda^r(\cdot)$ to the above to get

$$\begin{tikzcd} & R \arrow[dl] \arrow[d, "a_1 \cdots a_r"] \\ R^{\binom{s}{r}} \arrow[r, "\Lambda^r(\varphi)"'] & R \end{tikzcd}$$

③ Fix $c \geq 0$ and $\mathfrak{p} \in \operatorname{Spec}(R).$
TFAE:

(i) $\operatorname{Fitt}_c(M) \not\subseteq \mathfrak{p}.$

(ii) $\operatorname{im}(\varphi)_\mathfrak{p}$ contains a free summand of $R_\mathfrak{p}^r$ of rank $\geq r-c.$

(iii) $\nu_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq c.$

min'l # gens

Sketch. Can assume $(R, \mathfrak{p}, k)$ local. $\mathfrak{m} := \mathfrak{p}.$

$\operatorname{Fitt}_c(M) \not\subseteq \mathfrak{m} \iff \operatorname{Fitt}_c(R) = 0 \iff \operatorname{Fitt}_c(k \otimes_R M) \neq 0$

$\Updownarrow$ → first example

$\nu_R(M) \leq c \overset{\text{NAK}}{\iff} \nu_k(k \otimes_R M) \leq c$

④ Fix $c \geq 0$ and $\mathfrak{p} \in \operatorname{Spec} R.$ TFAE

(i) $\operatorname{Fitt}_{c-1}(M)_\mathfrak{p} = 0$ and $\operatorname{Fitt}_c(M)_\mathfrak{p} = R_\mathfrak{p}.$

(ii) $\operatorname{im}(\varphi)_\mathfrak{p}$ is a free summand of $R_\mathfrak{p}^r$ of rank $r-c.$

(iii) $M_p \cong (R_p)^c$.

Deduce from (3).

(5) $c \geq 0$. M is projective of rank c.

$\iff \mathrm{Fitt}_{c-1}(M) = 0$ and $\mathrm{Fitt}_c(M) = R$.

Hilbert-Birch Theorem

$R \to$ comm noetherian

Given $I \subseteq R$ with free resolution

$$0 \to R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \to 0.$$

Then, $\exists$ nzd $a \in R$ s.t. $I = a \cdot I_n(\varphi)$.

Moreover, if I is projective, then I is principal.

If $\mathrm{projdim}(I) = 1$, then $\mathrm{depth}(I_n(\varphi), R) \geq 2$.

Conversely, if $\varphi : R^n \to R^{n+1}$ is a matrix s.t. $\mathrm{depth}(I_n(\varphi), R) \geq 2$,

then $0 \to R^n \to R^{n+1} \to I_n(\varphi) \to 0$

is a free resolution.

Regular Local Rings are UFDs.

Thm. $M \to$ projective module

If M has a finite free resol^n, then it must have a free resol^n of length 1.

Say $0 \to F_s \to F_{s-1} \to \dots F_1 \to 0$.

$\downarrow$ M

M projective $\Rightarrow \partial(F_i)$ proj $\forall i \geq 1$ ... ∎

So, Projective + FFR $\Rightarrow$ stably free.
↳ finite free resolution

Corollary. $I \subseteq R$ proj + FFR $\Rightarrow$ I principal.

Proof. $0 \to R^n \to R^{n+1} \to I \to 0$.

I proj $\Rightarrow$ $I = (a)$. ∎

Thm. R comm. noe. domain s.t. each f.g. R-modules has FFR. Then, R is a UFD.

Corollary. Regular local rings are UFDs.
↳ not true otherwise

Proof. Suppose R is local. ($\because$ Regular.)
- Induction on dim R.
  - dim $R \leq 1$ is clear.

Pick $w \in m_R \setminus m_R^2$. Then, R/w is also RLR
and hence a domain.

$\therefore$ w is prime.
Suffices to verify R_w is a UFD.
Note dim $R_w <$ dim R.

Pick $p \in \operatorname{Spec}(R_w)$ with ht $= 1$.

Suffices to show p is principal.

Note. p has a FFR. (For p comes from R.)

It now suffices to prove it is projective.

(Earlier corollary.)

This can be tested locally. But localisations are RLRs of dim < dim R.

$\therefore$ UFD.

$\therefore \mathfrak{p}$ is locally free and hence projective.

For general R, again fix $\mathfrak{p}$ of ht $= 1$.

<u>WANT</u>: $\mathfrak{p}$ is principal.

Since $\mathfrak{p}$ has FFR, suffices to prove projective.

This is local. ∎

R $\to$ PID, M torsion module.

$$0 \to R^r \xrightarrow{\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}} R^r \to M \to 0 \qquad d_i \neq 0$$

$$\operatorname{Fitt}_0(M) = \left(\prod d_i\right) \subseteq \operatorname{ann}_R(M).$$

$$\operatorname{length}_R(M) = \operatorname{length}\left(R/\operatorname{Fitt}_0(M)\right)$$

Lecture 11 (15-02-2023)

Wednesday, February 15, 2023 1:18 PM

Complete Intersections

Cohen Structure Theory

$(R, \mathfrak{m}, k)$ local

- R is equicharacteristic if R contains a field (as a subring).
  - $\mathbb{Z} \longrightarrow R$ structure map.

    If $\operatorname{char} R = p > 0$ prime, then $\operatorname{char} k = p$.
  - When $\operatorname{char} k = 0$, $\mathbb{Q} \hookrightarrow R$.
- Mixed characteristic : $\mathbb{Z} \hookrightarrow R$ and $\operatorname{char} k = p > 0$.

  $\mathbb{Q} \not\hookrightarrow R$

When R is $\mathfrak{m}$-adically complete:

- If R equicharacteristic, then R contains a copy of k as a subring.

$$\begin{tikzcd} k \arrow[r, hook] \arrow[rr, bend right, "="'] & R \arrow[r, two heads] & k \end{tikzcd}$$

- R mixed characteris, then R contains a complete DVR $\mathcal{O}$ s.t.

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook] \arrow[rr, bend right, "\text{field of fractions}"'] & R \arrow[r, two heads] & k \end{tikzcd}$$

$$R = k[[x_1, \dots, x_n]]/I \qquad \text{(equi. case)}$$

$$R = \mathcal{O}[[x_1, \dots, x_n]]/I.$$

Thus, R is of the form S/I, where S is a regular local ring.

We call this a Cohen presentation of R.
(S/I, where S is an RLR.)

Say $R = S/I$ is a Cohen presentation.

$I \subseteq \mathfrak{m}_S$. Suppose $I \not\subseteq \mathfrak{m}_S^2$. Pick $x \in I \setminus \mathfrak{m}_S^2$.

$$\begin{tikzcd} S \arrow[rr, two heads] \arrow[dr, two heads] & & R \\ & S/xS \arrow[ur, two heads] & \end{tikzcd}$$

S/xS regular local

Thus, we can assume that $I \subseteq \mathfrak{m}_S^2$.

Equivalently,

$$\operatorname{edim} R = \operatorname{edim} S.$$

$$\operatorname{edim} R = \operatorname{rank}_k\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}\right) = \operatorname{rank}_k\left(\frac{\mathfrak{m}_S}{I + \mathfrak{m}_S^2}\right)$$

$$\operatorname{edim} S = \operatorname{rank}_k\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2}\right)$$

In this case, we say that S/I is a minimal Cohen presentation.

Defn. $(R, \mathfrak{m}, k)$ local is said to be a complete intersection (c.i.) if in some Cohen presentation

$$\hat{R} = S/I$$

$\mathfrak{m}$-adic

$\hat{R} = S/I$

$\mathfrak{m}$-adic completion of R

and I can be generated by a regular sequence in S.

[Not every c.i. is a quotient of a regular ring.]

Intrinsic characterisation?

Let $K^R :=$ Koszul complex on a minimal gen for $\mathfrak{m}_R$.

(Well defined up to iso of complexes (with multiplicative structure).)

- $\hat{R} \otimes_R K^R = K^{\hat{R}}$ $\quad (\because \mathfrak{m}\hat{R} = \mathfrak{m}_{\hat{R}}.)$

- $R \to \hat{R}$ induces

$$K^R \longrightarrow \hat{R} \otimes_R K^R = K^{\hat{R}}.$$

Then,

$$H(K^R) = H(\hat{R} \otimes_R K^R) \overset{\text{flatness of } R\to\hat{R}}{=} \hat{R} \otimes_R H(K^R)$$

Since $\mathfrak{m} H(K^R) = 0$, the map

$$H(K^R) \longrightarrow \hat{R} \otimes_R H(K^R)$$

is an iso.

- Thus $K^R \underset{\text{quasi}}{\xrightarrow{\sim}} K^{\hat{R}}$.

Say, $R = S/I$ is a minimal Cohen presentation.
$(I \subseteq \mathfrak{m}_S^2)$

Lemma. $\operatorname{rank}_k (H_i(K^R)) = \beta_i^S(R)$.

In particular, the RHS is independent of the min'l Cohen presentation.

Proof. Let $K^S \to$ Koszul complex on S.

S regular, so $K^S \xrightarrow{\sim} k$.

Since $I \subseteq \mathfrak{m}_S^2$,

$$R \otimes_S K^S = K^R.$$

(min'l gen set for $\mathfrak{m}_S$ gives a min'l gen set for $\mathfrak{m}_R$.)

$$\therefore H_*(K^R) = H_*(R \otimes_S K^S)$$
$$= \operatorname{Tor}_*^S(R, k).$$

$\because K^S \xrightarrow{\sim} k$

$\boxtimes$

$R = S/I \longrightarrow$ min'l Cohen presentation

$$\operatorname{ht} I \leq \nu_S(I)$$

$\nu_S(I) \to$ #gens of I as an S-module (Krull)

$\operatorname{ht} I = \operatorname{depth}_S(I, S)$, since S reg.

Recall: I is generated by a regular seq iff equality holds.

Thm, I is gen by regular sequence

$$\Leftrightarrow \nu_S(I) = \dim(S) - \dim(S/I) \quad \text{regular}$$

$$\beta_1^S(R) = \operatorname{edim}(S) - \dim(R)$$
$$= \operatorname{edim}(R) - \dim(R) \quad \because \text{min'l}$$

$\nu_S(I) = \beta_1^S(R) = \operatorname{rank}_k H_1(K^R)$

Therefore, if $R = S/I$ min'l Cohen presentation, then

I is gen. by a regular sequence

$$\iff \operatorname{rank}_k H_1(K^R) = \operatorname{edim}(R) - \dim(R).$$

Corollary. R is a local ring.
R is c.i. iff

$$\operatorname{rank}_k H_1(K^R) = \operatorname{edim}(R) - \dim(R).$$

Proof. All the numbers above are same for R and $\hat{R}$.

Remark. Say $\hat{R} = S/I$ is some Cohen presentation where $I = \langle \text{reg seq} \rangle$.

Then, we can reduce it to a min'l Cohen presentation where ideal $= \langle \text{reg seq} \rangle$.

Proof.

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & \hat{R} \\ & S/xS = S' \arrow[ur] & \end{tikzcd}$$

$x \in I \setminus \mathfrak{m}_S^2$.

Then, $x \in \mathfrak{m}_S I$ and so can be extended to a min gen set for I.

(x, y)

Then, y is a regular seq...

Theorem. R c.i. $\Rightarrow R_{\mathfrak{p}}$ c.i. $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Nontrivial!!!

This is clear if R itself has a Cohen presentation.

$R = S/I$, I gen. by reg. seq.

$\mathfrak{p} \in \operatorname{Spec} R \subseteq \operatorname{Spec} S$. $R_{\mathfrak{p}} \cong S_{\mathfrak{p}} / I S_{\mathfrak{p}} \ldots$

$(R, \mathfrak{m}, k)$:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R}^{\mathfrak{m}} \arrow[d, "?"] \\ R_{\mathfrak{p}} \arrow[r] & \hat{R_{\mathfrak{p}}}^{\mathfrak{p}R_{\mathfrak{p}}} \end{tikzcd}$$

not a localisation of $\hat{R}^{\mathfrak{m}}$.

Pick $\mathfrak{q} \in \operatorname{Spec} \hat{R}$ s.t. $\mathfrak{q} \cap R = \mathfrak{p}$.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R} \arrow[d] \\ R_{\mathfrak{p}} \arrow[r] & (\hat{R})_{\mathfrak{q}} \end{tikzcd}$$

flat + local

Theorem $\varphi : R \to S$ flat local map.
Then, S c.i. $\iff$ R and $S/\mathfrak{m}_R S$ are c.i.

Fact. R c.i. $\iff$ $R[\![x_1, \dots, x_n]\!]$ c.i.

$(R, \mathfrak{m}, k)$ local

$R = S/I$ min'l Cohen presentation (say it exists)

Write $\mathfrak{m} = (x_1, \dots, x_n)$ with $n = \operatorname{edim}(S)$.

$I = (f_1, \dots, f_c)$, min'l gen set.

$\mathfrak{m}_R = \underline{\bar{x}} \cdot R$.

K^R : $\longrightarrow \bigoplus^n R e_i \longrightarrow R \to 0$

K^R : $\quad \cdots \longrightarrow \bigoplus_{i=1}^{n} Re_i \longrightarrow R \to 0$

$\partial e_i = x_i$

Since $I \subseteq \mathfrak{m}_S^2$, can write

$$f_j = \sum_{i=1}^{n} S_{ij} x_i, \quad \text{where } S_{ij} \in \mathfrak{m}.$$

Let $z_j = \sum_{i=1}^{n} \overline{S_{ij}}\, e_i \quad \in \bigoplus_{i=1}^{n} Re_i.$

Note $\partial(z_j) = \sum_{i=1}^{n} \overline{S_{ij}}\, \overline{x_i} = \overline{f_j} = 0$

$\therefore z_1, \ldots, z_c$ are cycles.

Claim. The classes $[z_1], \ldots, [z_c] \in H_1$ form a k-basis.

Tate's result. $(R, \mathfrak{m}, k)$

$$R^n \xrightarrow{f} R \qquad \begin{array}{l} \mathrm{im}(f) = \mathfrak{m} \\ n = \mathrm{edim}(R) \end{array}$$

$K^R = (\Lambda^*(\Sigma R^n), \partial).$

$\rightsquigarrow K^R$ is a dg (= differential graded) R-algebra.
(Even graded -commutative.)

$\rightsquigarrow H(K^R)$ is a graded-commutative k-algebra.

Universal property gives: $\quad \chi^R : \Lambda_*(\Sigma H_1(K^R)) \longrightarrow H(K^R)$ map of k-algebras

$$H_1(K^R) = \Sigma k^c$$

Theorem (Tate) $\quad R$ is c.i. $\iff \chi^R$ is an isomorphism.

(Assmus) $\iff \Lambda^2 H_1(K^R) \twoheadrightarrow H_2(K^R).$

Lecture 12 (22-02-2023)

22 February 2023 13:26

$(R, m_R, k) \to$ local ring

R is c.i. $\equiv$ $\hat{R} \cong S/I$, where S regular
I gen by reg seq.
Can assume $I \subseteq m_S^2$.

Intrinsic characterisation:

$$\operatorname{rank}_k H_1(K^R) = \operatorname{edim}(R) - \dim(R)$$

(In general, $\geqslant$ holds.)

Key: $H_1(K^R) \cong I/m_S I$.

K^R is an exterior algebra with ∂ satisfying Leibniz rule.
K^R is a (commutative) differential graded R-algebra.

$\therefore H(K^R)$ is also graded-commutative k-alg. $\left(ab = (-1)^{|a||b|} ba \text{ for homog. } a, b.\right)$

By the universal prop. of exterior algebra,
$$H_1(K^R) \hookrightarrow H(K^R)$$
induces
$$\Lambda_k(\Sigma H_1(K^R)) \longrightarrow H(K^R) \quad \text{of graded } k\text{-algebra.}$$

- Image is the k-subalgebra of $H(K^R)$ generated by $H_1(K^R)$.

Theorem (Tate-Asmus) (R, m_R, k) local. TFAE:

1. R is complete intersection.
2. $\Lambda_k(\Sigma H_1(K^R)) \xrightarrow{\cong} H(K^R)$, i.e., $H(K^R)$ is the exterior algebra on $H_1(K^R)$.
3. The map above is surjective in deg two, i.e.,
$$H_1(K^R)^{\wedge 2} = H_2(K^R).$$

Can assume R is complete. Let $R = S/I$ min'l Cohen presentation.

Then $H_*(K^R) \cong \mathrm{Tor}_*^S(k, R)$ as k-algebras.

- S regular, so m_S is generated by a regular sequence.

Now, suppose S is any local ring and $I \subseteq J \subseteq m_S$ are ideals generated by regular sequences.

Want to compute $\mathrm{Tor}_*^S(S/J, S/I)$. (In the case we care about, $J = m_S$.)

Let $I = (\underline{a})$, where $\underline{a} = a_1, \dots, a_m \in m_S$ reg,
$J = (\underline{b})$, — " — $\underline{b} = b_1, \dots, b_n$ — " —.

$K(\underline{a}; S) \xrightarrow{\sim} S/I$ and
$K(\underline{b}; S) \xrightarrow{\sim} S/J$. } dg algebra maps

$$\mathrm{Tor}_*^S(S/J, S/I) = H_*\left[K(\underline{b}; S) \otimes_S S/I\right] \text{ as algebras.}$$
$$= H_*\left(K(\underline{b}; S/I)\right)$$

$$\begin{array}{ccc} K(\underline{b}; S) \otimes_S S/I & \xleftarrow{\sim} & K(\underline{b}; S) \otimes_S K(\underline{a}; S) \\ & & \downarrow \sim \\ & & S/J \otimes_S K(\underline{a}; S) = K(\underline{a}; S/J) \end{array}$$

map of dga

quasi iso $\because K(\underline{a}; S) \xrightarrow{\sim} S/I$ and $K(\underline{b}; S)$ fin. free

$K(\underline{a}; S/J) = K(\underline{0}; S/J)$ $\because I \subseteq J$

exterior algebra

$$H(K(\underline{0}; S/J)) = K(\underline{0}; S/J) = \wedge \underbrace{\Sigma (S/J)^m}_{H_1(K(\underline{0}; S/J))}$$

Therefore: $H(\underline{b}; S/I) \cong \wedge \Sigma H_1(\underline{b}; S/I)$

In summary: $\mathrm{Tor}^S(S/J, S/I) \cong \wedge \Sigma \, \mathrm{Tor}_1^S(S/J, S/I)$

$$\mathrm{Tor}_1^S(S/J, S/I) = \frac{I \cap J}{IJ} = I/IJ.$$

$\because I \subseteq J$

Thus:

$\because I \subseteq J$

Thus:

Lemma. If $I \subseteq J$ are gen. by a regular sequence, then

$$\mathrm{Tor}_*^S(S/J, S/I) \cong \Lambda\left(\Sigma\, I/IJ\right).$$

Moreover, I/IJ free S/J-module.

Specialising to our c.i. case, we get ① ⇒ ② of Tate-Assmus ...

② ⇒ ③ ok.

Proof of ③ ⇒ ①: Hypothesis:

$$\Lambda^2(\Sigma H_1(K^R)) \twoheadrightarrow H_2(K^R).$$

Equivalently,

$$\Lambda^2\, \Sigma\, \mathrm{Tor}_1^S(k,R) \twoheadrightarrow \mathrm{Tor}_2^S(k,R).$$

($R = S/I$ Cohen pres.)

Let $I = (\underline{a})$ with $\underline{a}$ min'l gen set.

Let $F \xrightarrow{\sim} R$ be a min'l free resolution of R over S.

$$F: \quad \cdots \to S^n \longrightarrow S^m \xrightarrow{(a_1 \cdots a_m)} S \longrightarrow 0$$

same as Koszul

$$\begin{tikzcd}
K(\underline{a}; S) \arrow[r, dashed] \arrow[d] & F \arrow[d, "\simeq"] \\
R = S/\underline{a}S \arrow[r, equal] & R
\end{tikzcd}$$

lift of identity on R.

A morphism of S-complexes.

WANT TO DEDUCE: $H_1(K(\underline{a}; S)) = 0$

$\therefore \underline{a}$ reg.

Can ensure that the lifting is id in degrees 0, 1.

$$\begin{tikzcd}
K(\underline{a}; S): \cdots \arrow[r] & S^{\binom{m}{2}} \arrow[r] \arrow[d, dashed, "*"] & S^m \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d, equal] & 0 \\
F: \cdots \arrow[r] & S^n \arrow[r] & S^m \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

ETP. $(*)$ is onto. Then a diagram chase shows $H_1 = 0$.

$\Rightarrow \underline{a}$ is reg.

$\Rightarrow R$ is c.i.

By NAK, suffices to prove

$$K_2(\underline{a}; S) \otimes_S k \longrightarrow F_2 \otimes_S k \quad \text{is onto}$$

Equivalently, surjectivity on H_2. ($\because$ differential becomes zero)

$$K(\underline{a}; S) \otimes_S k \longrightarrow F \otimes_S k.$$

In homology:

$$H_*(K(\underline{a}; k)) \longrightarrow H_*(F \otimes_S k) = \mathrm{Tor}_*^S(R, k)$$

$$\| $$

$$\Lambda \Sigma H_1(\underline{a}; k) \longrightarrow \mathrm{Tor}_*^S(R, k) \qquad (+)$$

Note. $H_1(\underline{a}; k) \xrightarrow{\cong} \mathrm{Tor}_1^S(R, k)$.

Hypothesis: Tor_2^S generated by Tor_1^S.

$\therefore (+)$ is onto. #

$(R, \mathfrak{m}_R, k)$ local ring.

$$\sup\{i \geq 0 : H_i(K^R) \neq 0\} = \mathrm{edim}(R) - \mathrm{depth}(R).$$

$\therefore H_1(K^R)^s = 0$ for $s > \mathrm{edim}(R) - \mathrm{depth}(R)$.

Theorem of Wiebe: R is c.i. $\Leftrightarrow H_1(K^R)^s \neq 0$ for $s = \mathrm{edim}(R) - \mathrm{depth}(R)$.

Proof $(\Rightarrow)$ By Tate.

Indeed, R c.i. $\Rightarrow H(K^R) = \Lambda \Sigma H_1(K^R)$

$H_1(K^R) \cong k^c$, $\quad c = \operatorname{edim} R - \dim R$
$= \operatorname{edim} R - \operatorname{depth} R.$

($\Leftarrow$) Can reduce to the case $\operatorname{depth}(R) = 0$.

Ek,
we can choose a min gen set $x_1, \dots, x_n$ $(n = \operatorname{edim} R)$
for m_R s.t. $x_1, \dots, x_d$ is a reg seq. on R, $d = \operatorname{depth} R$.

$$K(\underline{x}; R) = K(x_1, \dots, x_d; R) \otimes K(x_{d+1}, \dots, x_n; R)$$
$$\downarrow \simeq$$
$$\underbrace{R/(x_1, \dots, x_d)R}_{R'} \otimes_R K(x_{d+1}, \dots, x_n; R)$$

$$\therefore \; \underset{K^R}{\overset{\|}{K(\underline{x}; R)}} \xrightarrow{\simeq} \underset{K^{R'}}{\overset{\|}{K(x_{d+1}, \dots, x_n; R')}}$$

$\therefore H(K^R) \cong H(K^{R'})$ as k-algebras.

$\operatorname{edim} R - \operatorname{depth} = \operatorname{edim} R'$ (& $\operatorname{depth} R' = 0$).

R is c.i. $\iff$ R' is c.i.

Thus, can assume $\operatorname{depth} R = 0$. The hypothesis is

$$H_1(K^R)^{\operatorname{edim}(R)} \neq 0.$$

The desired conclusion is that R is c.i.

This hypothesis is equivalent to $\operatorname{Fitt}_0(M) \neq 0$.

Fitting ideals and Koszul homology

Say $J \subseteq R$ is an ideal. (R noe., possibly not local)

$$R^n \xrightarrow[(a_{ij})]{} R^m \xrightarrow[(r_1 \cdots r_m)]{} J \to 0 \qquad \text{presentation}$$

$$\mathrm{Fitt}_0(J) = I_{m\times m}((a_{ij})).$$

Koszul cx on $\underline{r} = r_1, \dots, r_m$.

$$K(\underline{r};R): \quad \cdots \to R^{\binom{m}{2}} \longrightarrow \overset{=\bigoplus Re_i}{R^m} \xrightarrow[(r_1 \cdots r_m)]{} R \to 0.$$

Cycles in $K_1(\underline{x};R)$ are precisely the syzygies of J.

$\left[\partial(\Sigma b_i e_i) = 0 \iff \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \operatorname{im}(a_{ij})\right.$

In particular

$$z_j := \sum_{i=1}^m a_{ij} e_i \quad \text{are cycles in } K_1(\underline{r};R)$$

for $j = 1, \dots, n$.

$$\langle z_{i_1} \wedge \cdots \wedge z_{i_m} : i_1 < i_2 < \cdots < i_m \rangle = I_{m\times m}((a_{ij}))\cdot K_m(\underline{r};R)$$

In summary: $H_1(\underline{r};R)^m = \mathrm{Fitt}_0(J)\cdot K_m(\underline{r};R)$.

Recall: $H_m(\underline{r};R) = \operatorname{ann}_R(J)$.

So, $H_1(\underline{r};R)^m \subseteq H_m(\underline{r};R)$ reflects

$\mathrm{Fitt}_0(J) \subseteq \operatorname{ann}(J)$.

One can restate (depth 0 case) of Wiebe as:

R is c.i. with $\dim R = 0$ $\iff$ $\mathrm{Fitt}_0(m) \neq 0$.

Lecture 13 (27-02-2023)

Monday, February 27, 2023 1:26 PM

Wiebe's Theorem

$S \to$ comm ring

$I \subseteq J$ where

$$I = (a_1, \dots, a_n),$$

$$J = (x_1, \dots, x_n), \quad \text{where } \underline{a}, \underline{x} \text{ are reg.}$$

We can write $\underline{a} = \mathcal{U}\, \underline{x}$.

Lemma. $(I : J) = I + (\det \mathcal{U})$, and $\det \mathcal{U} \notin I$.

Example. $R = k[x_1, \dots, x_n]$.

$$(x_1^{\ell}, \dots, x_n^{\ell}) \subseteq (x_1, \dots, x_n).$$

$$\mathcal{U} = \begin{bmatrix} x_1^{\ell-1} & & 0 \\ & \ddots & \\ 0 & & x_n^{\ell-1} \end{bmatrix}.$$

$$\left((x_1^{\ell}, \dots, x_n^{\ell}) : (x_1, \dots, x_n)\right) = (x_1^{\ell}, \dots, x_n^{\ell}) + x_1^{\ell-1} \cdots x_n^{\ell-1}.$$

Proof of Lemma: Equivalent:

$$\left(0 :_{S/I} J/I\right) = (\det \mathcal{U}) \cdot S/I.$$

Note $\quad (0 :_{S/I} J/I) = H_n(\underline{x}, S/I)$.

Also,

$$\begin{tikzcd} S^n \arrow[r, "(a_1,\dots,a_n)"] \arrow[d, "u"'] & S \arrow[d, equal] \\ S^n \arrow[r, "(x_1,\dots,x_n)"'] & S \end{tikzcd}$$

This induces $\quad \Lambda^* u : K(\underline{a}; S) \longrightarrow K(\underline{x}; S)$.

The top degree map is $K_n(\underline{a}; S) \xrightarrow{\det u} K_n(\underline{x}; S)$.

This induces $\quad \Lambda^* u : K(\underline{a}; S/I) \longrightarrow K(\underline{x}; S/I)$.

↳ all differentials 0 here

$\therefore H_n(*) = *$ for this

Need to prove $H_n(\Lambda^* u)$ is onto.

(The map is still $\xrightarrow{\det u}$.)

Moreover :

$$\begin{tikzcd} H_*(\underline{a}; S/I) \arrow[r] & H_*(\underline{x}; S/I) \\ \Lambda^* H_1(a; S/I) \arrow[r] \arrow[u, "\cong"] & \Lambda^* H_1(\underline{x}; S/I) \arrow[u, "\cong"', "\text{Tate}"] \end{tikzcd}$$

Suffices to now prove surjectivity at degree one.

$$H_1(\underline{a}; S/I) \longrightarrow H_1(\underline{x}; S/I)$$

$$\begin{tikzcd} & K(\underline{a}; S) \otimes K(\underline{x}; S) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ K(\underline{a}; S/I) & & K(\underline{x}; S/I) \end{tikzcd}$$

$$\begin{array}{ccc} K(\underline{a}; S/J) & & K(\underline{x}; S/I) \\ & \nwarrow & \nearrow \Lambda^* U \\ & K(\underline{a}; S/I) & \end{array}$$

both have zero diff.

$\therefore$ surjective on H^1

(or could've directly applied $H_n(-)$ here.)

Special Case

$R = S/I$, S is RLR, $I \subseteq m_S^2$, R is c.i.

(Assume R complete.)

- $\dim R = 0$
- $I = \underbrace{(a_1, \ldots, a_n)}_{\text{regular}} \subseteq \underbrace{(x_1, \ldots, x_n)}_{\text{regular}} = m_S$.

$n = \operatorname{edim} R$.

By lemma: $(0 :_R m_R) = (\det U)$.

$$\det U \neq 0$$

Def^n. (R, m_R, k) local.

$\operatorname{Soc}(R) = (0 :_R m_R) \cong \operatorname{Hom}_R(k, R)$.

socle

Note. m annihilates $\operatorname{soc}(R)$ by def^n.

$\therefore \operatorname{soc}(R)$ is a k-vector space. $\operatorname{rank}_k(\operatorname{soc}(R)) =: \operatorname{type}(R)$.

Corollary. R c.i. with $\dim R = 0$, then

$(0 :_R m_R) \subseteq$ any nonzero ideal.

(if $\dim \neq 0$, then $\operatorname{soc} = 0$ anyway.)

"c.i. rings are gorenstein"

Corollary. $\varphi : (R, m_R) \to T$.

Suppose R is c.i.

If $\varphi(\operatorname{soc} R) \neq 0$, then φ is an iso.

Weibe's Thm. (R, m_R) local

If $\operatorname{image}\left(\wedge^c H_1(K) \to H_c(K)\right) \neq 0$,

where $c = \operatorname{edim} R - \operatorname{depth} R$

then R is c.i.

We showed that we can reduce to $\operatorname{depth} R = 0$.

That is,

$$\wedge^n H_1(K^R) \to H_n(K^R) = Z_n(K^R) \quad \text{nonzero}$$

$$\Rightarrow R \text{ is c.i.} \quad \left(\begin{array}{l}\text{necessarily} \\ \dim R = 0\end{array}\right)$$

$$(\Leftarrow \quad \operatorname{Fitt}_0(m_R) \neq 0)$$

Can assume R is complete. Write

$R = S/I$ min'l Cohen presentation.

$$\frac{I}{m_S I} \leftrightarrow H_1(K^R).$$

Hypothesis, $\exists z_1, \ldots, z_n \in Z_1(K^R)$ s.t.

$$z_1 \wedge \cdots \wedge z_n \neq 0 \quad \text{in} \quad K_n^R.$$

Let $a_1, \dots, a_n \in I \setminus m_S I$ be some rep^n of $z_1, \dots, z_n$.

<u>Case 1.</u> Suppose $\dim(R) = 0$, i.e., $\text{height}(I) = n$.

Then, $\text{depth}_S(I, S) = n$ ($\because S$ regular).

<u>Exercise.</u> $\exists\, a_1', \dots, a_n' \in m_S I$ s.t.

$a_1 + a_1', \dots, a_n + a_n'$ is a reg. seq. in S.

$\therefore$ Can assume $a_1, \dots, a_n$ is a reg. seq.

$S/(\underline{a}) = R' \twoheadrightarrow R$ induce

$$\mathrm{Soc}(R') = H_n(K^{R'})$$

$$\begin{tikzcd}
\Lambda^n H_1(K^{R'}) \arrow[r] \arrow[d] \arrow[dr, "\neq 0"] & H_n(K^{R'}) \arrow[d, "\neq 0"] \subseteq R' \arrow[d] \\
\Lambda^n H_1(K^R) \arrow[r, "\neq", "0"'] & H_n(K^R) \subseteq R
\end{tikzcd}$$

$\Rightarrow$

image of this map contains $z_1 \wedge \dots \wedge z_n$

$\therefore\ \mathrm{soc}(R')R \neq 0$.

$\therefore\ R' = R$ by earlier part.

$\therefore\ R = S/(\underline{a})$ with $\underline{a}$ regular!

<u>Case 2</u> $\dim R > 0$ (will prove that this cannot be).

$z_1 \wedge \dots \wedge z_n \neq 0$ in $K_n^R \cong R$.

$\therefore \exists\, s \gg 0$ s.t. $z_1 \wedge \dots \wedge z_n \notin m_R^s$, by Krull int. thm.

$$\Lambda^n H_1(R) \longrightarrow H_n(R) \qquad z_1 \wedge \dots \wedge z_n \qquad R$$

$$\begin{tikzcd}
\Lambda^n H_1(R) \arrow[r] \arrow[d] \arrow[ddr, bend left, "\neq 0"'] & H_n(R) \arrow[d] & z_1\wedge\cdots\wedge z_n \arrow[dd, mapsto] & R \arrow[d] \\
\Lambda^n H_1(K^{R/m_R^{s+1}}) \arrow[r, "①"] \arrow[d] & H_n(R/m_R^{s+1}) \arrow[d, "②"] & & R/m_R^{s+1} \arrow[d] \\
\Lambda^n H_1(K^{R/m^s}) \arrow[r] & H_n(R/m^s) & \neq 0 & R/m_R^s
\end{tikzcd}$$

$\therefore$ ① $\neq 0$.

$\Rightarrow$ R/m_R^{s+1} is c.i by ①.

But ② $\neq 0 \Rightarrow R/m^{s+1} \to R/m^s$ is an iso (by socle corollary).

$\therefore m^s = m^{s+1}$.

But then NAK forces $m^s = 0$.

But then dim $= 0$. ∎

Wiebe.

$$\Lambda^i H(K^R) \longrightarrow H_i(K^R)$$

Tate : R c.i. $\Rightarrow$ iso for all i.

Assmus : Onto for $i = 2$ $\Rightarrow$ R c.i.

Wiebe : Non zero for $i =$ edim R $-$ depth R $\Rightarrow$ R c.i.

Bruns : Map is zero for $i >$ edim R $-$ dim R.

Lecture 14 (01-03-2023)

Wednesday, March 1, 2023 1:28 PM

$A \to$ any ring (possibly noncomm.)

$E \to$ A-module (left)

Defn. E is *injective* if $\mathrm{Hom}_A(-, E)$ is exact.

I.e., given any solid diagram

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[dl, dashed, "\tilde{f}"] \\ E & \end{tikzcd}$$

a dotted extension exists.

Thm. (Baer's Criterion)

E is injective $\iff$ $I \hookrightarrow A$ has the extension property for all left ideals I.

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & A \arrow[dl] & 1 \arrow[d, mapsto] \\ E & & x \end{tikzcd}$$

Thus: any map $I \to E$ is multiplication by some $x \in E$.

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$)

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[dl, dashed] \\ E & \end{tikzcd}$$

Consider all submodules U s.t.

$$N \le U \le M$$

$$\begin{tikzcd} & \\ E & \arrow[l, dashed, "f_U"'] \end{tikzcd}$$

$f \downarrow$

$\therefore$ Take max'l extension (U, f_U).

If $U = M$, done.

Else pick $x \in M \setminus U$.

Extend to $U + Ax \ldots$ ∎

Thus, enough to check extension along I.

Now, suppose R is a commutative domain.

① E injective $\Rightarrow$ E divisible. → Given $x \in E$, $r \in R \setminus \{0\}$, $\exists\, y \in E$ s.t. $ry = x$.

② Converse holds if R is a PID.

Equivalently,
$$\begin{tikzcd} R \arrow[r, hook, "r"] \arrow[d] & R \arrow[dl, dashed] \\ E & \end{tikzcd}$$
has extension property for all $r \neq 0$.

Example. Over $\mathbb{Z}$, $\mathbb{Q}$ is divisible and $\therefore$ injective.

Lemma. Any $\mathbb{Z}$-module can be embedded into an injective $\mathbb{Z}$-module.

Proof. $\mathbb{Q}$ divisible $\Rightarrow$ $\bigoplus \mathbb{Q}$ divisible

$\Rightarrow$ Any quotient of $\bigoplus \mathbb{Q}$ is also divisible

$\Rightarrow$ Any quotient of $\oplus \mathbb{Q}$ is also divisible

$\Rightarrow$ Any quotient of $\oplus \mathbb{Q}$ is injective.

Now, any free F can be embedded $F \hookrightarrow \oplus \mathbb{Q}$.

Now if $U \subseteq F$ submodule, then

$$F/U \hookrightarrow \oplus \mathbb{Q}/U.$$

Lemma. Let $A \longrightarrow B$ map of rings.

E : injective A-module.

Then,

$$\mathrm{Hom}_A(B, E)$$

viewed as a left B-module, is injective.

[If $f : B \longrightarrow E$ is A-linear, and $b \in B$ then

$$(b \cdot f)(x) = f(xb).$$

Check $b \cdot f$ is A-linear and this is an action.]

Proof. Need to check

$$\mathrm{Hom}_B(\_\_, \mathrm{Hom}_A(B, E)) \text{ is exact on mod-}B$$

But

$$\cong \mathrm{Hom}_A(\_\_, E).$$

Prop^n. $A \to$ any ring

Any A-module embeds into an injective A-module.

Proof. $\mathbb{Z} \longrightarrow A$ structure map.

$M \to A$-module

$M\downarrow_{\mathbb{Z}} \rightsquigarrow$ think of M as a $\mathbb{Z}$-module

$$M\downarrow_{\mathbb{Z}} \hookrightarrow E \quad \text{injective } \mathbb{Z}\text{-module}$$

$$\mathrm{Hom}_{\mathbb{Z}}(A, M\downarrow_{\mathbb{Z}}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, E).$$

A-modules

The map is also A-linear.

Suffices to embed

$$M \overset{\lambda}{\hookrightarrow} \mathrm{Hom}_{\mathbb{Z}}(A, M\downarrow_{\mathbb{Z}}).$$

$$\lambda(x)(a) := a\cdot x.$$

Essential Extensions

Defn. An extension $N \hookrightarrow M$ is essential if

$\forall\, U \overset{\neq 0}{\subseteq} M$ submodule

if $U \cap N \neq 0$.

It is proper if $N \neq M$.

Lemma. An A-module E is injective

$\iff$ E admits no proper essential extensions.

Proof. ($\Rightarrow$) E injective. If $E \overset{i}{\hookrightarrow} M$, essential then i split.

$F \oplus E = M$. But $F \cap E = 0$.

$\therefore F = 0$ by essentiality.

$\therefore M = E$.

($\Leftarrow$) Say E admits no proper essential inclusions.

Given
$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \\ E & \end{tikzcd}$$
, consider pushforward

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[d] \\ E \arrow[r, hook] & W \end{tikzcd} = \frac{M \oplus E}{\langle (n, f(n)) : n \in N \rangle}$$

In W, let $U \subseteq W$ be a max'l submodule s.t. $U \cap E = 0$.

But now
$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & W/U \\ & W \arrow[ur] & \end{tikzcd}$$
is injective and essential.

$\therefore$ Isomorphism.

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[d, "g"] \\ E \arrow[r, hook] \arrow[d, "\varphi"', "\cong"] & W \arrow[dl, "p"] \\ W/U & \end{tikzcd}$$

Define $M \longrightarrow E$ as $\varphi^{-1} \circ p \circ g$. Desired extension. $\blacksquare$

Def^n. M any A-module.

An injective hull of M is an essential extension
$$M \hookrightarrow E$$
with E injective.

(This E is unique up to iso.)

Prop^n. Let M be an A-module.

① Take $M \hookrightarrow I$ with I injective.
Then, any maximal essential extension of M in I is injective. (And hence an essential hull of M.)

② $M \hookrightarrow I$ with I injective, and $M \hookrightarrow E$ is an injective hull, then $\exists$ mono $E \hookrightarrow I$ s.t.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, hook] & E \arrow[d, hook] \\ & I \end{tikzcd}$$

commutes.

$\therefore E$ is a "minimal injective module containing M".

③ $M \hookrightarrow E$ and $M \hookrightarrow E'$ injective hulls, then $\exists$ comm diagram

$$\begin{tikzcd} & M \arrow[dl, hook] \arrow[dr, hook] & \\ E \arrow[rr, "\cong"'] & & E' \end{tikzcd}.$$

Proof. ②, ③ OK.

①. Let $M \hookrightarrow E \hookrightarrow I$.
E: max'l ess. ext^n of M in I.

ETP: E admits no proper essential extension.

Say $E \hookrightarrow U$ essential.

Then,

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, hook] & U \arrow[d, "i"] \\ & I \end{tikzcd}.$$

Moreover, $\ker(i) \cap E = 0$.
$\therefore \ker i = 0$.
$\Rightarrow i$ is injective.

n.t. the $E \subset U \subseteq I$ in I.

But the $E \subseteq U \subseteq I$ in I.

maximality $\Rightarrow E = U$. $\blacksquare$

($M \hookrightarrow E$ ess + $E \hookrightarrow U$ ess $\Rightarrow M \hookrightarrow U$ ess.)

Key Consequence.

$A \to$ any ring

$M \to$ any module.

Then, M admits a unique min'l injective resolution, which is unique up to iso (of complexes). $\blacksquare$

$$\begin{tikzcd}
 & E_A(M) \arrow[rr] \arrow[dr, two heads] & & E_A(M^1) \arrow[rr] \arrow[dr, two heads] & & E_A(M^2) \arrow[dr, two heads] & \\
M \arrow[ur, hook, "i^0"] & & M^1 \arrow[ur, hook, "i^1"] & & M^2 \arrow[ur, hook, "i^2"] & & \cdots
\end{tikzcd}$$

$M^1 = \operatorname{coker}(i^0)$

$$M \simeq 0 \to E^0 \to E^1 \to \cdots$$

$$\operatorname{Ext}^n_A(-, M) = H^n(\operatorname{Hom}_A(-, M)).$$

Remark. $\{E_i\}$ any family of injectives, $\prod E_i$ is injective.

Baer's Criterion $\Rightarrow$ If A is (left) Noetherian, then $\bigoplus E_i$ is injective.

Lecture 15 (13-02-2023)

Monday, March 13, 2023 1:24 PM

Structure of Injectives

R : M, N

$$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(P, N)).$$

where $P \xrightarrow{\sim} M$

E is injective $\iff \mathrm{Ext}^1_R(-, E) = 0$ on Mod R

$\iff \mathrm{Ext}^1_R(R/I, E) = 0$ for all ideals $I \subseteq R$.

$\iff \mathrm{Ext}^i_R(-, E) = 0$ on Mod R $\forall i \geq 1$

Note. $0 \to X \to Y \to Z \to 0$ is exact in Mod R

gives $0 \to \mathrm{Hom}(Z, E) \to \mathrm{Hom}(Y, E) \to \mathrm{Hom}(X, E) \to \mathrm{Ext}^1(Z, E) \to \cdots$

R commutative noetherian.

① $\{E_\lambda\}_\lambda$ injective $\Rightarrow \bigoplus_\lambda E_\lambda$ injective.

② Any injective can be written as a direct sum of indecomposable injectives.

indecomposable injectives.

③ The indecomposable injectives are precisely those of the form

$$E_R(R/\mathfrak{p})$$

for $\mathfrak{p} \in \operatorname{Spec} R$. (These are pairwise non-iso.)

④ $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion.

M is $\mathfrak{p}$-local if $M \xrightarrow{\sim} M_\mathfrak{p}$ is an iso.

$\Leftrightarrow$ $M \xrightarrow{r} M$ is an iso for $r \in R \setminus \mathfrak{p}$

$\Leftrightarrow$ R action on M factors through $R \to R_\mathfrak{p}$

every $x \in M$ is annihilated by $\mathfrak{p}^n$ $(n = n(x))$

i.e.,

$$\bigcup_{n \geq 0} (0 :_M \mathfrak{p}^n) = M$$

$$\|$$

$$\Gamma_{V(\mathfrak{p})}(M)$$

Lemma. $S := U^{-1}R$, where $U \subseteq R$ mult. closed.

$R \to S$. $\quad$ $R \to \hat{R}$ is an example.

① E injective R-module $\Rightarrow$ $S \otimes_R E$ injective S-module.

(Not true for arbit flat maps.)

② Any injective S-module is injective over R.

③ $M \hookrightarrow N$ is essential over R

$\Rightarrow$ $S \otimes_R M \hookrightarrow S \otimes_R N$ is essential.

In particular, $S \otimes_R E_R(M) = E_S(S \otimes_R M)$.

Proof. ② Holds for any flat map $R \to S$.

Let I be inj S-module.

$\operatorname{Ext}^1_R(M, I) \cong \operatorname{Ext}^1_S(S \otimes_R M, I) = 0$. ✓

$$\mathrm{Ext}^1_R(M, I) \cong \mathrm{Ext}^1_S(S \otimes_R M, I) = 0. \quad \checkmark$$

$\because$ I S-module and $R \to S$ flat

① Any f.g. S-module is of the form $S \otimes_R M$ with M f.g. over R.

("Clear denominators of generators.")

$$\mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R E) \cong \mathrm{Ext}^1_R(M, S \otimes_R E)$$
$$\cong S \otimes_R \mathrm{Ext}^1_R(M, E) = 0.$$

We used the following key property of localisation:

$R \to S$ flat

$S \otimes_R S \xrightarrow{\cong} S$ iso.

Defines "absolutely flat map"

③ WTS $U^{-1}M \hookrightarrow U^{-1}N$ is essential.

Equiv: $S\left(\frac{x}{u}\right) \cap U^{-1}M \neq 0$ for $\frac{x}{u} \neq 0$.

($x \in N$, $u \in U$.)

Can even just check when $u = 1$.

Of course, $x \neq 0$ in N.

Consider $\{ \mathrm{ann}_R(ux) : u \in U \} =: \Sigma$

Note $ux \neq 0$ since $\frac{x}{1} \neq 0$.

Noetherian $\Rightarrow$ Σ has maximal element(s).

(Proper ideals.)

Replace x by a suitable ux to ensure that $\mathrm{ann}_R(x)$ is a maximal el't of Σ.

$0 \neq M \cap Rx = (r_1 x, \ldots, r_n x), \quad r_i \in R.$

Say $r \in R$ is s.t.

$rx = 0$ in $U^{-1}N$.

$\Rightarrow urx = 0$ for some $u \in U$, i.e.,

$\Rightarrow r \in \operatorname{ann}_R(ux)$ - - - -

[always contains $\operatorname{ann}_R(x)$.

$\therefore r \in \operatorname{ann}_R(x)$.

Now, pick $r_i \in \{r_1, \ldots, r_n\}$ s.t. $r_i x \neq 0$.

Then, $r_i x \in U^{-1}M \cap S(\frac{x}{1}) \setminus \{0\}$. ∎

Lemma. Fix $\mathfrak{p} \in \operatorname{Spec} R$.

① $E_R(R/\mathfrak{p})$ is indecomposable.

② $\operatorname{Ass}_R(E_R(R/\mathfrak{p})) = \{\mathfrak{p}\}$.

③ $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion.

④ $\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(R/\mathfrak{p})) \cong k(\mathfrak{p})$.

Proof.

① A stronger property holds: any two nonzero modules in $E_R(R/\mathfrak{p})$ has nontrivial intersection.

($\equiv$ any submodule is indecomposable)

Let $M, N \neq 0$ be nonzero submodules.

$R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p})$ essential.

So, $M \cap R/\mathfrak{p} \neq 0$,

$N \cap R/\mathfrak{p} \neq 0.$

$(M \cap R/\mathfrak{p}) \cap (N \cap R/\mathfrak{p}) \neq 0$

nonzero ideals in domain

$\therefore \quad M \cap N \neq 0.$

This proves ①

$\cdot \quad \underbrace{R/\mathfrak{p} \hookrightarrow k(\mathfrak{p})}_{\text{ess}} \underbrace{\hookrightarrow}_{\text{ess}} E_{R_\mathfrak{p}}(k(\mathfrak{p}))$

$\hookrightarrow$ inj $/R_\mathfrak{p}$

$\Rightarrow$ inj $/R$

$\therefore E_R(R/\mathfrak{p}) = E_{R_\mathfrak{p}}(k(\mathfrak{p}))$ previous

$\cong R_\mathfrak{p} \otimes_R E_R(k(\mathfrak{p})).$

② follows from: $\forall M \subset \text{Mod } R$

$\text{Ass}_R(E_R(M)) = \text{Ass}_R(M)$

($\supseteq$) true since $M \hookrightarrow E_R(M)$.

($\subseteq$) Let $\mathfrak{p} \in \text{Ass}_R(E_R(M))$.

$R/\mathfrak{p} \hookrightarrow M.$ Let $U \cong R/\mathfrak{p}$ be the image.

$U \cap M \neq 0.$

$\therefore \text{Ass}(U \cap M)$ has an asso prime.

But $\text{Ass}(U \cap M) \subseteq \text{Ass}(U) = \{\mathfrak{p}\}.$

$\therefore \mathfrak{p} \in \text{Ass}(U \cap M) \subset \text{Ass}(M).$ ☒

This also implies $\mathfrak{p}$-power torsion.

④ May assume R local and $\mathfrak{p}$ max'l.

④ May assume R local and $\mathfrak{p}$ max'l.

NTP $\quad \mathrm{Hom}_R(k, E_R(k)) \cong k.$

The Hom is $\neq 0$ since $i : k \hookrightarrow E_R(k)$ is in the Hom.

Moreover, $\mathrm{Hom}_R(k, E_R(k))$ is a k-subspace of $E_R(k)$. But it's indecomposable by our earlier statement.

$\therefore$ it is k. ∎

Thm. Any injective can be written as a direct sum of indecomposable injectives.

Proof $E \to$ inj. R-module.

Pick $\mathfrak{p} \in \mathrm{Ass}_R(E)$. $\therefore \; R/\mathfrak{p} \hookrightarrow E$. This factors:

$$R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p}) \hookrightarrow E.$$

Now Zorn… ∎

This finishes proof all of the main properties!

Uniqueness?

Suppose

$$E \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, R} E(R/\mathfrak{p})^{\nu(\mathfrak{p})}.$$

$\neq 0$ iff

Then, $\nu(\mathfrak{p}) = \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E_\mathfrak{p})$ $\neq 0$ iff $\mathfrak{p} \in \operatorname{Ass}_R E$.

and hence independent of decomposition.

Key: $\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(R/\mathfrak{q})_\mathfrak{p}) = \begin{cases} k(\mathfrak{p}) & \mathfrak{p} = \mathfrak{q} \\ 0 & \text{else} \end{cases}$ by earlier

~~Say $\mathfrak{q} \neq \mathfrak{p}$.~~

~~If $\mathfrak{q} \not\subseteq \mathfrak{p}$, $E(R/\mathfrak{q})$ is $\mathfrak{q}$-pow torsion~~

~~but $\exists\, a \in \mathfrak{q} \setminus \mathfrak{p}$. so~~

~~a is inverted but kills $E(R/\mathfrak{q})$.~~

Use the following:

$\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M) \neq 0 \iff \mathfrak{p} \in \operatorname{Ass}(M)$.

$$M \hookrightarrow E_R(M) = \bigoplus_{\mathfrak{p} \in \operatorname{Ass}_R(M)} E(R/\mathfrak{p})^{\mu(\mathfrak{p})}$$

$$\mu(\mathfrak{p}) = \operatorname{rank}_{R_\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E_R(M)_\mathfrak{p})$$

$$= \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

In particular, if M is f.g., then $\mu(\mathfrak{p})$ is finite.

Defn. $M \in \operatorname{Mod} R$. The ith Bass number of M is wrt $\mathfrak{p}$

$$\mu_R^i(\mathfrak{p}; M) := \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

$$\mu_R^i(\mathfrak{p}; M) := \operatorname{rank}_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

Let $M \xrightarrow{\simeq} I$ be a min'l inj. resolution.

Then $$I^i \cong \bigoplus E(R/\mathfrak{p})^{\mu^i(\mathfrak{p}; M)}.$$

Lecture 16 (15-03-2023)

Wednesday, March 15, 2023 1:26 PM

Matlis Duality

$R \to$ comm noetherian

① $M \otimes_R \operatorname{Hom}_R(N, E) \xrightarrow{\text{natural map}} \operatorname{Hom}_R(\operatorname{Hom}_R(M, N), E)$. (+)

$x \otimes \alpha \longmapsto [f \mapsto \alpha\, f(x)]$

adjoint

$\operatorname{Hom}(M, N) \otimes_R M \otimes_R \operatorname{Hom}_R(N, E) \longrightarrow E$

$N \otimes \operatorname{Hom}_R(N, E)$

(+) is bijective when E is injective and M is f.g

Sketch. True when $M = R$.

$\therefore$ true when $M = R^{\oplus n}$.

Next, present M as

$G \to F \to M \to 0$ F, G finite free

Then,

$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(F, N) \to \operatorname{Hom}_R(G, N)$ is exact.

$\because E$ injective

$\operatorname{Hom}_R(\operatorname{Hom}_R(G,N), E) \to \operatorname{Hom}_R(\operatorname{Hom}_R(F,N), E) \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,N), E) \to 0.$

tensoring $\cong$ isom.s since $\cong$ $\Longrightarrow$ $\cong$

tensoring right exact

$$\begin{array}{ccccccc} \downarrow \cong & & \downarrow \cong & \Longrightarrow & \downarrow \cong & & \\ G \otimes_R \mathrm{Hom}_R(N, E) & \longrightarrow & F \otimes_R \mathrm{Hom}_R(N,E) & \longrightarrow & M \otimes_R \mathrm{Hom}_R(N,E) & \longrightarrow & 0 \end{array}$$

isoms since F, G fin free

$\blacksquare$

Corollary. If E and E' are injective $\Rightarrow$ $\mathrm{Hom}_R(E', E)$ flat.

Proof. $- \otimes_R \mathrm{Hom}_R(E', E) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(-, E'), E)$ on f.g. mod R.

But RHS is an exact functor.

$\therefore$ $- \otimes_R \mathrm{Hom}_R(E', E)$ is exact on fg mod R.

$\Longrightarrow$ —"— on Mod-R. $\blacksquare$

Thus, $\mathrm{Hom}_R(-, E) : \mathrm{Inj}\ R \longrightarrow \mathrm{Flat}\ R$.

Check. $\mathrm{Hom}_R(-, E) : \mathrm{Flat}\ R \longrightarrow \mathrm{Inj}\ R$.

---

② $(R, m, k) \twoheadrightarrow S \simeq R/I$.

Claim $\mathrm{Hom}_R(S, E_R(k)) \cong E_S(k)$. ← Injective over S.

$$= (0 :_{E_R(k)} I)$$ ← largest S-module of $E_R(k)$

$$\left(\begin{array}{ccc} k & \hookrightarrow & E_R(k) \\ & \searrow & \uparrow \\ & & (0 :_{E_R(k)} I) \end{array} \right)$$

[illegible] ring (R, m, k)

Fix a local (comm noe) ring $(R, \mathfrak{m}, k)$.

Fix $E := E_R(k)$.

$$(-)^\vee := \mathrm{Hom}_R(-, E).$$

all R-modules

$$\begin{array}{ccc} \mathrm{Mod}\ R & \xrightarrow{(\ )^\vee} & \mathrm{Mod}\ R. \\ \cup\!| & & \\ \mathrm{mod}\ R & \longrightarrow & ? \end{array}$$

f.g. R-modules

Consider $M \xrightarrow{\text{eval}} M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$

$x \longmapsto (f \mapsto f(x))$

Obs.

- $R^\vee = E$
- $R^{\vee\vee} = E^\vee = \mathrm{Hom}_R(E, E) = \mathrm{End}(E)$ ← flat, by earlier.

$R \xrightarrow{\text{eval}} \mathrm{End}_R(E)$ → ring homomorphism even

$r \longmapsto (E \xrightarrow{\cdot r} E)$

will see $\mathrm{End}_R(E) \cong \hat{R}$ as rings (and R-mod)

If M is f.g., we have a comm diagram

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E) \\ \cong\uparrow & & \uparrow\cong \\ M \otimes_R R & \longrightarrow & M \otimes_R \mathrm{End}_R(E) \end{array}$$

Lemma. ① When N has finite length, then

(a) $\ell_R(N) = \ell_R(N^\vee)$.

(R = field recovers usual duality)

Lemma ①

(a) $\ell_R(N) = \ell_R(N^\vee)$.

(b) $\beta_0^R(N) = \mu_R^0(N^\vee)$,

$\mu_R^0(N) = \beta_0^R(N^\vee)$.

(c) $N \xrightarrow{\cong} N^{\vee\vee}$.

(recovers usual duality)

$\beta_0^R(M) :=$ min'l # gens of N, or $\dim_k(M/\mathfrak{m}M)$

$\mu^0(M) := \operatorname{rank}_k \operatorname{Hom}_R(k, M)$

$= \operatorname{rank}_k \operatorname{Soc}_R(M)$

② When R is artinian,

$$(\ )^\vee : (\underset{\text{f.g.}}{\operatorname{mod}}\ R)^{op} \longrightarrow \operatorname{mod}\ R$$

is an equiv of categories.

(If we drop "artinian", it's an equiv on fin length R-mod.)

Proof. ① Recall $\operatorname{Hom}_R(k, E) \cong k$.

$\therefore k^{\vee\vee} \cong k$.

$\Rightarrow k \xrightarrow{\cong} k^{\vee\vee}$ ($\because$ nonzero and $\dim = 1$.)

$\therefore$ (a), (c) hold for k.

Now induce on $\ell_R(N)$.

$$0 \to k \hookrightarrow N \to N' \to 0.$$

$\ell_R(N') = \ell_R(N) - 1$.

Apply $(-)^\vee$

$$0 \to (N')^\vee \to N^\vee \to k \to 0.$$

$$\Rightarrow \ell_R(N^\vee) = \ell_R((N')^\vee) + 1$$ induction

$$= \ell_R(N') + 1$$

$$= \ell_R(N).$$

Similarly use

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] \arrow[d, "\cong"'] & N \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & k^{\vee\vee} \arrow[r] & N^{\vee\vee} \arrow[r] & N'^{\vee\vee} \arrow[r] & 0 \end{tikzcd}$$

for (c).

(b) $\beta_0^R(N) = \operatorname{rank}_k (k \otimes_R N)$

$= \operatorname{rank}_k \operatorname{Hom}_R(k \otimes N, E)$

$= \operatorname{rank}_k \operatorname{Hom}_R(k, N^\vee)$

$= \mu_R^0(N^\vee)$. ✓

Suppose R is $\mathfrak{m}$-adically complete.

Lemma $(R, \mathfrak{m}, k)$ complete local. Then

$$R \xrightarrow{\cong} \operatorname{End}_R(E).$$

Proof. For each $n \geq 1$, consider

$E_n := \operatorname{Hom}(R/\mathfrak{m}^n R, E) \subseteq E$, $E_n = E_{R/\mathfrak{m}^n R}(k)$

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{End}_R(E) \arrow[d, "\text{restriction}"] \\ \dfrac{R}{\mathfrak{m}^n R} \arrow[r, "\cong"'] & \operatorname{End}_{R/\mathfrak{m}^n R}(E_n) \end{tikzcd}$$

(only because ...)

$\because$ Artinian

We get $R \xrightarrow{\cong} \varprojlim_n \frac{R}{\mathfrak{m}^n R} \xrightarrow{\cong} \varprojlim_n \mathrm{End}(E_n) \xleftarrow{\cong} \mathrm{End}_R(E)$.

iso $\because$ R-complete

have to check

Follows since $E = \bigcup_{n \geq 0} E_n$.

Theorem. R complete local ring.

One has an equivalence of categories

$$\mathrm{mod}\ R \underset{(\ )^\vee}{\overset{(\ )^\vee}{\rightleftarrows}} \mathrm{art}\ R.$$

f.g. artinian

(In particular, E is Artinian.)

Proof. ① We show E is Artinian.

Say $E \supseteq M_0 \supseteq M_1 \supseteq \cdots$.

This gives $R = E^\vee \twoheadrightarrow M_0 \twoheadrightarrow M_1 \twoheadrightarrow \cdots$.

Surjections correspond to ascending chain of ideals

$\therefore$ Stabilise eventually.

$0 \to M_n \hookrightarrow M_{n+1} \to M_{n+1}/M_n \to 0$

$\Rightarrow M_n = M_{n+1} \quad \forall n \gg 0.$

dualising

$\Rightarrow \left(\frac{M_{n+1}}{M_n}\right)^\vee = 0.$

But M_{n+1}/M_n is $\mathfrak{m}$-torsion being a subquotient of E.

$\therefore$ if nonzero, then $k \hookrightarrow \frac{M_{n+1}}{M_n}$.

But then, $\left(M_{n+1}/M_n\right)^\vee \twoheadrightarrow k^\vee$. $\rightarrow\leftarrow$

But then, $\left(M_{n+1}/M_n\right)^\vee \twoheadrightarrow k^\vee$. $\rightarrow\leftarrow$

$\therefore$ E is Artinian.

This now implies $M^\vee$ artinian for all f.g. M.

Now, if M artinian, note

$$\begin{array}{ccc} \mathrm{soc}(M) & \hookrightarrow & M \\ \wr\Vert & & \\ k^\mu & & \end{array}$$

$\therefore$ $\mathrm{soc}(M) \hookrightarrow E^\mu$.

$\because M$ is artinian, $\mathrm{soc}(M) \hookrightarrow M$ is essential. $(*)$

$\therefore$ $M \hookrightarrow E^\mu$.

Now, dualise to get $R^\mu \twoheadrightarrow M^\vee$.

$\Rightarrow M$ is f.g.

For f.g. M, we have

$$\begin{array}{ccc} M & \longrightarrow & M^{\vee\vee} \\ \cong\uparrow & & \uparrow\cong \\ M \otimes_R R & \xrightarrow[\cong]{} & M \otimes_R \mathrm{End}_R(E) \end{array}$$

M f.g.

By Lemma

$\therefore$ top is $\cong$.

Need to check for artinian.

Okay for E. For general artinian, do copresentation.

$R \twoheadrightarrow k$ $\qquad$ mod k

$\mathrm{Hom}_k(-, k)$

mod $k \subseteq$ mod R

"Can we lift the vector space duality to mod R?"

Is there is an R-module I s.t.

$\mathrm{Hom}_R(-, I) \cong \mathrm{Hom}_k(-, k)$ on mod k.

exact on mod R

$I = E$ does the job, by Matlis duality.

Lecture 17 (20-03-2023)

Monday, March 20, 2023 1:24 PM

Matlis Duality

$(R, \mathfrak{m}, k)$. $E = E_R(k)$.

ex: $M \neq 0 \implies \operatorname{Hom}_R(M, E) \neq 0$.

i.e., E is a cogenerator for Mod R.

Ex. R comm noetherian (not necessarily local).

Then, $\bigoplus_{\mathfrak{m} \in \operatorname{Max}(R)} E_R(R/\mathfrak{m})$ is a cogenerator. $\rightsquigarrow$ turns out to be $\mathbb{Q}/\mathbb{Z}$

Key computation:

$$\hat{R} \xrightarrow{\cong} \operatorname{End}_R(E).$$

Matlis duality followed from ↑ and E being injective.

$A \rightarrow$ any ring (possibly non comm, non noe ...)

I an A-complex.

- I is **K-injective** if $\operatorname{Hom}_A(-, I)$ preserves quasi isomorphisms.

That is, if $f : X \rightarrow Y$ is a map of A-cx's s.t. $H(f)$ is bijective, then

$$\operatorname{Hom}_A(f, I) : \operatorname{Hom}_A(Y, I) \rightarrow \operatorname{Hom}_A(X, I)$$

is bijective in homology.

- I is **semi-injective** if $\operatorname{Hom}_A(-, I)$ takes

- I is semi-injective if $\mathrm{Hom}_A(-, I)$ takes $\{$mono + quasi iso$\}$ to $\{$epi + quasi iso$\}$.

FACT: I semi-injective $\iff$ I K-injective + I^n injective $\forall n$

FACT: Any complex has a semi-injective resolution:

$\forall M$: $M \xrightarrow{\sim} I.$ ← semi-injective

Can also assume $\to$ is one-one.

Any two such resolutions are homotopy equivalent.

M, N A-complexes.

$P_M \xrightarrow{\sim} M$ and $P_N \xrightarrow{\sim} N$ semi proj resol^{n}s.

Then,

$$
\begin{aligned}
R\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(P_M, P_N)& \\
\downarrow \simeq& \\
\mathrm{Hom}_A(P_M, N)&.
\end{aligned}
$$

Similarly, if $M \xrightarrow{\sim} I_M$ and $N \xrightarrow{\sim} I_N$ are semi-inj resol^{n}s, then

$$
\begin{array}{ccc}
\mathrm{Hom}_A(M, I_N) & \xrightarrow{\sim} & \mathrm{Hom}_A(P_M, I_N) \\
\uparrow \simeq & & \uparrow \simeq \\
\mathrm{Hom}_A(I_M, I_N) & & \mathrm{Hom}_A(P_M, N)
\end{array}
$$

$$\operatorname{Ext}^i(M,N) = H^i(R\operatorname{Hom}_A(M,N)).$$

R commutative noetherian

$$M \in D^b(\operatorname{mod} R)$$

$M \xrightarrow{\sim} I$ <u>minimal</u> injective resol^n.

<u>Minimality</u> (defn): induced differential on $\operatorname{Hom}_{R_p}(k(p), I_p)$ is zero for all $p \in \operatorname{Spec} R$.

In the case R local and $p = m$:

$\operatorname{Hom}_R(k, I)$:

$$0 \to I^a \to \cdots \to I^n \to I^{n+1} \to \cdots$$

$$\Big\downarrow \operatorname{Hom}_R(k, -)$$

$$0 \to \operatorname{Hom}_R(k, I^a) \to \cdots \to \operatorname{Hom}_R(k, I^n) \to \cdots$$

$$\parallel \qquad\qquad \parallel$$

$$0 \to \operatorname{soc}(I^a) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \operatorname{soc}(I^n) \to \cdots$$

<u>Minimality</u> means $\partial(\operatorname{soc}(I^n)) = 0$.

Minimality gives:

$$\operatorname{Ext}^i_{R_p}(k(p), M_p) = H^i(\operatorname{Hom}_{R_p}(k(p), I_p))$$

$$= \operatorname{Hom}_{R_p}(k(p), I^i_p)$$

$\because$ differentials $= 0$

$$\Rightarrow \operatorname{rank}_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), M_p) = \#\text{ copies of } E(R/p) \text{ in } I^i$$

$$\parallel$$

$$\mu_R^i(\mathfrak{p}, M) \qquad \text{(Recall Bass number)}$$

Ex. $M, N \in D^b(\mathrm{mod}\ R)$, then the R-module $\mathrm{Ext}_R^i(M, N)$ is f.g. $\forall i$.

In particular, $\mu_R^i(\mathfrak{p}, M) < \infty$.

- $\mathrm{inj\,dim}_R(M) = \underset{\text{sup or inf}}{?} \{ n : M \xrightarrow{\sim} I \text{ semi inje resol}^n \text{ with } I^i = 0 \ \forall i > n \}$

WANT injdim $(0) = +\infty$.

(Possibly incorrect for $M = 0 \ldots$)

$$= ? \{ n \mid \mu_R^i(\mathfrak{p}, M) = 0 \quad \forall i > n, \ \forall \mathfrak{p} \in \mathrm{Spec}\, R \}.$$

Theorem 1 Fix $M \underset{\neq 0}{\in} D^b(\mathrm{mod}\ R)$.

Then,

$$\mathrm{inj\,dim}_R M = \inf \{ n \mid \mu_R^i(\mathfrak{m}, M) = 0 \quad \forall i > n, \ \forall \mathfrak{m} \in \mathrm{Max}(R) \}.$$

In particular, if $(R, \mathfrak{m})$ is local, then

$$\mathrm{injdim}_R M = \sup \{ n : \mu_R^i(k, M) \neq 0 \}.$$

Recall: $\mathrm{depth}_R(M) = \inf \{ n : \mathrm{Ext}_R^n(k, M) \neq 0 \}$

So, $\mathrm{depth}_R M \leq \mathrm{inj\,dim}_R M$.

Theorem 2 $M \in \mathrm{Mod}\ R$.

Theorem 2 $M \in \mathrm{Mod}\ R$.

$$\mu_R^i(\mathfrak{m}, M) \neq 0 \quad \forall\ \mathrm{depth}_R M \leq i \leq \mathrm{injdim}_R(M).$$

Will prove this later.

Theorem 3. R commutative noetherian.
$M \in D^b(\mathrm{mod}\ R)$.

$\mathfrak{p} \subsetneq \mathfrak{q}$ in $\mathrm{Spec}\ R$ s.t. there are no primes between $\mathfrak{p}$ and $\mathfrak{q}$.

Then, if
$$\mu_R^i(\mathfrak{p}, M) \neq 0 \Rightarrow \mu_R^{i+1}(\mathfrak{q}, M) \neq 0.$$

(This gives Theorem 1.)

Proof. Can localise at $\mathfrak{q}$ to assume $(R, \mathfrak{m}, k)$ local with $\mathfrak{m} = \mathfrak{q}$ and $\dim(R/\mathfrak{p}) = 1$.

We prove:
$$\mu^{i+1}(\mathfrak{m}, M) = 0 \Rightarrow \mu^i(\mathfrak{p}, M) = 0.$$

Ie, $\mathrm{Ext}_R^{i+1}(k, M) = 0 \Rightarrow \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M) = 0$

$\Downarrow$

$\mathrm{Ext}_R^{i+1}(-, M) = 0$ on finite length R-modules.

(Say N has finite length ≥ 1. Can embed $0 \to k \hookrightarrow N \to N' \to 0$ with length N' = length $N - 1$. ...)

Pick $r \in \mathfrak{m} \setminus \mathfrak{p}$. We have an exact seq.

$$0 \to \frac{R}{\ } \xrightarrow{r} \frac{R}{\ } \to \frac{R}{\ } \to 0.$$

$$0 \to \frac{R}{\mathfrak{p}} \xrightarrow{r} \frac{R}{\mathfrak{p}} \to \frac{R}{(\mathfrak{p}, r)} \to 0.$$

1-dim'l

$\therefore$ 0 dim'l $\Rightarrow$ Artinian $\Rightarrow$ finite length

$$\therefore \mathrm{Ext}^i_R(R/\mathfrak{p}, M) \xrightarrow{r} \mathrm{Ext}^i_R(R/\mathfrak{p}, M) \to 0.$$

$\therefore \cdot r$ is surjective. By NAK, it is zero. $\blacksquare$

Now localise ...

Theorem. $(R, \mathfrak{m}, k)$ local. $M, N \in D^b(\mathrm{mod}\, R)$.

(Ischebeck) If $\mathrm{injdim}_R N < \infty$, then

$$\sup \mathrm{Ext}^*_R(M, N) = \mathrm{injdim}_R N - \mathrm{depth}_R M.$$

[Auslander - Buchsbaum for injectives]

Recall: $M \otimes_R \mathrm{Hom}_R(N, I) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), I)$.

iso when M f.g. and I injective. (modules)

This extends to

$M, N, I \to R$- complexes.

$$M \otimes^L_R \mathrm{RHom}(N, I) \to \mathrm{RHom}_R(\mathrm{RHom}_R(M, N, I))$$

is an iso

when $M, N \in D^b(\mathrm{mod}\ R)$,

$I \simeq$ bounded complex of injectives.

Back to Ischebeck:

Let $E = E_R(k)$. Then,

$\mathrm{RHom}_R(N, E) \simeq$ bounded cx of flat modules

[$\mathrm{Hom}_R(-, E)$ takes inj to flat]

$RHom_R(N, E) \simeq$ bounded cx of flat ...

[$Hom_R(-, E)$ takes inj to flat]

Now, we apply Auslander-Bu...

$$\operatorname{depth}\left(M \overset{L}{\underset{R}{\otimes}} RHom(N,E)\right) = \operatorname{depth}_R M - \sup H_*\left(k \overset{L}{\underset{R}{\otimes}} RHom_R(N,E)\right)$$

$$\text{"} \qquad\qquad\qquad\qquad\qquad\qquad \text{"}$$

$$\operatorname{depth}\left(Hom_R(RHom_R(M,N),E)\right) \qquad \operatorname{depth} M - \sup H_*\left(Hom(RHom_R(k,N),E)\right) \quad (1)$$

$X := RHom_R(M, N)$.

$H^i(X) = Ext_R^i(M, N)$ ← f.g.

$X^\vee := Hom_R(X, E)$. ⤵ E inj

$H_i(X^\vee) = Hom_R(H^i(X), E)$

$= Hom_R(Ext_R^i(M,N), E)$ ⤵ E cogenerator

$s := \sup\{i : H_i(X^\vee) \neq 0\} = \sup\{i : Ext_R^i(M,N) \neq 0\}$

$H_s(X^\vee)$ artinian

$\therefore \operatorname{depth} H_s(X^\vee) = 0$. So, $\operatorname{depth}_R(X^\vee) = -s$.

$\therefore$ LHS of (1) is $-\sup Ext_R^*(M,N)$.

Do similar for RHS. ☒

Now, specialising to $M = R$:

$$0 = \operatorname{inj\,dim}_R N - \operatorname{depth} R.$$

$0 \quad = \operatorname{injdim}_R \quad \cdots \quad p$

$$\Rightarrow \boxed{\operatorname{injdim}_R N = \operatorname{depth} R.} \qquad (\text{If } \operatorname{injdim}_R N < \infty.)$$

In fact: If $(R, \mathfrak{m}, k)$ has a f.g. module $N \neq 0$ of finite injective dim, then R is C.M.

So, $\operatorname{inj\,dim} N = \dim R$.

Lecture 18 (22-03-2023)

Wednesday, March 22, 2023 1:23 PM

Dualising Complexes

$R \longrightarrow$ comm. noetherian

Def^n. A complex $\omega_R^{\cdot}$ is a dualising complex if

① $\omega_R \in D^b(\text{mod } R)$, homology is nonzero in finitely many places and is f.g.

Not local

② $\text{injdim } \omega_R < \infty$, i.e., $\omega_R \simeq$ bdd complex of injectives,

③ $R \xrightarrow{\simeq} R\text{Hom}_R(\omega_R, \omega_R) =: R\text{End}_R(\omega_R)$.

local

quasi iso

[ω_R itself could be an unbounded complex.]

① ω_R dualising $\Rightarrow$ $\Sigma^n \omega_R$ dualising $\forall n$.

$\Downarrow$

(starting with any one ω_R, doing these processes get all dual.--)

$\omega_R \otimes_R L$ dualising

(L a f.g. proj R-module of rank 1.)

(L_p is free R_p module of rank 1 for all associated p.)

② ω_R dualising $\Rightarrow$ $U^{-1}\omega_R$ dualising over $U^{-1}R$ for any multiplicative closed $U \subseteq R$.

Theorem. (Bass, Murthy ~60s)

$M \in D^b(\text{mod } R)$. $\quad M_p$ perfect $/R_p$ $\forall p$

$\Longrightarrow M$

(perfect = in $D^b(\text{mod } R)$ and $\operatorname{pdim}_R < \infty$.)

Not so for $\operatorname{inj\,dim}_R$.

Def^n. Semidualising $\rightarrow$ if ① and ③ are satisfied.

Dualising complexes may not exist.

R is always a semidualising complex.

Local Duality

R, ω_R as above.

(Assume ω_R exists.)

$$(-)^{\dagger} := R\operatorname{Hom}_R(-, \omega_R) : D(\operatorname{Mod} R) \to D(\operatorname{Mod} R)$$

restricts to equivalences (contrapositive)

length$_R M < \infty$

$$\begin{array}{ccc}
D^{fl}(\text{mod } R) & \xrightarrow{\simeq} & D^{fl}(\text{mod } R) \\
\cap\!| & & \cap\!| \\
D^b(\text{mod } R) & \xrightarrow[\simeq]{(-)^{\dagger}} & D^b(\text{mod } R) \\
\cup\!| & & \cup\!| \\
\operatorname{Perf}(R) & \xrightarrow{\sim} & \operatorname{Ferp}(R)
\end{array}$$

[illegible]

$$\text{finite proj dim} \to \mathrm{Perf}(R) \xrightarrow{\sim} \mathrm{Ferp}(R) \leftarrow \text{finite injective dim}$$

Proof. $M \in D^b(\mathrm{mod}\ R)$, then $M^{\dagger} \in D^b(\mathrm{mod}\ R)$.
$$M^{\dagger} = R\mathrm{Hom}_R(M, \omega_R)$$

Because: $M, \omega_R \in D^b(\mathrm{mod}\ R)$

$\Rightarrow$ each $\mathrm{Ext}^i_R(M, \omega_R)$ f.g. $\forall i$.

Moreover, inj dim $\omega_R < \infty \Rightarrow \mathrm{Ext}^i$ vanishes for large $|i|$.

Now, $\dagger\dagger$ factors as:

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, "\simeq"'] & & R\mathrm{Hom}_R(R\mathrm{Hom}_R(M, \omega_R), \omega_R) \\
& M \otimes^{L}_{R} R\mathrm{Hom}_R(\omega_R, \omega_R) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

induced by $R \xrightarrow{\sim} R\,\mathrm{End}_R(\omega_R)$

$\because$ $M \in D^b(\mathrm{mod}\ R)$, inj dim$_R\ \omega_R < \infty$.

$\therefore$ top is $\simeq$

Similarly now check $\mathrm{Perf} \to \mathrm{Ferp} \to \mathrm{Perf}$.

Local rings $(R, \mathfrak{m}, k)$.

$\omega_R \to$ dualising complex.

$$D^b(\mathrm{mod}\ R) \xrightarrow[\simeq]{(\ )^{\dagger}} D^b(\mathrm{mod}\ R)$$

$$k \xrightarrow{\simeq} k^{\dagger\dagger} \quad \text{(quasi iso)}$$

$R\,\mathrm{Hom}_R(k, \omega_R) \simeq$ graded k-vector space with 0 diff

$\omega_R \xrightarrow{\sim} I$ min'l

$\parallel$

$\mathrm{Hom}_R(k, I) = \mathrm{soc}(I)$

$\Rightarrow \quad R\,\mathrm{Hom}_R(k, \omega_R) \simeq \mathrm{Ext}^*_R(k, \omega_R)$

$k \xrightarrow{\sim} R\mathrm{Hom}_R(\mathrm{Ext}^*_R(k, \omega_R), \omega_R)$

$\cong$

$\mathrm{Hom}_k(\mathrm{Ext}^*_R(k, \omega_R), k) \otimes_R \mathrm{Ext}^*_R(k, \omega_R)$

↳ has to be rank 1, i.e.

$\Rightarrow \mathrm{Ext}^*_R(k, \omega_R) \simeq \Sigma^a k.$

Normalising convention: $\mathrm{Ext}^*_R(k, \omega_R) = k$ $(a=0)$, i.e.,

$$\mathrm{Ext}^i_R(k, \omega_R) = \begin{cases} 0 & i \neq 0, \\ 1 & i = 0. \end{cases}$$

$\omega_R \simeq I^{\cdot}$ min inj. resolution. Thus,

$$0 \to I^{-\dim R} \;\cdots \longrightarrow I^{-2} \longrightarrow I^{-1} \to I^0 = E_R(k) \to 0$$

$I^{-2} = \bigoplus_{\dim(R/\mathfrak{p})=2} E(R/\mathfrak{p})$, $\quad I^{-1} = \bigoplus_{\dim(R/\mathfrak{p})=1} E(R/\mathfrak{p})$

← each $\mathfrak{p}$ shows up!! since localisation @ $\mathfrak{p}$ gives dualizing for $R_\mathfrak{p}$ (not normalized) and still minimal...

Every prime shows up exactly once, at location prescribed by its "co-height".

<u>Observe</u>. If

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{q} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{q}_1 \arrow[d, dotted, dash] & & \mathfrak{q}_1' \arrow[d, dotted, dash] \\
\mathfrak{q}_\ell \arrow[dr, dash] & & \mathfrak{q}_{\ell'}' \arrow[dl, dash] \\
 & \mathfrak{p} &
\end{tikzcd}$$

max'l chains, the $\ell = \ell'$.

Thus, if ω_R exists, then R is catenary!

Connection to Matlis Duality.

$(R, \mathfrak{m}, k)$ local.

ω_R normalised.

$\simeq I^\cdot$ min inj. resol^n

$M \in \operatorname{mod} R$, $\operatorname{length}_R M < \infty$.

$$M^\dagger = R\mathrm{Hom}_R(M, \omega_R) = \mathrm{Hom}_R(M, I^\cdot) = \mathrm{Hom}(M, E).$$

$\mathrm{Hom}_R(M, E(R/\mathfrak{p})) = 0$
$\forall \ \mathfrak{p} \neq \mathfrak{m}$
since the Hom is $\mathfrak{m}$-torsion and $\mathfrak{p}$-local

Theorem. $(R, \mathfrak{m}, k)$ ω_R normalised.

$M \in D^b(\operatorname{mod} R)$.

① $\inf H(M^\dagger) = \operatorname{depth}_R M$, ① $\inf H_*(M^\dagger) = \operatorname{depth}_R M,$

② $\sup H_*(M^\dagger) = \sup\{\dim(R/\mathfrak{p}) - \inf H_*(M)_\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} R\}$.

$$= \sup\{\dim_R H_i(M) - i : i \geq 0\}.$$

Definition of $\dim_R(M)$ for a complex M.

Thus,

$$\operatorname{amp} H_*(M^\dagger) := \sup H_*(M^\dagger) - \inf H_*(M^\dagger)$$
$$= \dim_R M - \operatorname{depth}_R M$$
$$=: \operatorname{cmd}_R(M).$$

Proof. ① $\operatorname{injdim} \omega_R = 0$ (∵ normalised)

By Ischebeck, $\sup \operatorname{Ext}_R^*(M, \omega_R) = \underbrace{\operatorname{injdim} \omega_R}_{0} - \operatorname{depth} M$

$$\Rightarrow \operatorname{depth} M = -\sup H^*(M^\dagger)$$
$$= \inf H_*(M^\dagger).$$

② Recall for any $X \in D^b(\operatorname{mod} R)$

$$\sup H_*(X) \underset{s}{=} \sup\{-\operatorname{depth} X_\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} R\}.$$

Note: $\operatorname{depth}(M^\dagger) = \inf H_*(M^{\dagger\dagger})$
$$= \inf H_*(M).$$

Also, $(\omega_R)_{\mathfrak{p}} = \sum^{\dim(R/\mathfrak{p})} \omega_{R_{\mathfrak{p}}}$

normalising

$$\operatorname{depth}_{R_{\mathfrak{p}}} R\operatorname{Hom}_R(M, \omega_R)_{\mathfrak{p}} = \operatorname{depth}_{R_{\mathfrak{p}}} \left(R\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\omega_R)_{\mathfrak{p}})\right)$$
$$= \operatorname{depth}_{R_{\mathfrak{p}}} R\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, \Sigma^{\dim R/\mathfrak{p}} \omega_{R_{\mathfrak{p}}})$$
$$= -\dim(R/\mathfrak{p}) + \operatorname{depth}_{R_{\mathfrak{p}}} R\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}})$$
$$= -\dim(R/\mathfrak{p}) + \inf H_*(M_{\mathfrak{p}}). \quad \blacksquare$$

$$0 \to I_d \to \cdots \longrightarrow I_0 \to 0$$

$$I_n = \bigoplus_{\mathfrak{p} : \dim(R/\mathfrak{p}) = n} E(R/\mathfrak{p}).$$

- M an R-module.

$$0 \to \operatorname{Hom}_R(M, I_d) \to \cdots \to \operatorname{Hom}_R(M, \underset{\overset{\shortparallel}{I_0}}{E}) \to 0$$

$$\operatorname{Hom}_R(M, E(R/\mathfrak{p})) = 0 \Leftarrow \dim R/\mathfrak{p} > \dim M.$$
$$\Downarrow$$
$$\operatorname{ann}(M) \not\subseteq \mathfrak{p}.$$

This gives one direction of = in result above....

Lecture 19 (27-03-2023)

Monday, March 27, 2023 1:27 PM

$(R, \mathfrak{m}, k)$ local ring

$\omega_R \longrightarrow$ dualising complex (normalised)

$$\mathrm{Ext}_R^*(k, \omega_R) \cong k.$$

Equivalently, $\mu_R^i(\mathfrak{m}, \omega_R) = \begin{cases} 0 & ; \ i \neq 0, \\ 1 & ; \ i = 0. \end{cases}$

Then, $\forall M \in D^b(\mathrm{mod}\ R)$,

$$\mathrm{depth}_R(M) = \inf H_*(M^\dagger),$$

$$\dim_R(M) = \sup H_*(M^\dagger).$$

$(-)^\dagger = \mathrm{RHom}_R(-, \omega_R) : D^b(\mathrm{mod}\ R)^{op} \xrightarrow{\sim} D^b(\mathrm{mod}\ R).$

Ex. Let $X \in D(R)$.

If $\mathrm{RHom}_R(-, X)$ induces an autoequivalence $D^b(\mathrm{mod}\ R)^{op} \xrightarrow{\sim} D^b(\mathrm{mod}\ R)$,

then $X \simeq \Sigma^n \omega_R$ for some n.

Note: $\omega_R^\dagger = R \Rightarrow \omega_R$ is a CM complex.

Elaboration of Local Duality:

$M \in \mathrm{mod}(R)$ that is CM, $\dim(M) =: d$.

Then, $\inf H_*(M^\dagger) = d = \sup H_*(M^\dagger).$

$$\Rightarrow \quad M^+ = \Sigma^d H_d(M^+).$$

Claim. $H_d(M^+)$ is also CM of dimension d.

Proof. $(H_d(M^+))^+ \simeq (\Sigma^{-d} M^+)^+ \simeq \Sigma^d M^{++}$
$$\simeq \Sigma^d M. \qquad \blacksquare$$

$$\therefore \ \Sigma^{-d}(\ )^+ : \left\{ \begin{array}{l} \text{CM modules of} \\ \text{dimension } d \end{array} \right\} \circlearrowleft \ \equiv$$

Poincare Series and Bass Series.

$M \in D^b(\text{mod } R).$

$$P_M^R(t) := \sum_{n \in \mathbb{Z}} \operatorname{rank}_k \operatorname{Tor}_n^R(k, M)\, t^n.$$

Poincare Series

$\hookrightarrow$ Generating series of Betti numbers of M

$$I_R^M(t) := \sum_n \operatorname{rank}_k \operatorname{Ext}_R^n(k, M)\, t^n$$

$$= \sum_n \mu_R^n(\mathfrak{m}, M)\, t^n. \qquad \text{Bass series.}$$

Obs.

$$\left.\begin{array}{l} \mu_R^n(\mathfrak{m}, M^+) = \operatorname{rank}_k \operatorname{Ext}_R^n(k, M^+) \\ \quad = \operatorname{rank}_k \operatorname{Ext}_R^n(M^+, k) \end{array}\right\} (*)$$

$$(*)\quad \mathrm{Ext}^n_R(k, M^+) \cong \mathrm{Hom}_{\mathcal{D}(R)}(k, \Sigma^n M^+)$$
$$\cong \mathrm{Hom}_{\mathcal{D}(R)}((\Sigma^n M^+)^+, k^+)$$
$$\cong \mathrm{Hom}_{\mathcal{D}(R)}(\Sigma^{-n} M^{++}, k)$$
$$\cong \mathrm{Hom}_{\mathcal{D}(R)}(\Sigma^{-n} M, k)$$

local duality

$$\mathrm{Tor}_n^R(k, M)^\vee \cong \mathrm{Ext}^n_R(M, k)$$

$$\therefore \quad \mu^n_R(\mathfrak{m}, M^+) = \beta^R_n(M) \quad \text{and}$$
$$\mu^n_R(\mathfrak{m}, M) = \beta^R_n(M^+).$$

Theorem. $P^R_M(t) = I_R^{M^+}(t)$.

No gaps theorem. $M \in \mathrm{mod}(R)$. Then,

$$\mu^i_R(\mathfrak{m}, M) \neq 0 \quad \forall \ \mathrm{depth}_R M \leq i \leq \mathrm{injdim}_R M.$$

Proof (Roberts). May assume M indecomposable.

Say $\mu^i_R(\mathfrak{m}, M) = 0$ for some $i > \mathrm{depth}_R(M)$.
(Choose i least.)

Then, $\beta^R_i(M^+) = 0$.

Consider $F \simeq M^+$ min free resol^n.

$$\to F_{i+1} \to \underset{\substack{\shortparallel \\ 0}}{F_i} \to F_{i-1} \to \cdots \to F_{\mathrm{depth}(M)} \to 0$$

i.e., $F = F_{\leq i-1} \oplus F_{\geq i+1}$.

i.e., $F = F_{\leq i-1} \oplus F_{\geq i+1}$.

Thus, $M \simeq (M^\dagger)^\dagger \simeq F^\dagger$
$$\simeq (F_{\leq i-1})^\dagger \oplus (F_{\geq i+1})^\dagger.$$

Thus, $H_0(M) \simeq H_0(F_{\leq i-1}{}^\dagger) \oplus H_0(F_{\geq i+1}{}^\dagger)$ — (*)

and $0 = H_n(M) \simeq H_n(F_{\leq i-1}{}^\dagger) \oplus H_n(F_{\geq i+1}{}^\dagger)$.

$\therefore H_n(F_{\leq i-1}{}^\dagger) = H_n(F_{\geq i+1}{}^\dagger) = 0$ for all $n \neq 0$

(*) and M indec $\Rightarrow$ one of the two has H_0 zero as well.

Use degree argument: $H_0(F_{\geq i+1}{}^\dagger) = 0$.

$\therefore H_n(F_{\geq i+1}{}^\dagger) = 0$ for all n.

$\Rightarrow F_{\geq i+1}{}^\dagger = 0$.

$\Rightarrow$ projdim $M^\dagger < i$

$\Rightarrow$ injdim $M < i$. $\boxtimes$

Existence of Dualising Complexes.

- R comm. noetherian.

ω_R dualising $\Rightarrow U^{-1}\omega_R$ dualising for $U^{-1}R$.

Lemma. If $R \to S$ finite map, then $\mathrm{Hom}_R(S, \omega_R)$ is a dualising complex for S.

Proof. Need to check three things:

① $\mathrm{Hom}_R(S, \omega_R) \in D^b(\mathrm{mod}\, S)$.

Pf. Viewed as R-module, $\mathrm{Hom}_R(S, \omega_R) \in D^b(\mathrm{mod}\, R)$.

$\Rightarrow$ —〃— $\in D^b(\mathrm{mod}\, S)$. ∎

② $\mathrm{injdim}_S\, R\mathrm{Hom}_R(S, \omega_R) < \infty$.

Pf. $R\mathrm{Hom}_S(\text{—}, R\mathrm{Hom}_R(S, \omega_R)) \simeq R\mathrm{Hom}_R(\text{—}, \omega_R)$. ∎

vanishes beyond $\mathrm{injdim}_R\, \omega_R$.

③
$$\begin{tikzcd} S \arrow[r, "\simeq"'] \arrow[dr, bend right, "(-)^{\dagger\dagger}"'] & R\mathrm{Hom}_S(R\mathrm{Hom}_R(S, \omega_R), R\mathrm{Hom}_R(S, \omega_R)) \arrow[d, "\wr", "\text{adjunction}"'] \\ & R\mathrm{Hom}_R(R\mathrm{Hom}_R(S, \omega_R), \omega_R) \end{tikzcd}$$

The diagram above commutes. ∎ ∎

Uniqueness: ω_R and ω_R' dualising (for R).

Consider
$$\omega_R \otimes^{L}_R R\mathrm{Hom}_R(\omega_R, \omega_R') \to \omega_R'.$$
evaluation

Claim. The above is a q. iso.

Proof. $\omega_R \otimes^{L}_R R\mathrm{Hom}_R(\omega_R, \omega_R') \to R\mathrm{Hom}_R(R, \omega_R') = \omega_R'$

$\because$ inj dim $\omega_R' < \infty$ $\simeq$ $\qquad\qquad$ $\simeq$ since ω_R dual

$\because$ inj dim $\omega_{R'} < \infty$ $\simeq$ $\searrow$ $\nearrow$ $\simeq$ since ω_R dual

$\omega_R \in D^b(\text{mod } R)$)

$$\mathrm{RHom}_R(\mathrm{RHom}_R(\omega_R, \omega_R), \omega_{R'}).$$

$\blacksquare$

Lemma. $\mathrm{RHom}_R(\omega_R, \omega_{R'}) \simeq \Sigma^n P$ with

P a rank one projective module.

Proof. $\mathrm{RHom}_R(\omega_R, \omega_{R'}) \in D^b(\text{mod } R)$.

Enough to check after localisation that

$\mathrm{RHom}_R(\omega_R, \omega_R') \simeq \Sigma^n R$ when R is local.

May assume ω_R and ω_R' are normalised.

$$\beta_i^R\left(\mathrm{RHom}_R(\omega_R, \omega_R')\right) = \mu_R^{\,i}(m, \omega_R)$$

$\because \omega_R'$ normalized dualising

$$= \begin{cases} 1 & ; \ i = 0, \\ 0 & ; \ \text{else}. \end{cases}$$

$\blacksquare$

GORENSTEIN

" R is Gorenstein if inj dim $R < \infty$. " (Only for dim $R < \infty$.)

($\equiv R$ is dualising.)

Example R regular with $\dim R < \infty$

Lemma. $x \in R$ not a zero divisor

R Gorenstein $\Rightarrow R/xR$ Gorenstein.

If $x \in \mathrm{Jac}(R)$, then $\Leftarrow$ holds too.

Proof $R \twoheadrightarrow R/xR =: S$. finite map.

R dualising $\Rightarrow$ $\mathrm{RHom}_R(R, S)$ dualising for S

$$\Sigma^{-1} S \cong$$

$0 \to R \xrightarrow{x} R \to 0$ free resol^n $\simeq S$

$\therefore \mathrm{RHom}_R(S, R) \simeq \mathrm{Hom}_R(0 \to R \xrightarrow{x} R \to 0, R)$

in $\mathcal{D}(R)$

$\simeq 0 \to \underset{\deg\ 0}{R} \xrightarrow{x} \underset{-1}{R} \to 0$

$\simeq \Sigma^{-1} S.$

$\therefore \mathrm{RHom}_R(S, R) \simeq \Sigma^{-1} S$ in $\mathcal{D}(R)$.

Also as S-complexes.

Converse is exercise. ∎

Thus, R regular $\Rightarrow$ c.i. $\Rightarrow$ Gorenstein.

Note: (R, m, k) local. Then,

R Gorenstein $\Leftrightarrow \hat{R}$ Gorenstein.

Theorem (R, m, k) local. Then,

Theorem. $(R, \mathfrak{m}, k)$ local. Then,

R Gorenstein $\iff$ R CM and $\operatorname{type}(R) = 1$.

Pf. ($\Rightarrow$) R dualising.

$\therefore$ $R\operatorname{Hom}_R(R, R) \simeq R$.

$\Rightarrow$ R CM.

depth = dim

Moreover, $\operatorname{type}(R) = \mu_R^d(\mathfrak{m}, R)$ $\because$ R dualising.

dfn

$= 1$

(Thus, $\Sigma^d R$ is the normalised dualising cx for R.)

($\Leftarrow$) Suppose R is CM and type 1.

Passing to $\hat{R}$, we can assume R has a dualising complex (normalised).

R C.M. $\Rightarrow$ $\operatorname{amp} H_*(\omega_R) = 0$.

ETP: $\omega_R \simeq R$.

$$\beta_i^R(\omega_R) = \mu_R^i(\mathfrak{m}, R)$$

$$\beta_d^R(\omega_R) = \mu_R^d(\mathfrak{m}, R) = 1.$$

$(d = \dim R)$

$(d = \dim ...$

$F \xrightarrow{\sim} \omega_R$ min free resolution.

Since $H_i(\omega_R) = 0$ for $i < d$:

$$\cdots \to F_{d-1} \to \overset{= R}{F_d} \to \cdots$$

$\Rightarrow H_d(\omega_R) \cong R/I$.

Also, $\operatorname{ann}_R H_d(\omega_R) = 0$.

$-$ $R \xrightarrow{\sim} \operatorname{RHom}_R(\omega_R, \omega_R)$.

$\Longrightarrow H_d(\omega_R) \cong R$. $\therefore$ $R \simeq \Sigma^{-d} \omega_R$. ∎

Theorem (Roberts) $\operatorname{type}(R) = 1 \Longrightarrow R$ Gorenstein. ∎

Corollary. (R, m, k) artinian.

R is Gorenstein $\Longleftrightarrow \operatorname{type}(R) = 1$

$\Leftrightarrow \operatorname{Soc}(R) \simeq k$. ∎

Lecture 20 (29-03-2023)

Wednesday, March 29, 2023 1:25 PM

Ex. Look up sharp equivalence and deduce from local duality.

Theorem If R is artinian, TFAE:

① R Gorenstein. (I.e., $\text{injdim}_R R < \infty$.)

② $R \simeq E_R(k)$. ②′ $E_R(k)$ free R-module.

③ $\text{rank}_k \operatorname{soc}(R) = 1$. (I.e., $\operatorname{soc}(R) \cong k$.)

Example. Say R a k-alg. of finite rank.

$$k \hookrightarrow R \twoheadrightarrow k.$$

$$E_R(k) \simeq \text{Hom}_k(R, k)$$

- injective
- socle $(\text{Hom}_k(R,k)) \simeq k$.

$\left(\begin{array}{l}\text{Hom}_R(-, \text{Hom}_k(R,k)) \\ \cong \text{Hom}_k(-, k).\end{array}\right)$

Theorem tells us R Gorenstein $\Rightarrow$ $\text{Hom}_k(R, k) \cong R$ as R-modules.

What is a basis of $\uparrow$? (Treat $k \subset R$ splitting.)

Say $A = k \oplus A_1 \oplus \cdots \oplus A_g$.

- k field, $\text{rank}_k(A) < \infty$
- A is k-alg; either commutative or graded commutative
  $a \cdot b = (-1)^{|a||b|} ba$ for homog. a, b

$$0 \le i \le g : \quad A_i \times A_{g-i} \longrightarrow A_g \quad \rightsquigarrow \text{"symmetric", bilinear pairing}$$

This gives $A_i \rightarrow \text{Hom}_k(A_{g-i}, A_g)$.

Prop^n. A is Gorenstein $\Leftrightarrow$ $\uparrow$ is a bijection for $\forall\ 1 \le i \le g-1$.

$\Leftrightarrow \text{Hom}_k(A, k)[\pm g] \cong A$ as A-modules.

$\Longleftrightarrow \mathrm{Hom}_k(A, k)[\pm g] \cong A$ as A-modules.

$\Longleftrightarrow A_g \cong k$ and the pairings are nondegenerate.

Local Cohomology

R Commutative noetherian.

$I \subsetneq R$ ideal.

M an R-module (not necessarily finite).

$$\Gamma_I(M) := \{ m \in M : I^n \cdot x = 0 \text{ for some } n \geq 0 \}.$$

↳ I-power torsion submodule of M

$$\cong \bigcup_{n \geq 0} \mathrm{Hom}_R(R/I^n, M)$$

$$= \operatorname{colim}_n (0 :_M I^n)$$

$$= \bigcup_{j \geq 0} \mathrm{Hom}_R(R/I_j, M), \quad \text{where}$$

$\cdots \subseteq I_{j+1} \subseteq I_j \subseteq \cdots$ is cofinal with $\langle I_n \rangle_n$.

Example. $I = (x_1, \ldots, x_c)$.

$$\{(x_1^s, \ldots, x_c^s)\}_{s \geq 0}.$$

char p: $\{(x_1^{p^e}, \ldots, x_c^{p^e})\}_{e \geq 0}$.

char p: $\{(x_1^{p^e}, \ldots, x_c^{p^e})\}_{e \geq 0}$.

- $\Gamma_I(M) = \ker\left(M \longrightarrow \prod_{\mathfrak{p} \notin V(I)} M_{\mathfrak{p}}\right)$

  i.e., $I \nsubseteq \mathfrak{p}$

- Γ_I is left-exact on Mod-R.

  (Using left-exactness of Hom and exactness of direct colim.)

  $0 \to M' \to M \to M'' \to 0$ induces

  $$0 \to \Gamma_I(M') \to \Gamma_I(M) \to \Gamma_I(M'') \to \cdots.$$

$R\Gamma_I(M) = \Gamma_I(iM)$ where $M \to iM$ is a semi-injective res^n.

One gets
$$R\Gamma_I : D(R) \to D(R).$$

$$H_I^n(M) := H^n(R\Gamma_I(M)).$$

Computing $R\Gamma_I$:

① $\Gamma_I(-) = \bigcup_n \operatorname{Hom}(R/I^n, -)$

$$\Rightarrow \boxed{H_I^i(M) = \operatorname{colim}_n \operatorname{Ext}_R^i(R/I^n, M).}$$

Corollary. $H_I^i(M) = 0$ for $i < \operatorname{depth}_R(I, M)$.

Moreover, $H^i_I(M) \neq 0$ for $i = \operatorname{depth}_R(I, M)$.

Prove using Auslander type arguments...

Corollary. $H_I^i(E) = 0$ $i \geq 1$, E injective.

In fact,

$$H^i_I(E(R/\mathfrak{p})) = \begin{cases} E(R/\mathfrak{p}) & ; i = 0, \mathfrak{p} \supseteq I \\ 0 & ; i = 0, \mathfrak{p} \not\supseteq I \\ 0 & ; i \geq 1 \end{cases}$$

Say $I = (x_1, \dots, x_c)$.

Define $I_s := (x_1^s, \dots, x_c^s)$.

$\{I_s\}_{s \geq 0}$ cofinal to $\{I^n\}_{n \geq 0}$.

we have $K(\underline{x}^s) = K(x_1^s, \dots, x_c^s)$

$$\downarrow$$

$$R/I_s.$$

This gives

$$\begin{tikzcd} \operatorname{Hom}_R(R/I_s, M) \arrow[r] \arrow[d] & \operatorname{Hom}_R(K(\underline{x}^s), M) \arrow[d] \\ \operatorname{Hom}_R(R/I_{s+1}, M) \arrow[r] & \operatorname{Hom}_R(K(\underline{x}^{s+1}), M) \end{tikzcd}$$

$$\rightsquigarrow$$

$$\operatorname*{colim}_s \operatorname{Hom}_R(R/I_{s+1}, M) \to \operatorname*{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M).$$

$$\operatorname{colim}_s \operatorname{Hom}_R(R/I_{s+1}, M) \to \operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M).$$

$M \to J$ inj. resol^n.

$$R\Gamma_I(M) \longrightarrow \operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), J)$$
$$\uparrow \simeq$$
$$\operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M).$$

We get natural maps

$$H^i_I(M) \longrightarrow \operatorname{colim}_s H^i(\underline{x}^s; M).$$
$$\parallel$$
$$H^i(\operatorname{Hom}_R(K(\underline{x}^s), M))$$

Theorem The above is an iso for all i and M.

Idea. For any module M, it is clear that the above is an iso for $i=0$.

ETP: They both vanish on injectives. (δ-functor...)
We know it for H^i_I.
Just need to check for RHS.
Can prove for injective hulls! Check. ∎

$$\operatorname{Hom}_R(K(\underline{x}^s), M) \simeq \operatorname{Hom}_R(K(\underline{x}^s), R) \otimes_R M$$

$$\operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), M) \simeq \left(\operatorname{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), R)\right) \otimes_R M.$$

$$K^\infty(\underline{x}) := \operatorname*{colim}_s \mathrm{Hom}_R(K(\underline{x}^s), R).$$

"stable Koszul complex"

$$K(x_1^s, \dots, x_c^s) = \bigotimes_{i=1}^c K(x_i^s).$$

$$\mathrm{Hom}_R(K(\underline{x}^s), R) = \mathrm{Hom}_R\Big(\bigotimes_{i=1}^c K(x_i^s), R\Big)$$

$$= \mathrm{Hom}_R\Big(\bigotimes_{i=1}^{c-1} K(x_i^s), \mathrm{Hom}_R(K(x_c^s), R)\Big)$$

$$= \mathrm{Hom}_R\Big(\bigotimes_{i=1}^{c-1} K(x_i^s), R\Big) \otimes_R \mathrm{Hom}_R(K(x_c^s), R).$$

$c=1$:

$$\begin{array}{c} K(x^{s+1}) \\ \downarrow \\ K(x^s) \end{array} \qquad \begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^{s+1}"] \arrow[d, "\cdot x"'] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^s"] & R \arrow[r] & 0 \end{tikzcd}$$

<u>Dualise</u>:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^s"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^{s+1}"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^{s+1}"] \arrow[d, equal] & R \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \end{tikzcd}$$

$\Big\}$ colim

$$0 \to R \longrightarrow R[\tfrac{1}{x}] \to 0$$

natural localisation

natural map

localisation

$$\operatorname*{colim}_s \operatorname{Hom}_R(K(\underline{x}^s), R) \cong \bigotimes_{i=1}^{c} \left(0 \to \underset{\text{deg } 0}{R} \to \underset{1}{R\left[\tfrac{1}{x}\right]} \to 0\right)$$

This gives:

$$\overset{\infty}{K}(\underline{x}) : 0 \to R \to \underbrace{\bigoplus_{i=1}^{c} R\left[\frac{1}{x_i}\right] \to \cdots \to R\left[\frac{1}{x_1 \cdots x_c}\right]}_{} \to 0.$$

Čech complex computing sheaf cohomology

of $\operatorname{Spec} R \setminus V(\underline{x}) = \bigcup_{i=1}^{c} D(x_i)$.

call this cover $\mathcal{U}$

$= \check{C}(X, \mathcal{U})$

$$0 \to \check{C}(X, \mathcal{U}) \hookrightarrow K^\infty(\underline{x}) \to R \to 0.$$

Corollary. $H_I^i(M) = 0$ for $i \geq c+1$.

M module.

Note $H_I^i(M) = H_J^i(M)$ if $\sqrt{I} = \sqrt{J}$.

$\therefore$ $H_I^i(M) = 0$ for $i > \operatorname{ara}(I)$

$$\operatorname{ara}(I) = \inf\left\{c : \exists x_1, \dots, x_c \text{ s.t. } \sqrt{(x_1, \dots, x_c)} = \sqrt{I}\right\}.$$

Lecture 21 (03-04-2023)

Monday, April 3, 2023 1:20 PM

R comm. noe. ring

$\mathfrak{a} \subseteq R$ ideal.

$\Gamma_{\mathfrak{a}}(-) : \text{Mod-}R \to \text{Mod-}R$

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \geq 0} \mathrm{Hom}_R(R/\mathfrak{a}^n, M)$$

$$= \bigcup_{n \geq 0} \mathrm{Hom}_R(R/\mathfrak{a}_n, M).$$

where $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ is cofinal to $(\mathfrak{a}^n)_{n \geq 0}$.

Induces: $R\Gamma_{\mathfrak{a}} : D(R) \to D(R)$,

$$R\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(iM)$$

injective resol^n

$$H^i_{\mathfrak{a}}(M) = H^i(R\Gamma_{\mathfrak{a}}(M))$$

i^{th} local cohomology of M supported on $\mathfrak{a}$ (rather $V(\mathfrak{a})$)

$(\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{b}} \Leftarrow \sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}})$

$$- R\Gamma_{\mathfrak{a}}(M) = \underset{n}{\mathrm{hocolim}}\ R\mathrm{Hom}_R(R/\mathfrak{a}^n, M)$$

homotopy colimit (usual colim DNE in derived...)

$\frac{R}{\mathfrak{a}^{n+1}} \twoheadrightarrow \frac{R}{\mathfrak{a}^n}$ induces

$$R\mathrm{Hom}\left(\frac{R}{\mathfrak{a}^n}, M\right) \to R\mathrm{Hom}\left(\frac{R}{\mathfrak{a}^n}, M\right)$$

- $\mathfrak{a} = (x_1, \dots, x_c)$

  $\mathfrak{a}_s = (x_1^s, \dots, x_c^s)$

  $K(\mathfrak{a}_s) :=$ Koszul c'x on $x_1^s, \dots, x_c^s$.

$$\begin{tikzcd} K(\mathfrak{a}_s) \arrow[r] & R/\mathfrak{a}_s \\ K(\mathfrak{a}_{s+1}) \arrow[u] \arrow[r] & R/\mathfrak{a}_{s+1} \arrow[u, two heads] \end{tikzcd}$$

$\exists$ canonical choice for this map

(e.g. $c=1$

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^s"] & R \arrow[r] & 0 \\ 0 \arrow[r] & R \arrow[u, "x"] \arrow[r, "x^{s+1}"'] & R \arrow[u, equal] \arrow[r] & 0 \end{tikzcd}$$

Dualising:

$$\begin{tikzcd} R\mathrm{Hom}_R(R/\mathfrak{a}_s, -) \arrow[r] \arrow[d] & \mathrm{Hom}(K(\mathfrak{a}_s), -) \arrow[d] \\ R\mathrm{Hom}_R(R/\mathfrak{a}_{s+1}, -) \arrow[r] & \mathrm{Hom}_R(K(\mathfrak{a}_{s+1}), -) \end{tikzcd}$$

Induces:

$$\operatorname*{hocolim}_s R\mathrm{Hom}_R(R/\mathfrak{a}_s, -) \to \operatorname*{hocolim}_s \mathrm{Hom}(K(\mathfrak{a}_s), -).$$

$\cong$

$R\Gamma_{\mathfrak{a}}(-)$

Thm. The above is an isomorphism of functors.

In cohomology:

Koszul cohomology

$$\mathrm{Ext}_R^*(R/\mathfrak{a}_s, -) \to H^*(\mathfrak{a}_s; -).$$

Induce $\quad \operatorname{colim} \mathrm{Ext}_R^*(R/\mathfrak{a}_s, -) \to \operatorname{colim} H^*(\mathfrak{a}_s; -)$

Induces
$$\begin{tikzcd} \operatorname{colim}_s \operatorname{Ext}^*_R(R/\mathfrak{a}_s, -) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \operatorname{colim}_s H^*(\mathfrak{a}_s; -). \arrow[d, equal] \\ H^*_{\mathfrak{a}}(-) \arrow[r, "\cong"'] & \operatorname{colim}_s H^*(\mathfrak{a}_s ; -). \end{tikzcd}$$

Content of the above theorem.

$$\operatorname{Hom}_R(K(\mathfrak{a}_s), M) \cong \operatorname{Hom}_R(K(\mathfrak{a}_s), R) \otimes_R M.$$

($\because K$ perfect)

$$\rightsquigarrow \operatorname{colim}_s \operatorname{Hom}_R(K(\mathfrak{a}_s), M) \cong \left[\operatorname{colim}_s \operatorname{Hom}(K(\mathfrak{a}_s), R)\right] \otimes M.$$

Explicit description

$$K^\infty(\underline{x}) := \quad 0 \to R \to \bigoplus_i R\left[\tfrac{1}{x_i}\right] \to \cdots \to R\left[\tfrac{1}{x_1 \cdots x_c}\right] \to 0.$$

Stable Koszul complex / Extended Čech complex.

Explicit desc above follows from examining $c=1$.

$(R, \mathfrak{m}, k)$ local. Will look at $\Gamma_{\mathfrak{m}}(-)$.

<u>Key properties</u>: $M \in D^b(\operatorname{mod} R)$.

① $H^i_{\mathfrak{m}}(M)$ are artinian $\forall i$. (Not true for general $\mathfrak{a}$.)

② $H^i_{\mathfrak{m}}(-) = 0$ on $\operatorname{Mod} R$ $\forall i \geq \operatorname{ara}(\mathfrak{m}) + 1$
$= \dim(R) + 1$

③ $\inf H^*_{\mathfrak{m}}(M) = \operatorname{depth}(M)$.

④ $\sup H^*_{\mathfrak{m}}(M) = \dim_R(M)$.

Will check using local duality.

<u>Exercise</u>: $\hat{R} = \mathfrak{m}$-adic completion.

$$H^*_{\mathfrak{m}}(M) = H^*_{\mathfrak{m}}(\hat{R} \otimes M)$$

$\forall M \in D^b(\text{mod } R)$

Use Čech complex description.

$\therefore$ May assume R complete and has a dualising complex.

So, assume $(R, \mathfrak{m}, k)$ local and ω_R a normalised dualising complex.

<u>Local duality</u>.

$$M \xrightarrow{\simeq} R\mathrm{Hom}_R(M^\dagger, \omega_R)$$

$M^\dagger = R\mathrm{Hom}_R(M, \omega_R)$.

$$R\Gamma_{\mathfrak{m}}(M) \xrightarrow{\simeq} R\Gamma_{\mathfrak{m}}(R\mathrm{Hom}_R(M^\dagger, \omega_R)) \overset{\simeq}{\leftarrow} R\mathrm{Hom}_R(M^\dagger, \underbrace{R\Gamma_{\mathfrak{m}}(\omega_R)}_{\text{can compute this explicitly:}})$$

$\because M^\dagger \in D^b(\text{mod } R)$

(Accept this for now.)

$$\omega_R \simeq 0 \to \bigoplus_{\dim(R/\mathfrak{p})=0} E(R/\mathfrak{p}) \to \cdots \to E(R/\mathfrak{m}) \to 0.$$

$$R\Gamma_{\mathfrak{m}}(\omega_R) \simeq 0 \to \Gamma_{\mathfrak{m}}\left(\oplus E(R/\mathfrak{p})\right) \to \cdots \to \Gamma_{\mathfrak{m}}(E(R/\mathfrak{m})) \to 0.$$

Recall: $\Gamma_{\mathfrak{p}}(E(R/\mathfrak{p})) \subseteq E(R/\mathfrak{p})$ [illegible]

Recall: $\Gamma_{\mathfrak a}(E(R/\mathfrak p)) = \begin{cases} E(R/\mathfrak p) & \text{if } \mathfrak a \subseteq \mathfrak p \\ 0 & \text{else} \end{cases}$

Thus, $R\Gamma_{\mathfrak m}(\omega_R) \simeq E(R/\mathfrak m)$.

Up shot:

$$\boxed{R\Gamma_{\mathfrak m}(M) \simeq R\,\mathrm{Hom}_R(M^{\dagger}, E(R/\mathfrak m))}$$

Theorem $H^i_{\mathfrak m}(M) \cong H_i(M^{\dagger})^{\vee}$

$$\cong \left[\mathrm{Ext}_R^{-i}(M, \omega_R)\right]^{\vee}.$$

$(-)^{\vee}$: matlis duality

Corollary. ① $M \in D^b(\mathrm{mod}\, R) \Longrightarrow M^{\dagger} \in D^b(\mathrm{mod}\, R)$

$\Longrightarrow \mathrm{Ext}_R^{-i}(M, \omega_R)$ is a f.g. R-mod $\forall i$

$\Longrightarrow \mathrm{Ext}_R^{-i}(M, \omega_R)^{\vee}$ artinian $\forall i$.

i.e., $H^i_{\mathfrak m}(M)$ artinian.

② $\inf H^*_{\mathfrak m}(M) = \inf H_*(M^{\dagger}) = \mathrm{depth}_R(M)$.

③ $\sup H^*_{\mathfrak m}(M) = \sup H_*(M^{\dagger}) = \dim_R(M)$.

Grothendieck's Nonvanishing Result

---------- x ----------

Justification of "accept this"

$$R\Gamma_{\mathfrak a}\, R\,\mathrm{Hom}_R(M,N) \longleftarrow R\mathrm{Hom}_R(M, R\Gamma_{\mathfrak a} N)$$

$$R\Gamma_{\mathfrak a}(N) \longrightarrow N$$

$$\begin{array}{ccc} R\Gamma_{\mathfrak{a}}(N) & \longrightarrow & N \\ \| & & \downarrow \simeq \\ \Gamma(iN) & \hookrightarrow & iN \end{array} \quad \leftarrow \text{inj res}^n$$

$$\begin{array}{ccc} R\mathrm{Hom}_R(M, R\Gamma_{\mathfrak{a}}(N)) & \longrightarrow & R\mathrm{Hom}_R(M,N) \\ & \searrow & \uparrow \\ & & R\Gamma_{\mathfrak{a}}\, R\mathrm{Hom}_R(M,N) \end{array}$$

$M \in D^b(\mathrm{mod}\ R)$

is 0 when $N \in D_-(\mathrm{Mod}\ R)$

i.e., $H_i(N) = 0$ $\forall i \gg 0$

Another perspective :

$$R\Gamma_{\mathfrak{a}}\, R\mathrm{Hom}_R(M,N) \simeq K^\infty \otimes_R R\mathrm{Hom}_R(M,N)$$

$$\simeq R\mathrm{Hom}_R(M, N \otimes_R K^\infty)$$

$\because K^\infty$ is a bdd c'x of flat mod

$- \ M \in D^b(\mathrm{mod}\ R)$ & $H_i(N) = 0$ $\forall i \gg 0$

$$F \otimes_R \mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(M, F \otimes_R N)$$

F flat R-module
M f.g.

Recall : $H^i_{\mathfrak{m}}(M) = H_i(M^\dagger)^\vee$.

Suppose R is Gorenstein.

Then, $\omega_R \simeq \Sigma^d R$, $\quad d = \dim(R)$.

$$H^i_{\mathfrak{m}}(M) \cong \left(H_i\left(R\mathrm{Hom}_R(M, \Sigma^d R)\right)\right)^\vee$$

$$\cong \left(H^{-i}\left(R\mathrm{Hom}_R(M, \Sigma^d R)\right)\right)^\vee$$

$$\cong \left(H^{-i}\left(R\mathrm{Hom}_R(M, \Sigma^d R)\right)\right)^{\vee}$$

$$\cong \left(H^{d-i}\left(R\mathrm{Hom}_R(M,R)\right)\right)^{\vee}$$

$$\cong \mathrm{Ext}_R^{d-i}(M,R)^{\vee}$$

Corollary. R Gorenstein, $M \in D^b(\mathrm{mod}\, R)$:

$$H^i_{\mathfrak{m}}(M) = \mathrm{Ext}_R^{d-i}(M,R)^{\vee}. \qquad \blacksquare$$

When R is C.M. with canonical module ω, then

$\Sigma^d \omega$ normalised dualising complex,

$$H^i_{\mathfrak{m}}(M) \cong \mathrm{Ext}_R^{d-i}(M,\omega)^{\vee}. \qquad \blacksquare$$

Remark. $R \longrightarrow S$, $N \in D(S)$, $\mathfrak{a} \subseteq R$.

$$H^*_{\mathfrak{a}}(N) \cong H^*_{\mathfrak{a}S}(N). \quad \text{(Immediate from Koszul description)}$$

Note: $\mathrm{Ext}_R(R/\mathfrak{a}^n, N)$ & $\mathrm{Ext}_S(S/\mathfrak{a}^n S, N)$ can be different.

Another perspective on $R\Gamma_{\mathfrak{a}}(-)$:

$$\Gamma_{\mathfrak{a}}(M) \hookrightarrow M \qquad (M \text{ module})$$

M is $\mathfrak{a}$-power torsion if $\Gamma_{\mathfrak{a}}(M) = M$.

$M \in D(R)$ is $\mathfrak{a}$-power torsion if $H_i(M)$ is $\mathfrak{a}$-power torsion $\forall i$

$\Gamma_{\mathfrak{a}} D(R) := \{ M \in D(R) \ : \ M \ \mathfrak{a}\text{-power torsion} \}$

→ triangulated sub category of $D(R)$

→ closed under coproducts

i.e. $\Gamma_{\mathfrak{a}} D(R)$ is a localising sub cat. of $D(R)$.

$$\Gamma_{\mathfrak{a}} D(R) \underset{R\Gamma_{\mathfrak{a}}}{\overset{\longrightarrow}{\dashleftarrow}} D(R)$$

$R\Gamma_{\mathfrak{a}}$ is right adjoint to the inclusion,

i.e., $\quad R\Gamma_{\mathfrak{a}} M \longrightarrow M \qquad \forall \ M \in D(R)$

and $\quad \mathrm{Hom}_{\mathcal{D}}(-, R\Gamma_{\mathfrak{a}} M) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(-, M)$

is bijective on $\mathfrak{a}$-torsion objects.

FACT: $\Gamma_{\mathfrak{a}} D(R) = \mathcal{L}_{\mathfrak{a}}(\kappa(\mathfrak{a}))$.

localising subcategory?

Dwyer & Greenlees: Complete modules & torsion modules

Lecture 22 (05-04-2023)

Wednesday, April 5, 2023 1:13 PM

Kähler diff. & derivations

- Majadas & Rodicio
  Smoothness, regularity, and complete intersection
- Matsumura's two books
- André-Quillen homology of comm. algebra

K → comm. ring (possibly non noetherian)

R → K-algebra

M an R-module

Defn A derivation $d: R \to M$ is a K-linear map satisfying $d(xy) = (dx)y + x\,dy$.

Remark. $d(1^2) = 2d(1) \Rightarrow d(1) = 0$.

Also, $d(K) = 0$.

$$\mathrm{Der}_K(R, M) \subseteq \mathrm{Hom}_K(R, M).$$

↑ R-submodule

($d: R \to M$ der, $f: M \to N$ linear $\Rightarrow f \circ d$ der.)

$\mathrm{Der}_K(R, \_) : \text{Mod-}R \to \text{Mod-}R$ functor.

Theorem This functor is representable.

That is, ∃ R-module Ω and a derivation $\delta: R \to \Omega$ s.t.

$$\mathrm{Hom}_R(\Omega, M) \to \mathrm{Der}(R, M)$$
$$f \longmapsto f \circ \delta$$

$$f \longmapsto f \circ \delta$$

is bijective for all M. That is,

$$\mathrm{Hom}_R(\Omega, -) \xrightarrow{\cong} \mathrm{Der}_K(R, -).$$

· Moreover, (Ω, δ) is unique (up to iso).

We write $\Omega =: \Omega_{R/K}$, $\leftarrow$ module of Kähler differentials

$\delta =: \delta_{R/K}$. $\leftarrow$ universal derivation

Proof $R^e := R \otimes_K R$. $\leftarrow$ enveloping algebra

we have a surjection
$$\begin{aligned} R^e &\xrightarrow{\mu} R \\ x \otimes y &\longmapsto xy. \end{aligned}$$

Put $J := \ker \mu$.

Then, $\Omega = J/J^2$ and
$$\begin{aligned} \delta : R &\to J/J^2 \\ x &\longmapsto x \otimes 1 - 1 \otimes x \end{aligned}$$
does the job.

Check: $J = \langle x \otimes 1 - 1 \otimes x \mid x \in R \rangle$

In fact, suffices to take x that generate R as a K-algebra.

Ex. $R = k[x_\lambda]_{\lambda \in \Lambda}$, then $J = (x_\lambda \otimes 1 - 1 \otimes x_\lambda : \lambda \in \Lambda)$.

$$R \underset{i_2}{\overset{i_1}{\rightrightarrows}} R \otimes_K R, \qquad \begin{aligned} i_1(r) &= r \otimes 1, \\ i_2(r) &= 1 \otimes r. \end{aligned}$$ $\nearrow$ R-algebra maps $\swarrow$

$i_2 \qquad\qquad i_2(r) = 1 \otimes r.$

$$J = \sum_{x \in R} R(x \otimes 1 - 1 \otimes x).$$

These give J two R-mod structures. Both same.

$$\begin{tikzcd} R \arrow[r, "i_1 - i_2"'] \arrow[dr, dashed, bend right, "\delta"'] & J \arrow[r, hook] \arrow[d] & R^e \\ & J/J^2 & \end{tikzcd}$$

<u>Check</u>: δ is a derivation.

$$J = \sum_{x \in R} R\,\delta x.$$

Now, if $d: R \to M$ is a derivation,

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, "d"'] & \Omega \arrow[dl, dashed, "f"] \\ M & \end{tikzcd}$$

$f(\delta x) := dx$ for $x \in R$. ☒

<u>Example</u>. $R = K[\{x_\lambda\}_{\lambda \in \Lambda}]$.

M.

Derivation $R \to M$.

$\updownarrow$

Assignments $\{x_\lambda\} \xrightarrow{d} M$ (map of sets only)

Given an assignment, extend by

$$p(\underline{x}) = \sum \left(\frac{\partial p}{\partial x_\lambda}\right) d x_\lambda$$

R linear maps $\bigoplus R\,dx_\lambda \longrightarrow M$.

$$\Omega_{R/K} = \bigoplus R\,dx_\lambda$$
$$\delta_{R/K}(x_\lambda) = dx_\lambda.$$

Jacobi - Zariski sequences:
(JZ)

$\varphi: R \longrightarrow S$ map of K-algebras.

We get a derivation $R \longrightarrow \Omega_{S/K}$.

$$R \xrightarrow{\delta} S \xrightarrow{\delta_{S/K}} \Omega_{S/K}$$

By the universal property we get

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\delta_{R/K}"'] & S \arrow[d, "\delta_{S/K}"] \\
\Omega_{R/K} \arrow[r, "R\text{-linear}"] \arrow[d, "\text{extend scalars}"'] & \Omega_{S/K}. \\
S \otimes_R \Omega_{R/K} \arrow[ur, "d\varphi"'] &
\end{tikzcd}$$

$d\varphi$ is S-linear

$$s \otimes \delta_{R/K}\, x \longmapsto s\, \delta_{S/K}(\varphi(x)).$$

First JZ sequence.

$$\boxed{S \otimes_R \Omega_{R/K} \xrightarrow{d\varphi} \Omega_{S/K} \longrightarrow \Omega_{S/R} \longrightarrow 0}$$

First JZ sequence.

$$\boxed{S\otimes_R \Omega_{R/K} \xrightarrow{d\varphi} \Omega_{S/K} \longrightarrow \Omega_{S/R} \longrightarrow 0}$$

Prop^n. The above sequence is exact.

Proof. $N \in \text{Mod-}S$. $\mathrm{Hom}_S(-,N)$ applied to above yields

$$0 \to \mathrm{Hom}_S(\Omega_{S/R}, N) \to \mathrm{Hom}_S(\Omega_{S/K}, N) \to \mathrm{Hom}_S(S\otimes_R \Omega_{R/K}, N).$$

(Suffices to check $\circlearrowleft$ is exact $\forall N$.) $\quad \cong$ adjunction $\mathrm{Hom}_R(\Omega_{R/K}, N)$

becomes

$$0 \to \mathrm{Der}_R(S,N) \longrightarrow \mathrm{Der}_K(S,N) \longrightarrow \mathrm{Der}_K(R,N).$$

Easy to see above is exact

IInd JZ sequence.

$$R \twoheadrightarrow S = R/I.$$

$$\mathrm{Der}_R(S, -) = 0.$$

Thus, $\Omega_{S/R} = 0$.

$$S\otimes_R \Omega_{R/K} \longrightarrow \Omega_{S/K} \longrightarrow 0 \quad \text{exact.}$$

Moreover,

Prop^n.

$$I/I^2 \xrightarrow{\bar{\delta}} S\otimes_R \Omega_{R/K} \longrightarrow \Omega_{S/K} \longrightarrow 0 \quad \text{is exact.}$$

(As a seq of S-modules.)

$$\begin{tikzcd}
R \arrow[r, "\delta"] & \Omega_{R/K} \arrow[dd, bend left] \\
I \arrow[u, phantom, "\cup"] \arrow[ur, "\delta|_I"] \arrow[dr, "j"'] & \\
 & S\otimes_R \Omega_{R/K}
\end{tikzcd}$$

$$\xrightarrow{j} S\otimes_R \Omega_{R/K}$$

Check: j is R-linear.

This gives

$$\underset{I/I^2}{S\otimes_R I} \xrightarrow{j} S\otimes_R \Omega_{R/K}$$

Proof of exactness: Apply $\mathrm{Hom}_S(-, N)$ again.

$$0 \to \mathrm{Der}_K(S,N) \to \mathrm{Der}_K(R,N) \to \mathrm{Hom}_S(I/I^2, N)$$

$$\mathrm{Hom}_S(I/I^2,N) = \mathrm{Hom}_S(S\otimes_R I, N) = \mathrm{Hom}_R(I, N).$$

restriction to I: $\mathrm{Der}_K(R,N) \to \mathrm{Hom}_R(I,N)$

Example. $S = \dfrac{K[x_1,\dots,x_n]}{(f_1,\dots,f_c)}$. $\hookleftarrow I$

$$0 \to \underset{(\underline{f})}{I} \hookrightarrow \overset{R}{K[\underline{x}]} \twoheadrightarrow S.$$

$$I/I^2 = \sum R[f_i]$$

$$I/I^2 \xrightarrow{j} \bigoplus_i S\,dx_i \to \Omega_{S/K} \to 0.$$

$$j(f_h) = \sum_{i=1}^{c} \frac{\partial f_h}{\partial x_i}\, dx_i\,.$$

$$S^c \xrightarrow[\left(\frac{\partial f_h}{\partial x_i}\right)]{} S \to \Omega_{S/K} \to 0$$

$\left(\frac{\partial f_h}{\partial x_i}\right) \rightsquigarrow$ Jacobian matrix.

$\left(\partial x_i\right) \rightsquigarrow$ Jacobian matrix.

This gives $\Omega_{S/k}$ for any algebra S.
(Didn't need any finiteness.)

Ex. $R := \dfrac{k[x,y]}{(y^2 - x^3)}$.

$$R \xrightarrow{\begin{pmatrix} -3x^2 \\ 2y \end{pmatrix}} R^2 \longrightarrow \Omega_{R/k} \longrightarrow 0$$

happens to be 1-1.

$R \rightsquigarrow$ k-algebra. $I \to$ R-module.

An (comm algebra) extension of R by I is an exact sequence of k-modules:

$$0 \to I \xrightarrow{i} E \xrightarrow[p]{} R \to 0, \quad \text{where}$$

- p is a map of k-algebras, and
- $e \cdot i(x) = p(e) \cdot x$ for $x \in I$, $e \in E$.

In particular, $i(x) \cdot i(y) = 0$.

Equivalently: $I \subseteq E$ ideal s.t. $I^2 = 0$ and induced R-algebra structure on $I = I/I^2$ is the one we started with.

"Square-zero deformation of R by I."

R and R-module M. Want diagram

$$0 \to M \xrightarrow[i]{} E \xrightarrow[p]{} R \to 0 \qquad \text{exact}$$

s.t. p map of k-alg and $i(M)^2 = 0$

Two such extensions are equivalent if $\exists\, \varphi$ map of k-alg.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, dashed, "\varphi"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & E' \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

$(\Rightarrow \varphi \text{ iso})$

$\mathrm{Exalgcom}(R/K;\, M)$ = Equiv. classes of commutative algebra extensions of R by M.

Trivial extension: $R \ltimes M$.

Underlying K-module: $R \oplus M$. (Happens to be R-mod.)

Multiplication: $(r, m)\cdot(s, n) = (rs,\ rn + ms)$.

$$0 \to M \to R \ltimes M \to R \to 0$$

$$(r, m) \longmapsto r$$

$$m \longmapsto (0, m)$$

An extension

$$0 \to M \to E \xrightarrow{p} R \to 0 \qquad \text{is split}$$

if $\exists$ k-algebra map $s: R \to E$ s.t. $ps = \mathrm{id}$.

Ex An extension is split $\Leftrightarrow$ equivalent to the trivial extension.

Splitting need not be unique.

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & M \arrow[d] \\
 & E \arrow[d, "p"] \\
S \arrow[ur, bend left, "s_1"] \arrow[ur, bend left=20, "s_2"'] \arrow[r, "\varphi"'] & R \arrow[d] \\
 & 0
\end{tikzcd}$$

extension

Let s_1 and s_2 be two liftings (k-alg. maps).

Then, $s_1 - s_2 : S \longrightarrow M$

$\hookrightarrow$ S-module via φ.

<u>Check</u> : $s_1 - s_2$ is a k-linear derivation

<u>Conversely</u>. If $s : S \to E$ is a lifting and $d : S \to M$ is a k-linear der, then $s + d$ is also a lifting.

Lecture 23 (10-04-2023)

Monday, April 10, 2023 1:23 PM

$K \to$ comm. ring, $R \to$ K-algebra.

$\exists$ complex of R-modules:

$$\mathcal{L}^{R/k} : \quad 0 \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to 0$$

(called the truncated cotangent complex) with

(Construction: Beginning of next lec.)

$\mathcal{L}_0, \mathcal{L}_1$ free. Set

$$D_i(R/k, M) := H_i(\mathcal{L}^{R/k} \otimes_R M)$$
$$D^i(R/k, M) := H^i(\operatorname{Hom}_R(\mathcal{L}, M)).$$

André-Quillen (co) homology

The following key properties hold:

① $D_0(R/k, M) = \Omega_{R/k} \otimes_R M$,

$D^0(R/k, M) = \operatorname{Der}_k(R, M)$.

② k notherian + R f.g. k algebra (or a localisation of such),

$\rightsquigarrow$ essentially of finite type

then $\exists$ representative of $\mathcal{L}^{R/k}$ in $D^b(\text{mod } R)$.

i.e., $D_i(R/k, M)$ and $D^i(R/k, M)$ are in mod R when $M \in \text{mod } R$.

③ $R \to S$ map of K-algebras

$N \in \text{Mod } S$, we get a l.e.s.

$$\to D_0(R/k, N) \to D_0(S/k, N) \to D_0(S/R, N) \to 0$$

$$\to D_1(R/k, N) \to D_1(S/k, N) \to D_1(S/R, N) \to$$

$$D_2(R/k,N) \longrightarrow D_2(S/k,N) \to D_2(S/R,N)$$

need not be 1-1

Hly: $0 \to D^0(S/R,N) \to D^0(S/k,N) \to \cdots \to D_2(S/k,N)$
$$\downarrow$$
$$D_2(R/k,N).$$

(A form of JZ seq. ↑)

④ $R \twoheadrightarrow S = R/I$.

$$D_0(S/R, -) = 0,$$
$$D_1(S/R, -) = I/I^2 \otimes_S -,$$
$$D^1(S/R, -) = \mathrm{Hom}_S(I/I^2, -).$$

⑤ $R \to$ poly. ring over k.

$$D_i(R/K, -) = 0 \qquad \text{for } i = 1, 2.$$
$$D^i(R/K, -) = 0$$

Remark: $R \to K$-algebra. Choose a presentation

$$0 \to I \to P \twoheadrightarrow R \to 0.$$

P: polynomial ring; $P \twoheadrightarrow R$: map of K-alg.

$M \in$ Mod-R

$$0 \to \mathrm{Der}_K(R,M) \to \mathrm{Der}_K(P,M) \to \mathrm{Hom}_R(I/I^2, M) \to D^1(R/K,M) \to 0$$

$$\mathrm{Hom}_R(I/I^2,M) \cong \mathrm{Hom}_P(I,M)$$

$I/I^2 \cong R \otimes_P I$, adjunction

Thm. $\exists$ isomorphism

$$D^1(R/K, M) \cong \mathrm{Exalcom}(R/k, M)$$

$$D^1(R/K, M) \cong \mathrm{Exalcom}(R/K, M)$$

natural wrt K, R, M.

Recall

Equivalence classes of s.e. sequences

$$0 \to M \xrightarrow{i} A \xrightarrow{\pi} R \to 0 \quad \text{s.t.}$$

- π is a K-alg map,
- $a \cdot i(m) = \pi(a) \cdot m$.

$$(i(M)^2 = 0)$$

NOTE. ① $R \xrightarrow{\varphi} S$ K-alg map,

$N \in \text{Mod-}S$.

Consider

$$0 \to N \to B \to S \to 0 \in \mathrm{Exalcom}(S/K, M).$$

Consider pull back: $\{(r, b) : \varphi(r) = \pi(b)\}$.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & R \times_S B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & N \arrow[r] & B \arrow[r, "\pi"'] & S \arrow[r] & 0 \end{tikzcd}$$

$\in \mathrm{Exalcom}(R/K, M)$

② $f : M \to M'$ R-linear.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[r, "\pi"] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r] & A' \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

$\in \mathrm{Exalcom}(R/K, M)$

$\in \mathrm{Exalcom}(R/K, M')$

$$A' = \frac{A \oplus M'}{(i(m) - f(m) \mid m \in M)}$$

Proof of Thm Pick

$$0 \to I \to P \to R \to 0.$$

P free K-alg.

$$0 \to \frac{I}{I^2} \to \frac{P}{I^2} \to R \to 0 \in \mathrm{Exalgcom}(R/K, I/I^2).$$

Defining maps: $D'(R/K, M) \longrightarrow \mathrm{Exalcom}(R/K, M)$.

Given $f : I/I^2 \to M$ R-linear,

$\left(\mathrm{Hom}(I/I^2, M) \twoheadrightarrow D'(R/K, M), \text{ by earlier} \right)$

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, "f"] & P/I^2 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & A \arrow[r] & R \arrow[r] & 0
\end{tikzcd} \in \mathrm{Exalcom}(R/K, M).$$

the image of $[f]$.

Check: well-defined, i.e., if f comes from a derivation, then image of f is split.

Conversely, given

Construct:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & A \arrow[r] & R \arrow[r] & 0 \\
& I \arrow[r, hook] & P \arrow[r, two heads] \arrow[u, dashed, "\varphi"] & R \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \in \mathrm{Exalcom}(R/K, M)$$

$\exists \varphi$ since P a free-algebra.

$$\varphi(I) \subset M \Rightarrow \varphi(I^2) \subseteq \varphi(I)^2 = 0.$$

$\therefore \varphi|_I : I \longrightarrow M$. Check P-linear, so defines a class in ...

Def^n $R \to K$-algebra, $I \trianglelefteq R$ ideal.

R is I-smooth (over K)

Defn. $R \to$ K-algebra, $I \subseteq K$ ideal.

R is **I-smooth (over K)**

(or **formally smooth in I-adic topology**)

if given a solid diagram

$$\begin{tikzcd} & & & R \arrow[dll, dashed, "\varphi'"'] \arrow[d, "\varphi"] & \\ 0 \arrow[r] & N \arrow[r] & E \arrow[r, "\pi"'] & A \arrow[r] & 0 \end{tikzcd} \in \mathrm{Exalcom}(A/K, M)$$

s.t. $\varphi(I^n)$ f $n \gg 0$,

$\exists$ dotted map $\varphi' : E \to R$ of K-algebras s.t.

$$\pi \varphi' = \varphi.$$

NB. for $n \gg 0$, $\varphi'(I^n) \subseteq N$, so $\varphi'(I^{2n}) = 0$.

This property of $\varphi(I^n) = 0$ $(n \gg 0)$ is saying "continuous in I-adic topology, where target has discrete topology"!

① $I \subseteq J \subseteq R$ ideals.

I-smooth $\Rightarrow$ J-smooth.

Thus, smooth $\Longrightarrow$ I-smooth $\forall I$.

smooth = 0-smooth

Ex. R poly /K. R smooth /K.

More generally, $U^{-1}R$ is smooth /K for any mult. closed $U \subseteq R$.

Lemma. $R/K \to$ I-smooth.

Lemma. $R/K \to I$-smooth.

$S \to K$-algebra, complete wrt J-adic topology, for some $J \trianglelefteq S$.

Then, any map $\varphi : R \to S/J$ s.t. $\varphi(I^n) = 0$ for $n \gg 0$ lifts to S:

$$\begin{tikzcd} & S \arrow[d] \\ R \arrow[r] \arrow[ur, dashed] & S/J . \end{tikzcd}$$

Proof.

$$\begin{tikzcd} & \vdots \\ & S/J^4 \arrow[d] \\ & S/J^2 \arrow[d] \\ R \arrow[r] \arrow[ur, dashed] \arrow[uur, dashed] & S/J \end{tikzcd}$$

$\rightsquigarrow \quad R \to \hat{S}^J \cong S.$

Theorem R/K. $I \subseteq R$ ideal. Consider the conditions

① R is I-smooth.

② $\operatorname{colim}_n D^1(R_n/K, M) = 0 \quad \forall \; R/I$-modules M,

where $R_n = R/I^n$.

②' $\operatorname{colim}_n D^1(R_n/K, M) = 0$ whenever $I^s M = 0$ for some s.

③ $D^1(R/K, M) = 0 \quad \forall \; R/I$-mod M.

③' $D_1(R/K, M) = 0 \quad$ ——"——

④ $D_1(R/K, R/I) = 0$ and $\Omega_{R/K} \otimes_R R/I$ is a projective R/I-module.

⑤ $P \to P/J = R$, P poly alg.

Then, $\Omega_{P/k} \otimes R/I$ projective

and $R/I \otimes_R J/J^2 \longrightarrow R/I \otimes_P \Omega_{P/k}$ is split-injective.

Then,

$$\begin{array}{ccccc} ① & \Longleftrightarrow & ② & \Longleftrightarrow & ②' \\ \Uparrow & & & & \\ ③ & \Longleftrightarrow & ③' & \Longleftrightarrow & ④ \Longleftrightarrow ⑤. \end{array}$$

If $I = 0$ or R noetherian.

then ① $\Rightarrow$ ③.

Proof. ②' $\Rightarrow$ ②. ✓

② $\Rightarrow$ ②'. Induction on s.

$$0 \to \frac{I^{s-1}M}{I^s M} \to \underset{\substack{\| \\ M}}{\frac{M}{I^s M}} \to \frac{M}{I^{s-1}M} \to 0.$$

} André. Q gives

$$D^1(R_n/k, I^{s-1}M/I^sM) \to D^1(R_n/k, M) \to D^1(R_n/k, M/I^{s-1}M).$$

by ②, this is 0 since killed by I

killed by I^{s-1}, use induction.

② $\Rightarrow$ ①: Given

$\varphi(I^n) = 0$

$$\begin{array}{ccccccccc} & & & & R & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & N & \to & E' & \to & R/I^n = R_n & \xrightarrow{\varphi} & 0 \\ & & \| & & | & & | & & \end{array}$$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & R/I^n \arrow[r] \arrow[d] \arrow[d, bend left, "\varphi"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r, "\pi"'] & A \arrow[r] & 0
\end{tikzcd} \quad \in \mathrm{Exalcomm}(A/k, N)$$

Note $\quad I^n N = 0.$

(2') $\Rightarrow \exists\, \ell > n$ s.t.

$$0 \longrightarrow N \longrightarrow E \longrightarrow R_\ell \longrightarrow 0 \quad \text{splits.}$$

But R then compose $\quad R \to R_\ell \to E.$

(1) $\Rightarrow$ (2) Similar ...

Lecture 24 (12-04-2023)

Wednesday, April 12, 2023 1:18 PM

Truncated Cotangent Complex

Consider $P \twoheadrightarrow R$, map of k-algebras,
P is poly. alg $/k$.

$J := \ker(P \twoheadrightarrow R)$.

Fix a presentation
$$0 \to U \xrightarrow{i} F \xrightarrow{\pi} J \to 0$$
over P, where F is free mod $/P$.

$$\Lambda^2 F \to F \quad \text{map of } P\text{-mod}$$
$$x \wedge y \longmapsto \pi(x)y - x\pi(y).$$ π of this is zero

$$U_0 = \mathrm{im}(\Lambda^2 F \to F) \subseteq U.$$
$$\begin{array}{c} U_0 \\ \cup\!| \\ JU \end{array}$$

$JU \subseteq U_0 \subseteq JF$

$$\begin{tikzcd}
0 \arrow[r] & \frac{U}{U_0} \arrow[r] & \frac{F}{JF} \arrow[r] \arrow[dr] & R \otimes_P \Omega_{P/k} \arrow[r] & 0 \\
& & & \frac{J}{J^2} \arrow[u, "d"'] &
\end{tikzcd}$$

complex of R-modules

This is the truncated cotangent complex.
(What we called $\mathcal{L}^{*}_{R/k}$.)
(well-defined...)

Observations:

① $H_0(\mathcal{L}^1_{R/k}) = \Omega_{R/k}$.

Observations

① $H_0(\mathcal{L}'_{R/k}) = \Omega_{R/k}$.

② $K \twoheadrightarrow R = K/J$. Then, $H_1(\mathcal{L}'_{R/k}) = J/J^2$.

③ If J is principal and NZD, then can take $F = P$ and $U = 0$ to get

$$H_2(\mathcal{L}'_{R/k} \otimes_R -) = 0.$$

Can extend to J gen. by reg. sequence.

Recall: $D_i(R/k, -) = H_i(\mathcal{L}'_{R/k} \otimes_R -)$,

$D^i(R/k, -) = H^i(\mathrm{Hom}_R(\mathcal{L}'_{R/k}, -))$.

Consider

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \frac{U}{U_0} \arrow[r] & R\otimes_P F \arrow[r] & R\otimes_P \Omega_{P/k} \arrow[r] & 0
\end{tikzcd}$$

← exact

← $\mathcal{L}_{R/k}$

For M an R-module, apply $\mathrm{Hom}_P(-, M)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(R\otimes_P \Omega_{P/k}, M) \arrow[r] \arrow[d] & \mathrm{Hom}_R(R\otimes_P F, M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(\frac{U}{U_0}, M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_P(J, M) \arrow[r] & \mathrm{Hom}_P(F, M) \arrow[r] & \mathrm{Hom}_P(U, M) \arrow[r] & \mathrm{Ext}^1_P(J, M)
\end{tikzcd}$$

computes André-Qui.

exact

This gives us:

Lemma. $D^2(R/K, M) \hookrightarrow Ext_P^1(J, M)$.

Proof of Thm from last lecture.

③ $\Leftarrow\Rightarrow$ ④ $\Leftarrow\Rightarrow$ ⑤

Consider a sequence

$$0 \to V \to E \to G \to W \to 0$$

free (G), exact (at G, W)

of R-modules.

For all M an R/I-module

$$0 \to Hom_R(W, M) \hookrightarrow Hom_R(W, G) \to Hom_R(W, E) \to Hom_R(V, M) \to 0$$

is exact here

$\Leftarrow\Rightarrow$ $\left[H_1(R/I \otimes_R \mathcal{L}) = 0 \text{ and } R/I \otimes_R W \text{ projective} \right]$

③ $\Rightarrow$ ①: $\quad R \twoheadrightarrow R_n = R/I^n$.

$$D^1(R_n/R, M) \to D^1(R_n/K, M) \to D^1(R/K, M)$$

exact

$\Downarrow$

$$\underset{n}{\operatorname{colim}}\, D^1(R_n/R, M) \to \underset{n}{\operatorname{colim}}\, D^1(R_n/K, M) \to D^1(R/K, M)$$

exact

$\underset{n}{\operatorname{colim}}\, D^1(R_n/R, M) = 0$

$$D^1(R_n/R, M) = Hom_R\left(\frac{I^n}{I^{2n}}, M\right) \underset{\because M \text{ is } R/I\text{-mod}}{\cong} Hom_R\left(\frac{I^n}{I^{n+1}}, M\right)$$

$$D^1(R_n/R, M) \to D^1(R_{n+1}/R, M) \quad \text{induced by}$$

$D^1(R_n/R, M) \longrightarrow D^1(R_{n+1}/R, M)$ induced by

$$\frac{I^{n+1}}{I^{n+2}} \xrightarrow{0} \frac{I^n}{I^{n+1}}$$

$\therefore \Rightarrow 0$

This gives middle zero. ✓

① ⟹ ② when R is noetherian.

$R \twoheadrightarrow R_n$.

$$0 \to \operatorname{colim}_n D^1(R_n/R, M) \to D^1(R/R, M) \to \operatorname{colim}_n D^2(R_n/R, M)$$

$$\begin{array}{ccc} D^2(R_n/R, M) & \hookrightarrow & \operatorname{Ext}^1_R(I^n, M) \\ & & \downarrow \\ & & \operatorname{Ext}^1_R(I^{n+1}, M) \end{array}$$

<u>Claim.</u> R noe. $\Rightarrow \operatorname{colim} \operatorname{Ext}^1_R(I^n, M) = 0$.

<u>Sketch</u> follows from

$$\lim \operatorname{Tor}_i^R(R/I^n, M) = 0$$ when R is noe.

$\forall i \geq 1$, M f.g.

Lecture 25 (17-04-2023)

Monday, April 17, 2023 1:21 PM

Theorem Let R be a K-algebra, $I \trianglelefteq R$.

① R is I-smooth over K.

② $\operatorname{colim} D^1(R_n/K, M) = 0 \quad \forall\, M \in \mathrm{Mod}(R/I)$.

②' $\operatorname{colim} D^1(R_n/K, M) = 0 \quad \forall\, M \in \bigcup_{s \geq 0} \mathrm{Mod}(R/I^s)$

③ $D^1(R/K, \_\_) = 0$ on $\mathrm{Mod}(R/I)$.

④ $D_1(R/K, R/I) = 0$ and $(R/I \otimes_R \Omega_{R/K}) \in \mathrm{Proj}(R/I)$.

⑤ In any presentation $P \twoheadrightarrow R = P/J$, P poly. alg. $/K$,

$$R/I \otimes_R J/J^2 \longrightarrow R/I \otimes_P \Omega_{P/K}$$

is split injective.

$① \Leftrightarrow ② \Leftrightarrow ②' \Rightarrow ③ \Leftrightarrow ④ \Leftrightarrow ⑤$

$\Leftarrow$ if R noe on $I = 0$.

N.B. if $\mathfrak{m} \in \max(R)$ and R noetherian, then

R is $\mathfrak{m}$-smooth$/K \Leftrightarrow D_1(R/K, R/\mathfrak{m}) = 0$.

AQ theory for fields

lemma. $k \hookrightarrow k(t) = \ell$ field ext^n.

① If ℓ transcendental $/k$,

$D_0(\ell/k, \ell) \cong \ell$ and $D_i(\ell/k, \ell) = 0$ for $i = 1, 2$.

② When ℓ is alg. and separable,

$$D_i(\ell/k, \ell) = 0 \quad \forall i.$$

③ When ℓ is alg. and insep.,

$$D_0(\ell/k, \ell) \cong \ell \cong D_1(\ell/k, \ell),$$

$$D_2(\ell/k, \ell) = 0.$$

N.B. V an ℓ-vector space:

follows from Künneth formula

$$\begin{cases} D_i(\ell/k, V) \cong D_i(\ell/k, \ell) \otimes_\ell V \\ D^i(\ell/k, V) \cong \mathrm{Hom}_\ell(D_i(\ell/k, \ell), V) \end{cases}$$

Proof of Lemma.

① $D_i(k[t]/k, \ell) \cong D_i(\ell/k, \ell) \quad \forall i.$

($\because$ $k[t] \hookrightarrow k(t)$ localisation)

(e.g. $D_i(U^{-1}R/R, -) = 0 \ldots \forall i, \ldots$)

we already computed for poly. rings.

② $k[x] \twoheadrightarrow \ell = \dfrac{k[x]}{(f)}$, $f \neq 0$.

$$\underset{\substack{\| \\ 0}}{D_2(k[x]/k, \ell)} \longrightarrow D_1(\ell/k, \ell) \longrightarrow \frac{(f)}{(f^2)} \longrightarrow \ell \otimes_k \overset{\substack{\ell \\ \cong}}{\Omega_{k[x]/k}} \longrightarrow \Omega_{\ell/k} \longrightarrow 0$$

$$f \longmapsto df$$

$$0 \longrightarrow D_1(\ell/k, \ell) \longrightarrow \overset{\substack{\ell \\ \cong}}{\frac{f}{f^2}} \longrightarrow \ell \longrightarrow \Omega_{\ell/k} \longrightarrow 0.$$

$$f \longmapsto df$$

$df \neq 0$.

ℓ separable $\iff df \neq 0$.

Note: in any case, $D_2 = 0$.

Corollary. $k \hookrightarrow \ell$ any field extension.
$$D_2(\ell/k, \ell) = 0.$$

Pf. $\ell = \operatorname{colim}_i \ell_i$, ℓ_i f.g. field ext^n of k.

$$D_2(\ell/k, \ell) = \operatorname{colim}_i D_2(\ell_i/k, \ell_i).$$

Enough to prove when ℓ is f.g. (as a field).

Then use ZJ to reduce to primitive ext^n.

Observation. $h \hookrightarrow k \hookrightarrow \ell$ f.g. field extensions.

$$\begin{aligned}\operatorname{rank}_\ell D_0(\ell/h, \ell) - \operatorname{rank}_\ell D_1(\ell/h, \ell) &= \left(\operatorname{rank}_k D_0(k/h, k) - \operatorname{rank}_k D_1(k/h, k)\right) \\ &\quad + \left(\operatorname{rank}_\ell D_0(\ell/k, \ell) - \operatorname{rank}_\ell D_1(\ell/k, \ell)\right).\end{aligned}$$

(Euler characteristic-ish)

Follows from JZ & that $D_2 = 0$ for field ext^{n}s.

Theorem. (Cartier's Equality)
$k \hookrightarrow \ell$ f.g.

$$\operatorname{trdeg}_k \ell = \operatorname{rank}_\ell D_0(\ell/k, \ell) - \operatorname{rank}_k \underbrace{D_1(\ell/k, \ell)}_{\text{imperfection module.}}$$

Proof. Using observation, reduce to primitive.

Then, transcendental gives $1 - 0 = 1$,

$0 - 0 = 0$.

Then, transcendental gives $1-0=1$,

alg gives $1-1$ or $0-0=0$.

Theorem. ℓ/k separable $\iff$ ℓ/k smooth

$\Updownarrow$ def^n $\iff D_1(\ell/k, \ell) = 0$.

follows from remark after Thm.

$$\begin{array}{l} \ell \\ \mid \text{ alg + separable} \\ k(\underline{t}) \\ \mid \text{ purely transcendental} \\ k \end{array}$$

Proof. Note: $D_1(\ell/k, \ell) = 0$

$\iff D_1(\ell_i/k, \ell_i) = 0 \quad \forall$ f.g. intermediate field ext^n ℓ_i.

($\Leftarrow$) colimit

($\Rightarrow$) JZ

Similar statement true for separability

Can assume ℓ/k f.g.

($\Rightarrow$) Assume ℓ/k separable.

$$k \underset{\substack{\text{purely}\\\text{transc.}}}{\hookrightarrow} k(\underline{x}) \underset{\substack{\text{sep.}\\\text{alg}}}{\hookrightarrow} \ell.$$

$$\underset{\substack{\| \\ 0}}{D_1(k(\underline{x})/k, \ell)} \to D_1(\ell/k, \ell) \to \underset{\substack{\| \\ 0}}{D_1(\ell/k(\underline{x}), \ell)}.$$

$\therefore D_1(\ell/k, \ell) = 0$.

($\Leftarrow$) Say $D_1(\ell/k, \ell) = 0$.

$\mathrm{rank}_\ell(\Omega_{\ell/k}) = n < \infty$ since ℓ f.g.

$\operatorname{rank}_\ell(\Omega_{\ell/k}) = n < \infty$ since ℓ f.g.

Pick $x_1, \dots, x_n \in \ell$ s.t. $\{dx_1, \dots, dx_n\}$ is an ℓ-basis for $\Omega_{\ell/k}$.

$k \hookrightarrow k(\underline{x}) \hookrightarrow \ell$. (Not claiming $\underline{x}$ transcendental.)

JZ sequence gives:

$$\begin{tikzcd}
D_1(\ell/k, \ell) \arrow[r] & D_1(\ell/k(\underline{x}), \ell) \arrow[r] & \ell \otimes_k \Omega_{k(\underline{x})/k} \arrow[r, two heads] & \Omega_{\ell/k} \arrow[r] & \Omega_{\ell/k(\underline{x})} \arrow[r] & 0 \\
D_1(k(\underline{x})/k, k(\underline{x})) \arrow[u] & & & & & \\
0 \arrow[u] & & & & &
\end{tikzcd}$$

$\Vert$ 0, by hypothesis

$\therefore$ is zero

Choice of $\underline{x}$ $\Rightarrow$ this map is onto

$\therefore$ is zero

$$\Rightarrow \Omega_{\ell/k(\underline{x})} = 0$$

$$\xRightarrow{\text{CARTIER}} \operatorname{trdeg}(\ell/k(\underline{x})) = 0$$

and $D_1(\ell/k(\underline{x}), \ell) = 0$.

· $\operatorname{rank}_{k(x)} \Omega_{k(x)/k} = n$

$\Rightarrow \operatorname{trdeg}(k(\underline{x})/k) = n$.

$\Rightarrow \underline{x}$ transc. $/k$.

So, we may replace k by $k(\underline{x})$ and assume ℓ/k is finite (algebraic) and $D_1(\ell/k, \ell) = 0$.

Now, for any $\alpha \in \ell$, look at

$$k \hookrightarrow k(\alpha) \hookrightarrow \ell.$$

$$D_1(\ell/k, \ell) = 0 \Rightarrow D_1(k(\alpha)/k, k(\alpha)) = 0$$

$$\Rightarrow \alpha \text{ is sep}/k. \quad \boxtimes$$

Smoothness and Regularity.

Eventual punch line:

Theorem. k field. (S, η, ℓ) local k-algebra.

Then,

S/k is η-smooth $\Leftrightarrow$ S/k is geometrically regular.

$S \twoheadrightarrow R = S/I$, where $I \trianglelefteq S$ is f.g. ideal.

Fix $\underline{a} = a_1, \ldots, a_n$ gen set for I.

$K := K(\underline{a}; S)$ Koszul cx.

$$0 \to Z_1(K) \to K_1 \overset{\partial}{\twoheadrightarrow} I, \qquad K_1 \cong S^n$$

$$\operatorname{Im}(K_1 \wedge K_1 \to K_1) = \partial(K_2)$$

So, the truncated cotangent complex is

$$\mathcal{L}^{\natural}_{R/S} = \quad 0 \to H_1(K) \to R^n \to 0$$

$$R^n = K(\underline{a}; R)$$

Thus:

Lemma. For any $M \in \operatorname{Mod} R$, one has an exact sequence

Remark: for any ...

sequence

$$0 \to D_2(R/S, M) \to H_1(K) \otimes_R M \longrightarrow M^n \longrightarrow D_1(R/S, M) \longrightarrow 0 \qquad (*)$$

$$D_1(R/S, M) \cong I/I^2 \otimes_R M$$

Corollary. (S, η, k) noetherian local. $S \twoheadrightarrow R$.

TFAE:

① $S \twoheadrightarrow R$ is a c.i., i.e., $\ker(S \twoheadrightarrow R)$ can be generated by a regular sequence.

② $D_2(R/S, -) = 0$.

③ $D_2(R/S, k) = 0$.

Proof. Choose a min gen set $\underline{a}$ for ker.

Then, $\underline{a}$ reg. sequence $\iff H_1(K) = 0$

$\Rightarrow D_2(S/R, M) = 0 \quad \forall M$. [use $(*)$]

$\therefore$ ① $\Rightarrow$ ②.

② $\Rightarrow$ ③ clear.

③ $\Rightarrow$ ①:

$$0 \to H_1(K) \otimes_R k \to k^n \xrightarrow{\cong} I/I^2 \otimes_R k \to 0.$$

$\uparrow$ iso since $\underline{a}$ min'l.

But then $H_1(K) = 0$. ∎

Corollary. (S, η, ℓ) local noetherian.

S is regular $\iff D_2(\ell/S, \ell) = 0$.

Lemma. $K \to S$ map of (noe.) regular local rings.

Then, $D_2(S/K, \ell) = 0$ when ℓ is the residue field of S.

Proof. Let η maximal ideal of S, and $s_1, \dots, s_n$ min gen set for η.

$$K \to K[x_1, \dots, x_n] \to S$$
$$x_i \longmapsto s_i$$

By J 2: $D_2(S/K, \ell) \cong D_2(S/K[\underline{x}], \ell)$.

Consider

$$\begin{tikzcd} K[\underline{x}] \arrow[r, "x_i \mapsto s_i"] \arrow[d, "\varepsilon"', "\underline{x} \mapsto 0"] & S \arrow[d] \\ K \arrow[r] & S \otimes_{K[\underline{x}]} {}_{\varepsilon}K \end{tikzcd}$$

∵ since resolves K over $K[\underline{x}]$

$$\mathrm{Tor}_i^{K[x]}(S, {}_{\varepsilon}K) = H_i(S \otimes_{K[\underline{x}]} \mathrm{Kos}(\underline{x}; K[\underline{x}])).$$
$$= H_i(\mathrm{Kos}(\underline{s}; S))$$

$\underline{s}$ is regular

$$= \begin{cases} \ell & ; \quad i = 0, \\ 0 & ; \quad i \neq 0. \end{cases}$$

Thus, $D_i(S/K[\underline{x}], \ell)$
$$\cong D_i(\ell/K, \ell)$$

So, $D_2(S/K, \ell) \cong D_2(\ell/K, \ell)$.

$K \twoheadrightarrow k$. residue field

Now use JZ again, noting that $D_2(\ell/k, \ell) = 0$
and $D_2(K/k, \ell) = 0$
since K regular.

This gives us result. ∎

Def^n. k field, S → k-alg (noetherian)
Then, S is geometrically regular $/k$ if for all finite extensions $k \hookrightarrow k'$, the ring $S \otimes_k k'$ is regular.

Note. $k \hookrightarrow k'$ finite $\Rightarrow$ $S \otimes_k k'$ is noetherian.

Geometrically regular $\Rightarrow$ regular ($k = k'$)

Converse holds if char $k = 0$.

Theorem 1. Say char $k = p > 0$. (S, η, ℓ) local k-algebra.
TFAE:

1. S/k geometrically regular.
2. S/k is η-smooth. $\Leftrightarrow D_1(S/k, \ell) = 0$.
3. $(S \otimes_k k')$ regular for any finite subfield $k \hookrightarrow k' \hookrightarrow k^{1/p}$.
4. $(S \otimes_k k^{1/p})$ is regular.
(⇒ noetherian and local)

Lecture 26 (19-04-2023)

Wednesday, April 19, 2023 1:20 PM

Thm A: k field. (S, η, ℓ) local noe. k-algebra. Assume ℓ/k separable. TFAE:

1. S/k is η-smooth.
2. S/k geometrically regular.
3. S regular.

Thm B: Say char $k = p > 0$. (S, η, ℓ) local k-alg. TFAE:

1. S/k η-smooth.
2. S/k geometrically regular.
3. $S \otimes_k k'$ regular for all $k \subseteq k' \subseteq k^{1/p}$ (finite)
4. $S \otimes_k k^{1/p}$ regular local (in particular, noetherian).

S/k geometrically regular means $S \otimes_k k'$ regular for all finite field extensions.

Example. $k \to$ not perfect. Pick $a \in k \setminus k^p$.

Take $\ell = \dfrac{k[x]}{(x^p - a)}$. Then, ℓ regular but $\ell \otimes_k \ell$ is NOT.

$\ell \otimes_k \ell \cong \dfrac{\ell[t]}{(t^p - a)} \to$ local, not domain!

Proof of Thm A. $k \hookrightarrow S \twoheadrightarrow \ell$

$$0 \to D_2(\ell/S; \ell) \to D_1(S/k, \ell) \to D_1(\ell/k, \ell)$$

$0 = D_2(\ell/k, \ell)$

$D_1(\ell/k, \ell) = 0$ since sep

$$D_1(\ell/k, \ell) \quad \rightsquigarrow \quad 0 \quad \text{since sep}$$

$\therefore \cong$

$\therefore$ ① $\Leftrightarrow$ ③.

② $\Rightarrow$ ③ OK.

① $\Rightarrow$ ② Take any finite ext^n $k \hookrightarrow k'$.

$$\begin{tikzcd} k \arrow[r, hook] \arrow[d, "\text{finite}"'] & S \arrow[r] \arrow[d, ", \therefore \text{finite}"] & \ell \arrow[dd, bend left] \\ k' \arrow[r] & (S \otimes_k k') = T \arrow[d] & \\ & T_{\eta'} \arrow[r, two heads] & \ell' \end{tikzcd}$$

ℓ' residue field

Fix $\eta' \in \mathrm{Max}(S \otimes_k k')$

To prove: $T_{\eta'}$ is regular.

$$D_1(S/k, \ell')$$

flat base change

$$\cong D_1(S \otimes_k k'/k', \ell') = 0$$

Because $D_1(T_{\eta'}/T, -) = 0$

$$\cong D_1(T_{\eta'}/k', \ell') = 0$$

$$\Downarrow$$

(① $\Rightarrow$ ② didn't use separability.)

$T_{\eta'}$ is η'-smooth $/k'$

$\Downarrow$ ① $\Rightarrow$ ③

$T_{\eta'}$ regular. ∎

Proof of Thm B. ① $\Rightarrow$ ② $\Rightarrow$ ③ OK.

③ $\Rightarrow$ ④: $k^{1/p} = \mathrm{colim}_i\, k_i$, where $k \underset{\text{finite}}{\subseteq} k_i \subseteq k^{1/p}$.

<u>Claim</u>. $(S\otimes_k k_i)$ local (and noe., $\because$ finite ext^n of S).

<u>Lemma</u>. For any subfield $k \subseteq k' \subseteq k^{1/p}$,
S local $\Rightarrow (S\otimes_k k')$ local.

↳ Follows from Lemma. $\ell_i :=$ residue field of $S\otimes_k k_i$.

$$S\otimes_k k_i \twoheadrightarrow \ell_i$$

Take colimits:

$$\underset{\substack{i \\ \| \\ S\otimes_k k^{1/p}}}{\operatorname{colim}(S\otimes_k k_i)} \to \operatorname{colim}_i(\ell_i) =: \ell'.$$

Regularity $\Rightarrow D_2(\ell_i/S\otimes_k k_i, \ell_i) = 0$

$\Rightarrow D_2(\ell'/(S\otimes_k k^{1/p}), \ell') = 0$

Remains to prove: $S\otimes_k k^{1/p}$ is noetherian!

$$\begin{tikzcd}
k \arrow[r] \arrow[d, hook] & S \arrow[d] & \\
k_i \arrow[r] \arrow[d, hook] & (S\otimes_k k_i) \arrow[r] & S^{1/p} \cong S \\
k_i^{1/p} & &
\end{tikzcd}$$

<u>Claim:</u> This is flat.

$$\text{depth } S = \text{depth}(S^{1/p}) = \overset{\text{Aus. Buchs.}}{\text{depth}_S(S^{1/p})} = \text{depth}(\overset{\substack{\dim S \\ \|}}{S\otimes_k k_i}) - \sup \operatorname{Tor}_*(\ell_i, S^{1/p}).$$

<u>Lemma</u>. $(A, m_A, k_A) \to (B, m_B, k_B)$ map of

Lemma. $(A, m_A, k_A) \to (A, m_B, k_B)$ map of noetherian local rings. $N \in \text{mod } B$.
Then,
$$\operatorname{fd}_A N = \sup \operatorname{Tor}_*^A (k_A, N).$$

Using lemma, we get the claimed flatness.

Thus, colim gives
$$S \otimes_k k^{1/p} \longrightarrow S \quad \text{is flat.}$$
It is also faithful ($\because$ local homomorphism).
But now since S was Noe., so is $S \otimes_k k^{1/p}$.

Lemma. $(A, m_A, k_A) \longrightarrow (B, m_B, k_B)$ hom of local rings.
If map is flat & B noetherian, then A is noetherian.

$$\begin{tikzcd}
k \arrow[r, hook] \arrow[d, hook] & S' \arrow[d] \arrow[dr] & \\
k^{1/p} \arrow[r] & S \otimes_k k^{1/p} \arrow[r, "\text{``Relative Frobenius''}"'] & S^{1/p}
\end{tikzcd}$$

ASIDE.

$$\begin{tikzcd} \varphi : R \arrow[r] \arrow[d, "F"'] & S \arrow[d] & \\ R \arrow[r] & (S \otimes_R F_* R) \arrow[r] & S \end{tikzcd}$$

φ local map.
R, S noetherian

$F^{R/S}$ Relative Frob.

Radu & André: $R \to S$ geo. regular
$\Leftrightarrow F^{R/S}$ flat.

④ $\Rightarrow$ ①:

$$\begin{tikzcd} k \arrow[r, hook] \arrow[d, hook] & S \arrow[d, hook] & \\ k^{1/p} \arrow[r] & \underbrace{S \otimes_k k^{1/p}}_{T} \arrow[r, two heads] & \ell' \end{tikzcd}$$

residue field

T regular $\Leftrightarrow D_2(\ell'/T, \ell') = 0$.

WANT to deduce: $D_1(S/k, \ell') = 0$.

$$\cong D_1(T/k^{1/p}, \ell')$$

Lemma. $k \hookrightarrow k^{1/p} \hookrightarrow A$

Then, $D_1(A/k, \_) \to D_1(A/k^{1/p}, \_)$

is the zero map.

Lemma.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "F"'] & R \arrow[d, "F"] \\ K \arrow[r] & R \end{tikzcd}$$

M an R-mod.

Then, $D_i(R/K, F_* M) \xrightarrow{0} D_i(R/K, M)$.

Using this, finish proof of Thm B...

Def^n. $\varphi : (R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, \ell)$ map of local rings.

φ is geometrically regular at $\mathfrak{n}$ if

φ flat + k-algebra $S/\mathfrak{m}S$ is geometrically regular.

Theorem (Grothendieck) φ as above. TFAE:

① φ is geometrically regular at $\mathfrak{n}$.

② S/R is $\mathfrak{n}$-smooth (i.e., $D_1(S/R, \ell) = 0$).

Proof. ① $\Rightarrow$ ② follows Thm B and base change.

$$\begin{array}{ccccc} R & \longrightarrow & S & & \\ \downarrow & & \downarrow & & \\ k & \longrightarrow & S/\mathfrak{m}S & \longrightarrow & \ell \end{array}$$

$\operatorname{Tor}_i^R(S, k) = 0 \quad \forall i \geq 1.$

$$D_1(S/R, \ell) \cong D_1\left(\tfrac{S/\mathfrak{m}S}{R}, \ell\right) \underset{\text{Thm B}}{\cong} 0.$$

② $\Rightarrow$ ① Key point is to prove map is flat.

For then, we reduce to Thm B again.

Proof of Cohen's Structure Theorem only uses ① $\Rightarrow$ ②.

$(S,\eta,\ell) \to$ complete noetherian local ring.

CST. S is a quotient of a regular local ring.

Proof. $p = \operatorname{char} \ell \overset{>0}{.}$ We have

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & S \\ \mathbb{Z}_{(p)} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & (S,\eta,\ell) \arrow[d] \\ \mathbb{F}_p \arrow[r, hook] & \ell \end{tikzcd} \qquad (*)$$

For any noetherian local ring (R, m, k),
$\exists$ noe. local ring (R', m', ℓ) s.t. the following:

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[d] & (R', m', \ell) \arrow[d] \\ k \arrow[r, hook, "\text{field ext}^n"'] & \ell \end{tikzcd}$$

flat & $mR' = m'$
$\Downarrow$
$\dim R = \dim R'$
R regular $\Leftrightarrow R'$ regular.

"Gonflement" (Inflation)

Inflate $(*)$: geometrically regular $\Rightarrow m_c$ smooth $\Rightarrow$ lift.

$\mathbb{Z}_{(p)} \to (C, m_c, \ell) \qquad (S, \eta, \ell)$

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r, hook] \arrow[d] & (C, m_c, \ell) \arrow[d] & (S, \eta, \ell) \arrow[d] \\
\mathbb{F}_p \arrow[r, hook] & \ell \arrow[r, equal] & \ell
\end{tikzcd}$$

Now, we can lift $(C, m_c, \ell) \to (S, \eta, \ell)$.

Can make it surjective...

Lecture 27 (24-04-2023)

Monday, April 24, 2023 1:24 PM

Thm. k field of char $= p > 0$.

(S, m_S, k_S) local noetherian k-algebra.

TFAE:

① S/k is m_S-smooth.

② S/k geometrically regular.

③ $S \otimes_k k'$ is regular for all $k \subseteq k' \subseteq k^{1/p}$, $k \subseteq k'$ finite.

④ $S \otimes k^{1/p}$ is regular (in particular, noetherian).

Proof of ④ ⇒ ①: $T := S \otimes_k k^{1/p}$. (It is local.)

$$\begin{tikzcd} k \arrow[r, hook] \arrow[d, hook] & S \arrow[d] & \\ k^{1/p} \arrow[r, hook] & T \arrow[r] & k_T \end{tikzcd}$$

T regular $\equiv$ $D_2(k_T/T, k_T) = 0$.

WTS: $D_1(S/k, k_T) = 0$

JZ:

$$\begin{tikzcd} 0 \arrow[r] & D_2(k_T/T, k_T) \arrow[r] \arrow[d, equal] & D_1(T/k^{1/p}, k_T) \arrow[r, "①"] \arrow[dr, phantom, "\circlearrowleft"] & D_1(k_T/k^{1/p}, k_T) \\ & 0 & D_1(S/k, k_T) \arrow[r] \arrow[u, "\cong"] & D_1(k_T/k, k_T) \arrow[u, "0"] \end{tikzcd}$$

$\therefore$ ① is zero. $\therefore$ $D_1(T/k^{1/p}, k_T) = 0$. ⊛

Corollary. (MacLane's Criteria) k char $p>0$.

$k \hookrightarrow l$ field ext^n.

l/k separable $\iff$ $l \otimes_k k^{1/p}$ is a field.

Theorem. (Grothendieck)

$R \to S$ local map of noetherian local rings.

TFAE:

① S/R is m_S-smooth. $\iff D_1(S/R, k_S) = 0$.

② S/R is flat & the k_R-algebra $S/m_R S$ is geometrically regular.

Proof. ② $\Rightarrow$ ① ✓

① $\Rightarrow$ ② Crucial point is to prove

$D_1(S/R, k_S) \Longrightarrow S/R$ flat.

NTP: $\mathrm{Tor}_1^R(S, k_R) = 0$.

[$R \to S$ local noetherian, $S \in \mathrm{mod}\, S$;

N flat $/R$ $\iff$ $\mathrm{Tor}_1^R(N, k_R) = 0$]

$\to$ Some spectral sequence ... relating AQ to Tor.

Aliter.

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d] \\ k_R \arrow[r] & S \otimes_R^L k_R \end{tikzcd}$$

derived pushout

$H_0(A) =: B \simeq S/m_R S$

$S \otimes_R^L k_R = A \rightsquigarrow$ simplicial model

$D_1(S/R, k_S) = 0 \iff D_1(A/k_R, k_S) = 0$

Because $H_0(A) \to B$ is an iso.

$$D_i(B/A, k_S) = 0 \qquad i \le 1$$

$$D_2(B/A, k_S) \cong H_1(A) \otimes_B k_S.$$

$$\begin{array}{ccccccccc}
D_2(B/k_R, k_S) & \to & D_2(B/A, k_S) & \to & D_1(A/k_R, k_S) & \to & D_1(B/k_R, k_S) & \to & 0 \\
\| & \Rightarrow & \| & & \| & \Longrightarrow & \| & & \| \\
0 & & 0 & & 0 & & 0 & & D_1(B/A, k)
\end{array}$$

$$\therefore\ H_1(A) \otimes_B k_S = 0$$

$$\| $$

$$\mathrm{Tor}_1^R(S, k_B) \otimes_B k_S \qquad \Rightarrow \quad \mathrm{Tor}_1^R(S, k_B) = 0. \qquad \blacksquare$$

Cohen's Structure Theorem

$A \to$ local noetherian.

An A-algebra C is a **Cohen A-algebra** if

① C is noetherian, local, complete,

② $A \to C$ flat and local, ($\Rightarrow A \hookrightarrow C$, check!)

③ $m_C = m_A C$.

$$\boxed{\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \\
k_A \arrow[r] & \dfrac{C}{m_A C} = k_C
\end{tikzcd}}$$

Thus, C/A is m_C-smooth

$\iff$ k_C / k_A is separable.

Theorem (Bourbaki). Fix A local noe.

Theorem (Bourbaki). Fix A local noe.

Given any extension of fields $k_A \hookrightarrow \ell$, $\exists$ flat noe. local A-algebra that realises this extension:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & C \arrow[d] \\ k_A \arrow[r, hook] & \ell = C/m_C \end{tikzcd}.$$

[Moreover, this is unique if $k_A \hookrightarrow \ell$ is separable.]

Proof. Existence: Check Bourbaki. (Do for finite, then transfinite ind...) (Then check limit is noe + local.)

Uniqueness ℓ/k_A separable.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d] \\ \ell \arrow[r, equal] & \ell \end{tikzcd}$$

lifts since complete + smooth (separable)

$D_1(C/A, \ell) = 0$ forces lifts to be unique.

But now do usual business... ☒

Let S be a complete local ring. $p = \operatorname{char} k_S \geq 0$.

Then, we have

$$\mathbb{Z} \longrightarrow \mathbb{Z}_{(p)} \longrightarrow S.$$

Say C is a Cohen algebra $/\mathbb{Z}_{(p)}$.

(a) $p = 0 \Rightarrow \mathbb{Z}_{(p)} = \mathbb{Q}$, so C is field.
$(\because \mathfrak{m}_C = \mathfrak{m}_{\mathbb{Q}} C = 0.)$

(b) $p \neq 0 \Rightarrow C$ must be a (complete) DVR s.t. $pC = \mathfrak{m}_C$.

Theorem. S complete local ring.

① There exists a complete DVR C with $C/\mathfrak{m}_C \cong k_S$. and a surjective local map

$$C[\![y_1, \dots, y_n]\!] \twoheadrightarrow S.$$

Moreover, if S is equicharacteristic, (≡ contains a field) one can take $C = k_S$.

Proof.

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & S \arrow[d] \\ k = \mathbb{F}_p \text{ or } \mathbb{Q} \arrow[r, hook] & k_S \end{tikzcd}$$

separable

Choose a Cohen algebra that realizes $k \hookrightarrow k_S$.

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r, hook, "\mathfrak{m}_C"] \arrow[d] & C \arrow[r, "\text{smooth}"] \arrow[d] & S \arrow[d] \\ k \arrow[r, hook] & k_S \arrow[r, equal] & k_S \end{tikzcd}$$

$\because$ sep

Suppose $\mathfrak{m}_S = (s_1, \dots, s_n)$. We can define

$$\begin{tikzcd} & C[\![y_1, \dots, y_n]\!] \arrow[d] \\ C \arrow[ur] \arrow[r] & S \end{tikzcd} \qquad \begin{matrix} y_i \\ \downarrow \\ s_i \end{matrix}$$

Claim: $C[\![y]\!] \to S$ is surjective.

Can check on level of associated graded modules.
(o [illegible])

Can check on level of associated graded modules. (Bourbaki)

But then it is clear.

When S is equichar, then $k_S \hookrightarrow S$.

Can take $C = k_S$ then. $\blacksquare$

② When S is a domain, one can find

$$C[\![y_1, \ldots, y_m]\!] \hookrightarrow S$$

s.t. ext^n is module-finite.

<u>Extensions</u> :

③ (Gabber) In the context of ②, one can ensure

$$\text{Frac}(C[\![y]\!]) \hookrightarrow \text{Frac}(S)$$

is also separable.

④ (Avramov, Foxby, B. Herzog)

Given $R \longrightarrow S$ local map of noe. local rings.

If S is complete, $\exists$

flat + regular fibre

$$\begin{tikzcd} & R' \arrow[d] \\ R \arrow[ur] \arrow[r] & S \end{tikzcd}$$

($R'/\mathfrak{m}_R R'$ is reg.)

"Cohen factorisation" of maps

(Not so hard to prove this...)