Model Categories

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Table of Contents

Table of Contents

Notations

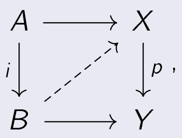
1. **C** will denote a category.
2. f, g will denote morphisms in a category.
3. Given a ring R, $\mathbf{Ch}(R)$ will denote the category of nonnegatively graded chain complexes over R, i.e., objects are chain complexes of the form

$$\cdots \to M_2 \to M_1 \to M_0,$$

and the morphisms are the obvious ones.

Definition 1 (Lift)

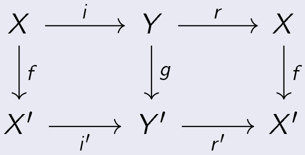
Given a commutative diagram of the form



a lift is a map $B \to X$ such that the resulting diagram commutes.

Definition 2 (Retract)

f is said to be a retract of g if there is a commutative diagram



such that ri and $r'i'$ are the appropriate identity maps.

Definition 3

A model category is a category $\mathbf{C}$ with three distinguished classes of maps:

1. weak equivalences ($\xrightarrow{\sim}$),
2. fibrations ($\twoheadrightarrow$), and
3. cofibrations ($\hookrightarrow$),

each of which is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called an acyclic fibration (resp. acyclic cofibration).

Additionally, we require the model category axioms **MC1** - **MC5** to be satisfied, which are stated on the next slide.

Model Category Axioms

MC1 Finite limits and colimits exist in **C**.

MC2 Let f and g be maps such that gf is defined. If two of of the three maps f, g, gf are weak equivalences, then so is the third.

MC3 If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.

MC4 Given a commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

, a lift exists in either of the following two situations: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

MC5 Any map f can be factored in two ways $f = pi = qj$, where i is a cofibration, p is an acyclic fibration, j is an acyclic fibration, and q is a fibration.

Fibrant and Cofibrant objects

By **MC1**, a model category **C** has both an initial object $\varnothing$ and a final object $*$. More generally, all finite products and coproducts exist. Similarly, pushouts and pullbacks exist as well.

Definition 4

An object $A \in \mathbf{C}$ is said to be cofibrant if $\varnothing \to A$ is a cofibration, and fibrant if $A \to *$ is a fibration.

An example

The category $\mathbf{Ch}(R)$ can be given the structure of a model category by defining a map $f : M \to N$ to be

1. a weak equivalence if f induces an isomorphism on homology groups,
2. a cofibration if for each $k \geqslant 0$, the map $f_k : M_k \to N_k$ is a monomorphism with a *projective* R-module as its cokernel,
3. a fibration if for each $k \geqslant 1$, the map $f_k : M_k \to N_k$ is an epimorphism.

Note that $\varnothing$ and $*$ are both the zero chain complex. The cofibrant objects in $\mathbf{Ch}(R)$ are the chain complexes M such that each M_k is projective. On the other hand, every object is fibrant.

The homotopy category $\mathrm{Ho}(\mathbf{Ch}(R))$ is equivalent to the category whose objects are these cofibrant chain complexes and whose morphisms are ordinary chain homotopy classes of maps.

Another example

The category **Top** of topological spaces can be given the structure of a model category by defining a map $f : X \to Y$ to be

1. a weak equivalence if f is a homotopy equivalence,
2. a cofibration if f is a closed Hurewicz cofibration,
3. a fibration if f is a Hurewicz fibration.

In this case, the homotopy category $\text{Ho}(\textbf{Top})$ is the usual homotopy category of topological spaces.

Yet another example

The category $\mathbf{Top}'$ of topological spaces can be given yet another structure of a model category by defining a map $f : X \to Y$ to be

1. a weak equivalence if f is a weak homotopy equivalence,
2. a cofibration if f is a retract of a map $X \to Y'$ in which Y' is obtained from X by attaching cells,
3. a fibration if f is a Serre fibration.

In this case, the homotopy category $\mathrm{Ho}(\mathbf{Top}')$ is the usual homotopy category of CW-complexes.

Some constructions

Given a model category $\mathbf{C}$, we may construct some new model categories.

Example

The opposite category $\mathbf{C}^{\mathrm{op}}$ is quite naturally a model category by keeping the weak equivalences the same and switching fibrations with cofibrations.

Example

If A is an object of $\mathbf{C}$, $A \downarrow \mathbf{C}$ is the category in which an object is a map $f : A \to X$ in $\mathbf{C}$. A morphism in this category from $f : A \to X$ to $g : A \to Y$ is a map $h : X \to Y$ such that $hf = g$. (For example, $* \downarrow \mathbf{Top}$ is the category of pointed spaces.)
This has the structure of a model category by defining h to be a weak equivalence, fibration, or cofibration according to whether it was so in $\mathbf{C}$. An object X of $* \downarrow \mathbf{Top}$ is cofibrant iff the basepoint of X is closed and nondegenerate.

Definition 5

Let $i : A \to B$ and $p : X \to Y$ be maps such that

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

has a lift for any choice of horizontal arrows (that make the diagram commute). Then, i is said to have the left lifting property (LLP) with respect to p, and p is said to have the right lifting property (RLP) with respect to i.

Proposition 6

Let $\mathbf{C}$ be a model category.

1. The cofibrations in $\mathbf{C}$ are precisely the maps which have the LLP with respect to acyclic fibrations.
2. The acyclic cofibrations in $\mathbf{C}$ are precisely the maps which have the LLP with respect to fibrations.
3. The fibrations in $\mathbf{C}$ are precisely the maps which have the RLP with respect to acyclic cofibrations.
4. The acyclic fibrations in $\mathbf{C}$ are precisely the maps which have the RLP with respect to cofibrations.

This shows that the axioms for model category are overdetermined in some sense: more precisely, if $\mathbf{C}$ is a model category, then given just the classes of weak equivalences and fibrations is enough to determine the class of cofibrations.

Cobase change

Given a pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & C \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ A & \xrightarrow[i']{} & P \end{array},$$

the map i' is the cobase change of i (along j). Similarly, one may define base change (using pullback diagrams).

Proposition 7

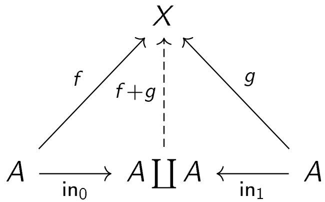
Let **C** be a model category.

1. The classes of cofibrations and acyclic cofibrations are closed under cobase change.
2. The classes of fibrations and acyclic fibrations are closed under base change.

Table of Contents

Overview

In this section, $\mathbf{C}$ is some fixed model category, and A and X are some objects of $\mathbf{C}$. We wish to exploit the model category axioms to construct a notion of homotopy relations on the set $\mathrm{Hom}_{\mathbf{C}}(A, X)$. We define *left homotopy* and *right homotopy* and see that these two coincide in the most important case – namely if A is cofibrant and X is fibrant.



In particular, taking $X = A$ and $f = g = \mathrm{id}_A$ gives us the folding map

$$A \coprod A \xrightarrow{\mathrm{id}_A + \mathrm{id}_A} A.$$

Conversely, given a map $f : A \coprod A \to X$, we get two structure maps $f_0 = f \circ \mathrm{in}_0$ and $f_1 = f \circ \mathrm{in}_1$. Note that $f_0 + f_1 = f$.

Cylinder objects

Definition 8

A cylinder object for A is an object $A \wedge I$ of $\mathbf{C}$ together with a diagram

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the folding map $\mathrm{id}_A + \mathrm{id}_A : A \coprod A \to A$. A cylinder object $A \wedge I$ is called

1. a good cylinder object, if $A \coprod A \to A \wedge I$ is a cofibration, and
2. a very good cylinder object, if in addition the map $A \wedge I \to A$ is a (necessarily acyclic) fibration.

If $A \wedge I$ is a cylinder object for A, we will denote the two structure maps $A \to A \coprod A \to A \wedge I$ by i_0 and i_1. Note that the composition $A \xrightarrow{i_0} A \wedge I \xrightarrow{\sim} A$ is id_A. Thus, i_0 and i_1 are weak equivalences, by **MC2**.

By **MC5**, at least one very good cylinder object exists for every A.

Proposition 9

If A is cofibrant and $A \wedge I$ is a good cylinder object for A, then the maps $i_0, i_1 : A \to A \wedge I$ are acyclic cofibrations.

Proof.

Suffices to prove that i_0 is a cofibration.
$A \coprod A$ is defined by the pushout diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & A \\ {\scriptstyle \text{cofibration}}\downarrow & & \downarrow{\scriptstyle \text{in}_0} \\ A & \underset{\text{in}_1}{\longrightarrow} & A \coprod A \end{array} .$$

Thus, in_0 is a cofibration, being a cobase change of one.
As i_0 is the composition, $A \xrightarrow{\text{in}_0} A \coprod A \hookrightarrow A \wedge I$, we are done. □

Left homotopy

Definition 10

Two maps $f, g : A \to X$ in $\mathbf{C}$ are said to be left homotopic (written $f \overset{\ell}{\sim} g$) if there exists a cylinder object $A \wedge I$ for A such that the sum map $f + g : A \coprod A \to X$ extends to a map $H : A \wedge I \to X$. Such a map H is said to be a left homotopy from f to g (via the cylinder object $A \wedge I$).
The left homotopy is said to be good or very good if $A \wedge I$ is so.

Note that in the above case, we have $f = Hi_0$ and $g = Hi_1$.

Remark 11

Suppose $f \overset{\ell}{\sim} g$. If f is a weak equivalence, then $f = Hi_0$ shows that H is weak equivalence. In turn, $g = Hi_1$ is a weak equivalence.
Thus, f is a weak equivalence iff g is so.

Left homotopy

Proposition 12

If $f \overset{\ell}{\sim} g : A \to X$, then there exists a good left homotopy from f to g. If X is fibrant, then there exists a very good left homotopy from f to g.

The first statement follows simply by factoring $A \coprod A \to A \wedge I$ as a product of a cofibration and an acyclic fibration.

Proposition 13

If A is cofibrant, then $\overset{\ell}{\sim}$ is an equivalence relation on $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

The proof of this is not tough and just works out by considering pushouts cleverly. Reflexivity and symmetry follow even if A is not cofibrant. A being cofibrant ensures that the maps $A \to A \wedge I$ are acyclic cofibrations, which lets one prove transitivity by consider pushouts of two mapping cylinders.

Left homotopy

Definition 14

$\pi^\ell(A, X)$ denotes the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A, X)$ under the equivalence relation generated by left homotopy.

Note that we define the above even if A is not cofibrant.

Proposition 15

If A is cofibrant and $p : Y \overset{\sim}{\twoheadrightarrow} X$ is an acyclic fibration, then

$$p_* : \pi^\ell(A, Y) \to \pi^\ell(A, X), \quad [f] \mapsto [pf]$$

is a (well-defined) bijection.

Left homotopy

Proposition 16

Suppose $h : A' \to A$ is map, and $f \overset{\ell}{\sim} g : A \to X$ with X fibrant. Then, $fh \overset{\ell}{\sim} gh : A' \to X$.

Proposition 17

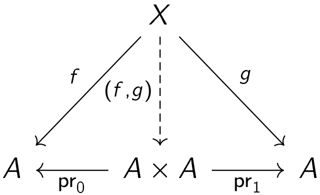
If X is fibrant, then the composition in $\mathbf{C}$ induces a map:

$$\pi^{\ell}(A', A) \times \pi^{\ell}(A, X) \to \pi^{\ell}(A', X)$$
$$([h], [f]) \mapsto [fh].$$

Path objects

We may now consider the dual notions. Since $\mathbf{C}^{\mathrm{op}}$ is a model category as described earlier, all the propositions go through by duality.

We have the notation for a product as:



A path object for X is an object X^I together with a diagram

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

that factors $(\mathrm{id}_X, \mathrm{id}_X) : X \to X \times X$. The path object is said to be good if p is a fibration and very good if $X \to X^I$ is additionally a (necessarily acyclic) cofibration.

Right homotopy

Definition 18

Two maps $f, g : A \to X$ in $\mathbf{C}$ are said to be right homotopic (written $f \overset{r}{\sim} g$) if there exists a path object X^I for X such that the product map $(f, g) : A \to X \times X$ lifts to a map $H : A \to X^I$. Such a map H is said to be a right homotopy from f to g (via the path object X^I). The right homotopy is said to be good or very good if X^I is so.

Definition 19

$\pi^r(A, X)$ denotes the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A, X)$ under the equivalence relation generated by right homotopy.

Right homotopy

Proposition 20

If X is fibrant, then $\overset{r}{\sim}$ is an equivalence relation on $\mathrm{Hom}_{\mathbf{C}}(A, X)$. If $i : A \overset{\sim}{\hookrightarrow} B$ is an acyclic cofibration, then composition with i induces a bijection $i^* : \pi^r(B, X) \to \pi^r(A, X)$.

Proposition 21

If A is cofibrant, then composition in $\mathbf{C}$ induces a map $\pi^r(A, X) \times \pi^r(X, Y) \to \pi^r(A, Y)$.

Relation between left and right homotopy

Proposition 22

Let $f, g : A \to X$ be maps.

1. If A is cofibrant and $f \overset{\ell}{\sim} g$, then $f \overset{r}{\sim} g$.
2. If X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{\ell}{\sim} g$.

Proof.

Fix a good left homotopy $H : A \wedge I \to X$. Let j be the map $A \wedge I \to A$. Choose a good path object $X \xrightarrow{q} X^I \overset{p}{\twoheadrightarrow} X \times X$. By **MC4**, we may find a lift $K : A \wedge I \to X^I$ in

$$\begin{array}{ccc} A & \xrightarrow{qf} & X^I \\ {\scriptstyle i_0}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle p} \\ A \wedge I & \xrightarrow[(fj,H)]{} & X \times X \end{array} .$$

The composite $Ki_1 : A \to X^I$ is the desired right homotopy. □

Homotopic maps

Definition 23

If A is cofibrant and X is fibrant, we will denote the identical right homotopy and left homotopy equivalence relations on $\mathrm{Hom}_{\mathbf{C}}(A, X)$ by $\sim$, and say that two maps related by this relation are homotopic. The set of equivalence classes is denoted $\pi(A, X)$.

Theorem 24

Suppose that $f : A \to X$ is a map in $\mathbf{C}$ between objects A and X which are both fibrant and cofibrant. Then, f is a weak equivalence if and only if f has a homotopy inverse, i.e., if and only if there exists a map $g : X \to A$ such that gf and fg are homotopic to the respective identity maps.

Weak equivalence $\Rightarrow$ homotopy inverse

Proof.

Suppose f is a weak equivalence and factor f as $A \overset{q}{\underset{\sim}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$. Thus, p is also a weak equivalence. Using **MC4** on the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle q}\big\downarrow{\scriptstyle \sim} & \overset{r}{\nearrow} & \big\downarrow \\ C & \longrightarrow & * \end{array}$$

gives us a left inverse $r : C \to A$ of q.
Recall the bijection $q^* : \pi^r(C, C) \to \pi^r(A, C)$. We have

$$q^*([qr]) = [qrq] = [q].$$

Thus, $[qr] = [\mathrm{id}_C]$. Thus, r is a homotopy inverse for q. Dually, there is a homotopy inverse s for p. Then, rs is a homotopy inverse for $f = pq$.

Homotopy inverse $\Rightarrow$ weak equivalence

Proof.

Suppose f has a homotopy inverse and factor f as $A \overset{q}{\underset{\sim}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$. By **MC2**, it suffices to prove that p is a weak equivalence. Note that C is fibrant and cofibrant (since A and X are so). Fix $g : X \to A$ and a homotopy $H : X \wedge I \to X$ between fg and id_X. Use **MC4** to get a lift $H' : X \wedge I \to C$ in the diagram

$$\begin{array}{ccc} X & \xrightarrow{qg} & C \\ {\scriptstyle i_0}\downarrow & \overset{H'}{\nearrow} & \downarrow{\scriptstyle p} \\ X \wedge I & \xrightarrow[H]{} & X \end{array} .$$

Let $s := H'i_1$, so that $ps = pH'i_1 = Hi_1 = \mathrm{id}_X$. By the previous part, q has a homotopy inverse, say r. We have $fr = pqr \sim p$. By the homotopy H' we also have $s \sim qg$ and thus,

$$sp \sim qgp \sim qgfr \sim qr \sim \mathrm{id}_C .$$

Thus, sp is a weak equivalence.

Homotopy inverse $\Rightarrow$ weak equivalence

Proof.

We have shown that sp is a weak equivalence. Now, looking at the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\mathrm{id}_C} & C & \xrightarrow{\mathrm{id}_C} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle sp} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[s]{} & C & \xrightarrow[p]{} & X \end{array}$$

shows that p is a retract of sp, and hence by **MC3** that the map p is a weak equivalence. □

Table of Contents

Overview and Definitions

We use the machinery from the previous section to construct the *homotopy category* $\mathrm{Ho}(\mathbf{C})$. We define the following six categories associated to $\mathbf{C}$:

1. $\mathbf{C}_c$ - full subcategory of $\mathbf{C}$ whose objects are cofibrant objects,
2. $\pi\mathbf{C}_c$ - subcategory of $\mathbf{C}_c$ whose morphisms are right homotopy classes,
3. $\mathbf{C}_f$ - full subcategory of $\mathbf{C}$ whose objects are fibrant objects,
4. $\pi\mathbf{C}_f$ - subcategory of $\mathbf{C}_f$ whose morphisms are left homotopy classes,
5. $\mathbf{C}_{cf}$ - full subcategory of $\mathbf{C}$ whose objects are fibrant and cofibrant,
6. $\pi\mathbf{C}_{cf}$ - subcategory of $\mathbf{C}_{cf}$ whose morphisms are homotopy classes.

Q, R

For each object X in $\mathbf{C}$, we can apply **MC5** to $\varnothing \to X$ and $X \to *$ to obtain

$$p_X : QX \overset{\sim}{\twoheadrightarrow} X \quad \text{and} \quad i_X : X \overset{\sim}{\hookrightarrow} RX$$

with QX cofibrant and RX fibrant. If X it itself cofibrant (resp. fibrant), then we let $QX = X$ (resp. $RX = X$).

Q, R

Proposition 25

Given a map $X \xrightarrow{f} Y$ in $\mathbf{C}$, there exists a map $QX \xrightarrow{\widetilde{f}} QY$ such that

$$\begin{array}{ccc} QX & \xrightarrow{\widetilde{f}} & QY \\ {\scriptstyle p_X}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes. $\widetilde{f}$ is unique up to left (and right) homotopy, and is a weak equivalence if and only if f is. If Y is fibrant, then $\widetilde{f}$ depends up to left (and right) homotopy only on the left homotopy class of f.

Thus, if $f = \mathrm{id}_X$, then $\widetilde{f} \overset{r}{\sim} \mathrm{id}_{QX}$. Similarly, if $f : X \to Y$ and $g : Y \to Z$ and $h = gf$, then $\widetilde{h} \overset{r}{\sim} \widetilde{g}\widetilde{f}$. Thus, we can define a functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ sending $X \mapsto QX$ and $X \xrightarrow{f} Y$ to $[\widetilde{f}] \in \pi^r(QX, QY)$.

Q, R

Dually, we have the existence of a map $\overline{f}$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle p_Y} \\ RX & \xrightarrow[\overline{f}]{} & RY \end{array}$$

commutes and we have the desired uniqueness properties.

As before, this gives us a functor $R : \mathbf{C} \to \pi\mathbf{C}_f$ sending $X \mapsto RX$ and $f \mapsto [\overline{f}] \in \pi^{\ell}(RX, RY)$.

Proposition 26

Restriction of Q to $\pi\mathbf{C}_f$ induces a functor $Q' : \pi\mathbf{C}_f \to \pi\mathbf{C}_{cf}$.
Restriction of R to $\pi\mathbf{C}_c$ induces a functor $R' : \pi\mathbf{C}_c \to \pi\mathbf{C}_{cf}$.

Homotopy category

Definition 27

The homotopy category $\mathrm{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ is the category with the same objects as $\mathbf{C}$, and with

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y) := \mathrm{Hom}_{\pi\mathbf{C}_{cf}}(R'QX, R'QY) = \pi(RQX, RQY).$$

There is a functor $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ which is the identity on objects and sends a map $X \xrightarrow{f} Y$ to the map $R'Q(X) \xrightarrow{R'Q(f)} R'Q(Y)$.

If each of X and Y is both fibrant and cofibrant, then by construction, the map $\gamma : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$ is surjective and induces a bijection $\pi(X, Y) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$.

Homotopy category

Proposition 28

Given a morphism f in $\mathbf{C}$, $\gamma(f)$ is an isomorphism iff f is a weak equivalence. Any morphism in $\mathrm{Ho}(\mathbf{C})$ can be written as a composition of maps of the form $\gamma(f)$ and $\gamma(g)^{-1}$.

Definition 29

Let $\mathbf{C}$ be a category, and $W \subseteq \mathbf{C}$ a class of morphisms. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be a localisation of $\mathbf{C}$ with respect to W if

1. $F(f)$ is an isomorphism for each $f \in W$,
2. whenever $G : \mathbf{C} \to \mathbf{D}'$ is a functor carrying elements of W into isomorphisms, there exists a unique functor $G' : \mathbf{D} \to \mathbf{D}'$ such that $G'F = G$.

Homotopy category

Proposition 30

Let $\mathbf{C}$ be a model category and $W \subseteq \mathbf{C}$ the class of weak equivalences. Then, $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ is a localisation of $\mathbf{C}$ with respect to W.

Chain complexes

As stated earlier, $\mathbf{Ch}(R)$ can be given the structure of a model category.

If A is an R-module and $n \geqslant 0$, we let $K(A, n)$ denote the chain complex which is A in degree n and 0 elsewhere.
These are the chain complex analogues of Eilenberg-Mac Lane spaces.

Proposition 31

For any two R-modules A and B and nonnegative integers n and m, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}(R))}(K(A, m), K(B, n)) \cong \mathrm{Ext}_R^{n-m}(A, B).$$