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MA 839

Advanced Commutative Algebra

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Lecture 0 (07-01-2021)

07 January 2021 08:08

A Quick Intro.

Setup : A ring is commutative with 1.

Let M be an R-module.

Observation : ① If M is cyclic, (say $M = \langle x \rangle = \{ax : a \in R\}$),

we get an R-linear map $R \twoheadrightarrow M$ which is onto.

$a \mapsto ax$

Then, $M \cong R/I$ where I is the kernel.

In this case, $I = \mathrm{ann}_R(x)$.

Thus, if M is cyclic, then M is a quotient of R.

② Suppose $\exists x, y \in M$ s.t. $M = \langle x, y \rangle = \{ax + by \mid a, b \in R\}$.

$= \{ax + by \mid (a, b) \in R^{\oplus 2}\}$

Then, we get an onto R-linear map $\underbrace{Re_1 \oplus Re_2}_{} \xrightarrow{\varphi} M$

$e_1 \mapsto x$

$e_2 \mapsto y$

} extend this

$\{e_1, e_2\}$ is a basis

this lets us extend the map

In particular, $M \cong R^2/\ker \varphi$.

Q. Is it necessary that we can actually write

$$M \cong \frac{R}{\langle\ \rangle} \oplus \frac{R}{\langle\ \rangle} ?$$

This has a positive answer: ① R is a field

② R is a PID

CAUTION: We won't include fields as PID.

That is, when we say "PID", we exclude fields !!

③ Suppose M is a finitely generated (f.g.) R-module.

(That is, suppose $M = \langle x_1, \ldots, x_n \rangle$.)

Then, M is a quotient of $R^{\oplus n}$.

way to get

Then, M is a quotient of $R^{\oplus}$.

way to get this

Define $R^{\oplus n} \overset{\varphi}{\twoheadrightarrow} M$ by $e_i \longmapsto x_i$.

$M \simeq R/\ker\varphi$.

④ In general, consider a free module with "M as basis", call it $F(M)$. Then $F(M)$ maps onto M.

Slightly more general: If $A \subset M$ is a generating set, i.e., $M = \langle A \rangle$, then $F(A)$ maps onto M.

Thus, M can be written as a quotient of a free-module.

To Summarise : If M is an R-module, then M can be written as a quotient of a free R-module.
Moreover, if M is f.g., then the free module can be assumed to have finite rank.

Free resolution of M (over R):

Let F be a free R-module mapping onto M with kernel K.
That is, $\varphi : F \twoheadrightarrow M$ is onto R-linear and $\ker\varphi = K$.

Now, $\exists$ a free R-module G and an onto map $\psi : G \to K$

We capture this in the following diagram

← compose ψ with $K \hookrightarrow F$

$$\begin{tikzcd}
G \arrow[rr, dashed, "\psi"] \arrow[dr, "\psi"'] & & F \arrow[r, "\varphi"] & M \arrow[r] & 0 \\
& K \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

means φ is onto

means the map $K \to F$ is 1-1

Note that $\operatorname{im}\psi = K = \ker\varphi$.

Thus, we have $G \xrightarrow{\psi} F \xrightarrow{\varphi} M \longrightarrow 0$.

① φ is onto and $\ker\varphi = \operatorname{im}\psi$.
② G and F are free R-modules.

Note that we can repeat the above process with K instead of F.

Change notation: $F_0 := F$, $F_1 := G$, $K_0 := K$, $\varphi_0 := \psi$, $\varphi_1 := \varphi$.

$$\begin{tikzcd}
 & K_1 \arrow[dr] & & & & & \\
F_2 \arrow[ur, "\varphi_2"] \arrow[rr, "\varphi_2"'] & & F_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\varphi_1"'] & & F_0 \arrow[r, "\varphi_0"] & M \arrow[r] & 0 \\
 & & & K_0 \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 & &
\end{tikzcd}$$

Thus, we get free modules $\{F_n, \varphi_n : F_n \to F_{n-1}\}$ such that $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$ written as

$$\cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

F_is are free, φ_0 is onto & $\ker \varphi_{n-1} = \operatorname{im} \varphi_n$, $n \geq 1$

Often, we drop the 'n' and call

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to \underbrace{F_1 \xrightarrow{\varphi_1} F_0 \to 0}_{\operatorname{im} \varphi_1 = K,}$$ as an

this is not exact here. φ_1 not onto (nec.)

R free resolution of M.

Q: ① If M is f.g.:
Can we get F_is so that $\operatorname{rank}(F_i) < \infty$ $\forall i$.

② If yes, are $\operatorname{rank}(F_i)$s independent of construction?

③ Can you describe the maps?

④ Give explicit bases for F_is s.t. the maps are "described nicely"?

Q. If two modules have "isomorphic" free resolutions, are they isomorphic?

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_3"] & F_2 \arrow[r, "\varphi_2"] \arrow[d, "\wr"', "\psi_2"] & F_1 \arrow[r, "\varphi_1"] \arrow[d, "\wr"', "\psi_1"] & F_0 \arrow[r] \arrow[d, "\wr"', "\psi_0"] & 0 \\
\cdots \arrow[r, "\varphi_3'"] & F_2' \arrow[r, "\varphi_2'"] & F_1' \arrow[r, "\varphi_1'"] & F_0' \arrow[r] & 0
\end{tikzcd}$$

$M \simeq F_0 / \operatorname{im} \varphi_1$

$M' \simeq F_0' / \operatorname{im} \varphi_1'$

$$\varphi_1' \psi_1 = \psi_0 \varphi_1 \qquad (*)$$

Claim. $\psi_0(\operatorname{im}\varphi_1) = \operatorname{im}\varphi_1'$

($\subseteq$) clear by (*)

($\supseteq$) clear again since $\varphi_1' = \psi_0 \varphi_1 \psi_1^{-1}$

Thus, $M_0 \simeq F_0/\operatorname{im}\varphi \simeq \dfrac{\psi_0(F_0)}{\psi_0(\operatorname{im}\varphi)} = \dfrac{F_0'}{\operatorname{im}\varphi_1'} \simeq M_0'$

Lecture 1 (11-01-2021)

11 January 2021 11:07

Free modules:

(Free modules)

As usual : R is a (commutative) ring (with 1).
M is an R-module.

Def^n. ① Let $A \subset M$. A is said to be a generating set of M (as an R-module) if
$\forall x \in M$, $\exists x_1, \ldots, x_n \in A$ and $(a_1, \ldots, a_n) \in R^n$ s.t.

$$x = a_1 x_1 + \cdots + a_n x_n .$$

(Note that A need not be finite.)

Notation : $M = \langle A \rangle$

If $A = \{x_1, \ldots, x_n\}$ is finite, then $M = \langle x_1, \ldots, x_n \rangle$ and M is said to be finitely generated.

②ⓐ Let $x_1, \ldots, x_n \in M$. We say $\{x_1, \ldots, x_n\}$ is linearly independent (over R) if for $(a_1, \ldots, a_n) \in R^n$,

$$a_1 x_1 + \cdots + a_n x_n = 0 \Rightarrow (a_1, \ldots, a_n) = 0 \text{ in } R^n .$$

ⓑ A subset $A \subset M$ is linearly independent (over R) if every finite subset of A is linearly independent (over R).

③ A subset $A \subset M$ is a basis of M (over R) if $M = \langle A \rangle$ (over R) and A is linearly independent (over R).

④ M is free (over R) if M has a basis (over R).

REMARKS ① Not every R-module has a basis.
② A minimal generating set need not be lin. indep.
③ A maximal lin indep. set need not be a gen. set.

Q. If every R-module has a basis, is R a field?
(Yes. Take a non-field ring R and any non-trivial ideal $I \subsetneq R$.
Then, R/I has no lin. indep. set over R.)

Q. If an R-module M has a basis, does every basis have the same cardinality?

Ans. Yes. This is called the Invariant Basis Number (IBN) property of R.

Remark. This is not true if R is non-commutative. (That is, we can find a counterexample of a non-commutative ring.)
If R is a division ring, then again we have IBN.

Def^n. If M has a finite basis, say B, then we define
$$\text{rank}(M) := |B|.$$
If M is free with an infinite basis, $\text{rank}(M) := \infty$.

} well-defined, by IBN

(Rank)

(When we do say "rank", we will usually mean "finite rank.")

EXAMPLES. ① $R^{\oplus n}$ is a free R-module of rank n
$M_{m \times n}(R)$ of rank mn
$R[x]$ of rank ∞

② Let A be a non-empty set and
$$\mathcal{F}_0(A, R) = \{ f : A \to R \mid f(a) = 0 \text{ for all but fin. many } a \in A \}.$$

Then, $\mathcal{F}_0(A, R)$ is an R-module under pointwise operations.
In fact, $\mathcal{F}_0(A, R)$ is a free R-module with basis $\{\chi_a\}_{a \in A}$, where

$$\chi_a(b) = \begin{cases} 0 & ; \ b \neq a \\ 1 & ; \ b = a \end{cases}$$

To see where the above set is generating, given any $f \in \mathcal{F}_0(A, R)$, we can write

$$f = \sum_{a \in A} f(a)\, \chi_a .$$

↑ the sum is actually finite since $f(a)=0$ for all but finitely many.
(it is to be understood that 0s are ignored.)

Q. What if we take $\mathcal{F}(A,R)$? (All functions.)

Universal Property of free modules: (Free R-module on A)

Def^n: Given a non-empty set A, a free R-module on A is a pair $(F(A), e)$ where (i) $F(A)$ is an R-module,
(ii) $e : A \to F(A)$ is a (set) function

satisfying:

Given an R-module M and a function $f : A \to M$, there exists a unique R-linear $\tilde{f} : F(A) \to M$ making the following diagram commute.

$$\begin{tikzcd} & F(A) \arrow[dd, dashed, "\tilde{f}"] \\ A \arrow[ur, "e"] \arrow[dr, "f"'] \arrow[r, phantom, "\circlearrowright"] & \\ & M \end{tikzcd}$$

(That is, $\tilde{f} e = f$.)

Remarks. ① Given $A = \emptyset$, a free R-module on A exists, and is unique up to isomorphism.
Moreover, $e : A \to F(A)$ is one-one and $F(A)$ is free with basis $\{e_a\}_{a \in A}$, where $e_a := e(a)$.

② If M is a free R-module, then $M \cong F(B)$, where B is a(ny) basis of M.

Thus, an R-module M is free iff $M \cong F(A)$ for some A.

What the universal property is really saying is that: given a free R-module M with basis A, every R-linear

$$M \to N$$ R-module

is completely determined by its action on A.

(The above is in the sense that given any assignment of values on A, we do get an R-linear map.)

EXAMPLE: Given an R-module M, such that $M = \langle A \rangle$, we can write M as a quotient of $F(A)$.
(What we did last lec.)

Lecture 2 (12-01-2021)

12 January 2021 08:35

Weyl Algebra

Ex. $\Bbbk$ is a field, $\Bbbk[x_1, \dots, x_d]$

$\partial_1, \dots, \partial_d$ → partial diff op.

$A_d(k) = k[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$ D-modules

↑ non-comm. How would you define product?

Tensor Product

(Tensor product)

Tensor product (of two modules) essentially converts the study of bilinear maps to linear maps.

Defn. Given R-modules M and N, the tensor product of M and N (over R) is a pair (T, θ), where T is an R-module, $\theta : M \times N \to T$ is R-bilinear satisfying:

Given (K, φ) where K is an R module, $\varphi : M \times N \to K$ is R-bilinear, there exists a unique R-linear map $\tilde{\varphi} : T \to K$ making the following diagram commute

$$\begin{tikzcd} & T \arrow[d, dashed, "\tilde{\varphi}"] \\ M \times N \arrow[ur, "\theta"] \arrow[r, "\varphi"'] \arrow[r, phantom, shift left=3, "\circlearrowright"] & K \end{tikzcd}$$

, i.e., $\tilde{\varphi}\theta = \varphi$.

(We are using "with", but can use "and" once we prove $M \otimes_R N \simeq N \otimes_R M$.)

Thm. A tensor of M with N exists and is unique, up to isomorphism.

Uniqueness follows by universal property.

Notation: $M \otimes_R N$

Construction:

Want

$$\begin{tikzcd} M \times N \arrow[rr, dashed, "\theta"] \arrow[dr, "\varphi"'] & & T \arrow[dl, dashed, "\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

$\theta(x_1+x_2, y) = \theta(x_1,y)+\theta_2(x,y)$

Step 1: Let $F = F(M \times N)$, the free module on the set $M \times N$.

We get a map $e : M \times N \longrightarrow F(M, N)$

$$(x, y) \longmapsto e_{(x,y)}$$

$\{e_{(x,y)} : x \in M, y \in M\}$ is a basis for F.

Let G be the submodule of F generated by

- $e_{(x_1+x_2, y)} - e_{(x_1, y)} - e_{(x_2, y)}$
- $e_{(x, y_1+y_2)} - e_{(x, y_1)} - e_{(x, y_2)}$
- $e_{(ax, y)} - a\, e_{(x,y)}$
- $e_{(x, ay)} - a\, e_{(x,y)}$

$\forall\, x, x_1, x_2 \in M, \ \forall\, y, y_1, y_2 \in M, \ \forall\, a \in R$

Step 2: Define $T = F/G$. Let $\pi : F \twoheadrightarrow T$ be the natural map.

Set $\pi(e_{(x,y)}) =: x \otimes y$.

Note that

$$\left.\begin{aligned} (x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\ (ax) \otimes y = a(x \otimes y) &= x \otimes (ay) \end{aligned}\right\} \begin{aligned} &\forall\, x, \ldots \in M \\ &\forall\, y, \ldots \in N \\ &\forall\, a \in R \end{aligned}$$

Consider $\theta = \pi e : M \times N \longrightarrow T$

$$(x,y) \longmapsto x \otimes y$$

$$\begin{tikzcd} & \overset{e(x,y)}{F(M \times N)} \arrow[dr, "\pi"] & \\ \underset{(x,y)}{M \times N} \arrow[ur, "e"] \arrow[rr, dashed, "\theta"] & & \underset{x \otimes y}{T} \end{tikzcd}$$

Step 3. Now, suppose we are given a bilinear $\varphi : M \times N \to K$. ($K$ is some R-module.)

$$\begin{tikzcd}
M \times N \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists ! \tilde{\varphi}"] \\
& K &
\end{tikzcd}$$

Q1: Is there even a func. $\tilde{\varphi}$?

Q2: Is it R-linear?

Q3: Is it unique?

Note also

$$\begin{tikzcd}
& F \arrow[dd, dashed] & \\
M \times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists ! \tilde{\varphi}"] \\
& K &
\end{tikzcd}$$

Note we have a set map $M \times N \xrightarrow{\varphi} K$ which induces an R linear map $\bar{\varphi} : F \to K$. (UMP of free modules)

$$\begin{tikzcd}
& F \arrow[dd, dashed, "\bar{\varphi}"] & \\
M \times N \arrow[ur, dashed, "e"] \arrow[rr] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\exists ! \tilde{\varphi}"] \\
& K &
\end{tikzcd}$$

We now want to show that $\bar{\varphi}$ factors through T. It would suffice to show that $G \subseteq \ker \bar{\varphi}$.

Using bilinearity of φ, it follows that all our (four types of) generators of G are in $\ker \bar{\varphi}$.

Thus, $\bar{\varphi}$ factors through quotient. That is, $\exists !$ R-linear $\tilde{\varphi} : T \to K$ s.t.

$$\begin{tikzcd}
F \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi"'] & K \\
T \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

commutes. That is, $\bar{\varphi} = \tilde{\varphi} \pi$.

$$\begin{tikzcd}
& F \arrow[dr, dashed, "\pi"] \arrow[dd, dashed, "\bar{\varphi}"] & \\
M \times N \arrow[ur, dashed, "e"] \arrow[rr, "\theta"] \arrow[dr, "\varphi"'] & & T = F/G \arrow[dl, dashed, "\tilde{\varphi}"] \\
& {} &
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[dr, "\varphi"'] & {} \arrow[d, dashed] & {} \arrow[dl, dashed, "\tilde{\varphi}"] \\ & K & \end{tikzcd}$$

Can now verify $\tilde{\varphi}\ \theta = \varphi$. (Use commutation of diff. triangles.)
Can also verify that $\tilde{\varphi}$ is unique R-linear such.

Basic Properties:

(1) ["Identity"] $R \otimes_R M \simeq M$

(2) [Commutativity] $M \otimes_R N \simeq N \otimes_R M$

(3) [Associativity] $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$

(4) [Distributivity] $M \otimes_R (N \oplus L) \cong (M \otimes_R N) \oplus (M \otimes_R L)$

Q. How does get an R-linear map $M \otimes_R N \to L$?
A. Give an R-bilinear map $M \times N \to L$.
Pretty much the only way. $x \otimes y$ would be 0 even if $x, y \neq 0$.
Thus, checking "well-defined" ness would become quite difficult.

Q. Let M be an R-module. $R \subset S$ subring.
Can you identify a natural S-module on $S \otimes_R M$?
(Base change)

Lecture 3 (14-01-2021)

14 January 2021 09:31

Distributivity : Given R-modules L, M, N

$$L \otimes_R (M \oplus N) \cong (L \otimes_R M) \oplus (L \otimes_R N)$$

How do we show? Want something like :

$$x \otimes (y, z) \longmapsto (x \otimes y, x \otimes z)$$

Note that elements of this form only <u>GENERATE</u> the tensor.

We now need to show the above map is well-defined. (As a function)
To do so, we go back to $L \times (M \oplus N)$ and use the universal property.

This ϕ is well defined. Every elt. here is uniquely written in the given form.

$$\begin{tikzcd}
L \times (M \oplus N) \arrow[r, "\phi"] \arrow[d, "\theta"'] & (L \otimes_R M) \oplus L \otimes_R N \\
L \otimes (M \oplus N) \arrow[ur, dashed, "\tilde{\phi}"'] &
\end{tikzcd}$$

$$(x, (y, z)) \longmapsto (x \otimes y, x \otimes z)$$

Note that ϕ is R-bilinear, thus an R-linear map $\tilde{\phi}$ (as indicated) which makes the diagram commute does exist.

To now show that is an isomorphism, we construct an inverse

$$\psi : (L \otimes_R M) \oplus (L \otimes_R N) \longrightarrow L \otimes_R (M \oplus N)$$

verify these are well defined (Again universal property)

$$(x \otimes y, 0) \longmapsto x \otimes (y, 0)$$

$$(0, x \otimes 3) \longmapsto x \otimes (0, 3)$$

Can verify now that ψ is the two-sided inverse of $\tilde{\phi}$.

inverse of $\tilde{\phi}$.

REMARK. Suppose M and N are R-modules.

① If $x = 0 \in M$, then $x \otimes y = 0 \quad \forall y \in N$.
However if $x \otimes y = 0$ for some $x \in M$, $y \in N$, we cannot conclude $x = 0$ or $y = 0$.

Example: Take $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$.

In fact, look at $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ ← this is the zero module.

Thus
$x \otimes y = 0 \ \forall y \in N \not\Rightarrow x = 0$.

② $M \otimes_R N$ is generated by $\{x \otimes y : x \in M, y \in N\}$ as R-module.
In particular, if M and N are f.g., then so is $M \otimes_R N$.

Take fin. gen. sets S_M and S_N. Then
$M \otimes_R N = \langle x \otimes y : x \in S_M, y \in S_N \rangle$.

③ If M and N are free, then so is $M \otimes_R N$. Identify a basis.

For finite rank: write
$$M = \overbrace{R \oplus \cdots \oplus R}^{m}$$
$$N = \underbrace{R \oplus \cdots \oplus R}_{n}$$

Then,
$$M \otimes_R N = (R \oplus \cdots \oplus R) \otimes_R (R \oplus \cdots \oplus R)$$
↳ distribute and use $R \otimes_R R \cong R$.

④ It is possible that $M \neq 0 \neq N$ but $M \otimes_R N = 0$.
(See 1)

Q. Given a simple tensor $x \otimes y$, how can we deterime if it's 0?
Concrete ex: Is $2 \otimes 3 \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ non-zero?

Q. Is it possible that $M \otimes_R M = 0$ even if $M \neq 0$?
Yes. Take $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$.

Yes. Take $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$.

$$\left(\frac{a}{b}+\mathbb{Z}\right)\otimes\left(\frac{c}{d}+\mathbb{Z}\right) = \left(\frac{a}{db}+\mathbb{Z}\right)\otimes\underbrace{(c+\mathbb{Z})}_{0} = 0.$$

Tensor Algebra

(Tensor Algebra)

The tensor algebra of M

$$R\oplus M\oplus T_2(M)\oplus T_3(M)\oplus\cdots$$
$$\parallel$$
$$T(M) = R\oplus M\oplus(M\otimes M)\oplus(M\otimes M\otimes M)\oplus\cdots$$

$T(M)$ is clearly an additive group. One can define multiplication by "concatenation."

$$\underset{\substack{\uparrow\\ m^{th}\text{ piece}}}{(x_1\otimes\cdots\otimes x_m)}\cdot\underset{\substack{\uparrow\\ n^{th}\text{ piece}}}{(y_1\otimes\cdots\otimes y_n)} = \underset{\substack{\uparrow\\ (m+n)^{th}\text{ piece}}}{x_1\otimes\cdots\otimes x_m\otimes y_1\otimes\cdots\otimes y_n}.$$

Elements of $T(M)$ are written as formal sums: $\underset{\substack{\uparrow\\ R}}{z_0}+\underset{\substack{\uparrow\\ M}}{z_1}+\cdots+\underset{\substack{\uparrow\\ M^{\otimes n}\\ \parallel\\ T_n(M)}}{z_n}$

Identify $T(R^{\otimes n})$.

Some quotients of $T(M)$:

① Symmetric algebra

(Symmetric Algebra)

Given $x, y \in M$, $x\otimes y \neq^{*} y\otimes x$.

($\neq^{*}$: not necessarily equal)

Define $$\mathrm{Sym}(M) = \frac{T(M)}{\langle x\otimes y - y\otimes x \mid x, y \in M\rangle} = R\oplus M\oplus \mathrm{Sym}_2(M)\oplus\cdots$$

$\mathrm{Sym}(M)$ is now a commutative algebra.

R and M not affected. Only $M^{\otimes 2}$ onwards.

② Exterior algebra. [Wedge (M)]

(Exterior algebra)

$$\Lambda(M) = \frac{T(M)}{\langle x\otimes y + y\otimes x\rangle}$$

$\langle x \otimes y + y \otimes x \rangle$

$$\begin{aligned} &\parallel \\ R \oplus M &\oplus \Lambda^2(M) \oplus \cdots \end{aligned}$$

Q. What are $\mathrm{Sym}(R^{\oplus n})$ and $\Lambda(R^{\oplus n})$?

(Trivial extension)

③ $$R \ltimes M = \frac{T(M)}{\langle x \otimes y \mid x, y \in M \rangle}$$

(Trivial extension or idealisation.)

In this algebra, M is an ideal, with $M^2 = 0$.
This is called an idealisation of M.

Q. What is $R \ltimes R^{\oplus n}$?

Lecture 4 (18-01-2021)

18 January 2021 10:33

Base change:

(Base change or extension of scalars)

Let R and S be rings and $\varphi : R \longrightarrow S$ a ring homomorphism. Then, we say that S is an R-algebra "via φ", i.e., S has an R-module structure defined by

$$a \cdot x = \varphi(a)x \qquad \forall a \in R \ \forall x \in S.$$

Two key examples : ① R is a subring of S (φ is 1-1)

② S is a quotient of R (φ is onto)

(and their compositions)

Example. If $I \subset R[x_1, \dots, x_d]$ is an ideal, then

$$S = \frac{R[x_1, \dots, x_d]}{I} \quad \text{is an } R\text{-algebra}$$

(via the natural maps $R \hookrightarrow R[x_1, \dots, x_d] \twoheadrightarrow S$)

A consequence of the Hilbert Basis Theorem:

Thm. Every finitely generated algebra over a Noetherian ring is a Noetherian ring.

(Not saying Noetherian as an R-module! Hilbert's Basis Thm does not give that!)

Note. If S is an R-algebra (via φ) and M is an S-module, then M has a natural R-module structure (via φ).

Q: What about the reverse? If M is an R-module, is there a "natural" way to induce an S-module structure on it?

No, in general. Take $\mathbb{Z}$ as a $\mathbb{Z}$ module and $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$.

Is there any "good" $\mathbb{Q}$-module structure on $\mathbb{Z}$?

Example. Let M be an R-module and $I \trianglelefteq R$ an ideal, $A \subset R$ m.c.s.

① When is M an R/I-module? } induced multiplication
② When is M an R_A-module?

In general, we can create modules over R/I and R_A starting from M: They are M/IM and M_A, respectively.

Obs. $M/IM \simeq R/I \otimes_R M$ and $M_A \simeq R_A \otimes_R M$
(Isomorphic as R-modules.)

Base change (or extension of scalars):

Let S be an R-algebra (via φ) and M be an R-module. The R-module

$$S \otimes_R M$$

has a natural S-module structure defined by

$$a(b \otimes x) := (ab) \otimes x. \qquad \forall a,b \in S \ \forall x \in M$$

(Note that the above is only being defined for simple tensors.)

Note: ① If $M = \langle x_1, \ldots, x_n \rangle$ over R, then

$$S \otimes_R M = S\langle 1 \otimes x_i \ : \ 1 \leq i \leq n \rangle.$$

② If $M \cong R^{\oplus n}$, $S \otimes_R M \cong S^{\oplus n}$ as R-modules.

$\otimes$ distributes over $\oplus$

In fact, $S \otimes_R M \cong S^{\oplus n}$ as S-modules as well.

Q. Given an R-linear map $\psi: M \to N$, will this induce an S-linear map $\overline{\psi}: S \otimes_R M \to S \otimes_R N$?

Does this help in the above?

③ If $M = R[x]$, then $S \otimes_R M \cong S[x]$ as S-modules.

④ If M is a free R-module, then $S \otimes_R M$ is a free S-module.

Example of base change : "Mod p test for irreducibility of a polynomial in $\mathbb{Z}[x]$"

Recall the test : Given $f = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$

If we can find a prime p s.t. $f \bmod p$ is irred in $(\mathbb{Z}/p\mathbb{Z})[x]$, then f is irred.*

(*Need to take care of degree not falling.)

This is an example of base change with $R = \mathbb{Z}$ and $S = \mathbb{Z}/p\mathbb{Z}$.

Complexes and Homology

(Complexes and Homology)

Example Construct a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & & & \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[rr] \arrow[dr] & & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\
 & & & & & 2\mathbb{Z} \arrow[ur, hook, "\ker"'] & 1 \arrow[r, mapsto] & \bar{1} & \\
 & & & & x \arrow[r, mapsto, "\wr\ 1\text{-}1"'] & 2x & & &
\end{tikzcd}$$

Q1. What is a generating set for $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$?

Ans. $\{\bar{1}\}$ ← singleton.

Thus, we map one copy of $\mathbb{Z}$ onto $\mathbb{Z}/2\mathbb{Z}$.
That is,

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ 1 &\longmapsto \bar{1} \end{aligned}$$

Q2. What is ker ?

Ans. It is $2\mathbb{Z}$.

Q3. What can map onto $2\mathbb{Z}$?

Q3. What can map onto $2\mathbb{Z}$?

Ans. $\mathbb{Z}$. $x \mapsto 2x$

Q4. What is ker?

It is 0. The map is 1-1. This gives the diagram

Note that we could have also written

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

since $2\mathbb{Z}$ itself is a free $\mathbb{Z}$-module. The reason we did not do this is because we are sticking to writing free R-modules as copies of R.

Q. Let R be a ring and $a \in R$. Is the following a free resolution of $R/\langle a \rangle$ over R?

$$0 \to R \xrightarrow{a} R \to R/\langle a\rangle \to 0$$

$$x \mapsto ax$$

(finite free resolution) Called a finite free resolution.

Obs. • Note that if some ker is free, we can stop there. ↑ (That is basically saying that $R^{\oplus n} \to \ker$ will be 1-1.)

• The above does happen if R is a PID and M is f.g. So, in that case, the resolution stops right away as above. (At Stage 1.)

Stage 1

$$\cdots \to 0 \to 0 \to 0 \to R^{\oplus m} \to R^{\oplus n} \to M \to 0$$

Stage 0 $(m \leq n)$

Thus, every f.g. module has a finite free resolution of "length" 1.

(Recall that a submodule of a f.g. free module over a PID is free.)

• Question of "optimality" of free resolution

We know $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

We know $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$

is a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$.

So is

$$0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \xrightarrow{\begin{bmatrix} 2 & 0 \end{bmatrix}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$f \mapsto e_2 \qquad e_1 \mapsto 2$$

$$e_2 \mapsto 0$$

Note that we can go on and create an extra copy of $\mathbb{Z}$ and make it longer.

$$0 \to \mathbb{Z}g \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 0 \end{bmatrix}} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

$$e_1 \longmapsto 2$$

$$f_1 \longmapsto e_2 \longmapsto 0$$

$$g \longmapsto f_2 \longmapsto 0$$

Q. If $\operatorname{rank}(F_i) \geq \operatorname{rank}(F_{i-1})$, does that mean non-optimal?

I don't think so. If a column is 0, then yes. But otherwise, don't think so.

Q. What is a free resolution of R over itself

$$0 \to R \xrightarrow{id} R \to 0.$$

(Dropping the module: $0 \to R \to 0$)

In fact, the above is true for any free R-module F.

$$0 \to F \to F \to 0.$$

(Dropping the module $0 \to F \to 0$.)

Both are free resolutions of length 0.

Q. If every f.g. R-module has a free resolution of length 0, what is R?

Ans. length $0 \iff M$ is free.

Every R-module has a basis iff R is a field.
(is free)

Q. If every f.g. R-module has a free resolution of length 1, what is R?

Ans. Field or PID.

One way implication is obvious. (Field or PID $\Rightarrow$ length 1)

($\Leftarrow$) Suppose I is an ideal. Let R/I have a resolution of length 1.
↳ f.g.

$$0 \to F_1 \to F_0 \to R/I \to 0$$

We also have

$$0 \to I \to R \to R/I \to 0.$$

Have to conclude I is free. That would imply I is principal.

INCOMPLETE.

Back to example: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is a $\mathbb{Z}$-free resolution of $\mathbb{Z}/2\mathbb{Z}$.

Tensor with $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}$. ← We get a sequence of $\mathbb{Z}/6\mathbb{Z}$ modules via base change.

① Note that $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}/6\mathbb{Z}$ module. ($\bar{1}_{\mathbb{Z}/6\mathbb{Z}} \mapsto \bar{1}_{\mathbb{Z}/2\mathbb{Z}}$ gives a ring homo)

② $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}/(6\mathbb{Z}+2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

↳ Should be generated by $\{\bar{1} \otimes \bar{1}\}$ (recall tensor generated by tensor of gen.)

③ $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$

Thus, tensoring the free resolution with $\mathbb{Z}/6\mathbb{Z}$ gives

$$0 \to \mathbb{Z}/6\mathbb{Z} \xrightarrow[\bar{2}]{?} \mathbb{Z}/6\mathbb{Z} \xrightarrow{?} \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\bar{1} \longmapsto \bar{1}$$

$$\bar{1} \longmapsto \bar{2}$$

Is this a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/6\mathbb{Z}$?

No! The map $\bar{1} \mapsto \bar{2}$ is not injective.

$3\mathbb{Z}/6\mathbb{Z}$ is the kernel!

↪ not free ☹

After base change, the free resolution does not remain a resolution.

Lecture 5 (19-01-2021)

19 January 2021 11:31

Recall: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is a $\mathbb{Z}$-free resolution of $\mathbb{Z}/2\mathbb{Z}$.

Tensoring with $\mathbb{Z}/6\mathbb{Z}$ does not give a $\mathbb{Z}/6\mathbb{Z}$-free resolution of $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

It gives a free "complex" with "homology".

Note. If S is an R-algebra (via φ), $M \xrightarrow{\psi} N$ is R-linear, then ψ induces a natural S-linear map $S \otimes_R M \longrightarrow S \otimes_R N$.

Def^n. A complex of R-modules is a sequence (finite or countable) of R modules with maps between them

$$\cdots \to M_{n+1} \xrightarrow{\varphi_{n+1}} M_n \xrightarrow{\varphi_n} M_{n-1} \to \cdots$$ such that

$$\ker \varphi_n \supset \operatorname{im} \varphi_{n+1}, \quad \text{i.e.,} \quad \varphi_n \circ \varphi_{n+1} = 0 \quad \forall n.$$

(Finite / infinite in one or both directions)

The complex is exact at n^{th} stage if $\ker \varphi_n = \operatorname{im} \varphi_{n+1}$.

$\mathbb{Z}$-modules: $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$ (dropping the modulos)

$\mathbb{Z}/6\mathbb{Z}$-modules: $0 \xrightarrow{\varphi_2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\varphi_0} 0$
(after tensoring)

$\bar{1} \longmapsto \bar{2}$; $\bar{1} \longmapsto 0$

0th stage

$\ker \varphi_0 = \mathbb{Z}/6\mathbb{Z}$
$\underbrace{\operatorname{im} \varphi_1 = 2\mathbb{Z}/6\mathbb{Z}}_{\text{NOT EXACT!}}$

1st stage

$\ker \varphi_1 = 3\mathbb{Z}/6\mathbb{Z}$
$\underbrace{\operatorname{im} \varphi_2 = 0}_{\text{NOT EXACT!}}$

In both cases, we do have $\operatorname{im} \subset \ker$. Thus, it is indeed a complex

Q: Note that $0 \to M \xrightarrow{\varphi} N \to 0$ is always a complex. When is it exact?

A: Precisely when φ is an iso. (0^{th} stage exact $\Leftrightarrow$ φ is onto
1^{st} stage exact $\Leftrightarrow$ φ is 1-1)

Remark. $0 \to M \xrightarrow{\varphi} N$ is exact $\Leftrightarrow$ φ is 1-1
$M \xrightarrow{\varphi} N \to 0$ is exact $\Leftrightarrow$ φ is onto

Def^n. A complex $\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \cdots$ is exact if it is exact at each stage n.

Remark. Thus, if it is not exact, then it is not exact at some n. That is, $\operatorname{im} \varphi_{n+1} \subsetneq \ker \varphi_n$. Since we do have containment, the quotient $\ker \varphi_n / \operatorname{im} \varphi_{n+1}$ makes sense. $\operatorname{im} \varphi_{n+1} = \ker \varphi_n \Leftrightarrow \ker \varphi_n / \operatorname{im} \varphi_{n+1} = 0$.

Def^n. Given a complex $C_\cdot$, we define the n^{th} homology of $C_\cdot$ as

$$H_n(C_\cdot) = \frac{\ker \varphi_n}{\operatorname{im} \varphi_{n+1}}.$$

Thus, the homology is an "obstruction" to the complex being exact!

(Familiar examples: $\ker \phi$ is an obstruction to ϕ being 1-1.
〃 $\operatorname{coker} \phi$ for onto. $[G, G]$ for G being abelian.)

Ex. Find the homologies in the previous examples.

Example. If $F_\cdot : \cdots \to F_2 \to F_1 \to F_0 \to 0$ is a free resolution of M, then

$$H_n(F_\cdot) = \begin{cases} 0 & ; \quad n > 0, \\ M & ; \quad n = 0. \end{cases}$$

(This was a complex, by construction; similar reason for $n>0$ stages being exact.)

Remark If S is an R-algebra, $C_\cdot$ is an exact complex of R-modules, $S \otimes_R C_\cdot$ is a complex but not necessarily exact.

Not^n. An exact complex $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is called a short exact sequence (s. e. s).

This gives φ is 1-1, ψ is onto.
↳ can think of L as a submodule of M
$\operatorname{im} \varphi = \operatorname{Ker} \psi$ gives $M/L \cong N$.

Properties of submodules and quotients can be transferred to s.e.s.s usually.
E.g. M is Noe (Art) $\Leftrightarrow$ N and L are Noe. (Art.)

Functorial properties of $\otimes$

① Fix an R-module K. Define $T(M) = K \otimes_R M$ for all R-modules M.

② Note: For any R-module M, $T(M)$ is also an R-module.

③ Given $\varphi: M \to N$, R-linear, we get an induced map
$$T(\varphi) : T(M) \to T(N)$$

[$T(\varphi) : K \otimes_R M \longrightarrow K \otimes_R N$ defined by
$$[T(\varphi)](x \otimes y) = x \otimes \varphi(y).$$] need to verify

(Not^n: $T(\varphi) = \mathrm{id} \otimes \varphi$ or φ_*.)

This assignment T has the following properties:

This assignment T has the following properties:

④ (a) $T(\psi\circ\varphi) = T(\psi)\circ T(\varphi)$

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \arrow[r, "\psi"] & L \\ T(M) \arrow[r, "T(\varphi)"'] & T(N) \arrow[r, "T(\psi)"'] & T(L) \end{tikzcd}$$

(b) $T(\mathrm{id}_M) = \mathrm{id}_{T(M)}$

(c) $T(0) = 0$ (0 module or 0 map? Yes.)

(d) $T(M\oplus N) = T(M)\oplus T(N)$

T above is a functor by (1) - (4). Denoted by $K\otimes_R -$.
It is a covariant functor, since arrows are in same direction.

Furthermore, $T(0)=0$ show that if $C_\cdot$ is a complex of R-modules, then so is $T(C_\cdot)$.
(Since compositions go to compositions and 0 goes to 0.)

Q. Does $K\otimes_R -$ preserve exactness?
No! We already have examples. Easy one: $0\to\mathbb{Z}\xrightarrow{2}\mathbb{Z}$ is exact.
Apply $\mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}} -$ to get answer.

Note that if S is an R-algebra (via φ), then base change (i.e., $S\otimes_R -$) gives a functor from R-modules to S-modules.
(Covariant or contravariant.)

Q. If $0\to L\xrightarrow{\varphi} M\xrightarrow{\psi} N\to 0$ is a s.e.s. of R-modules and K is a fixed R-module, what can you say about

$$K\otimes_R\left(0\to L\xrightarrow{\varphi} M\xrightarrow{\psi} N\to 0\right).$$

Examples of functors: ① Localisation : R-modules $\to$ R_A - modules.
② Forgetful functors : Group/ Rings /etc. $\to$ Set
③ Identity /inclusion functor
④ Linearisation : Set $\to$ R-modules ($A\mapsto F(A)$)

⑤ Fundamental group : Top. $\longrightarrow$ Grp

$\text{Top} \longrightarrow \text{Ring}$

$X \longmapsto \{\text{continuous } f : X \to \mathbb{R}\}$

⑥ $\mathbb{k}$ a field.

Field extensions of $\mathbb{K} \longrightarrow \mathbb{k}$-vector spaces

Is this a functor?

(Note that we have to think of the morphisms as well.)

Lecture 6 (21-01-2021)

21 January 2021 09:30

Q. What can we say about $K \otimes_R \left(0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0 \right)$?

A. We do get a complex

$$0 \to K \otimes_R L \xrightarrow{\varphi_*} K \otimes_R M \xrightarrow{\psi_*} K \otimes_R N \to 0$$

$\varphi_* = id_K \otimes \varphi$, $\psi_* = id_K \otimes \psi$

Do we have exactness at all three points?

① Is φ_* 1-1?

② Is ψ_* onto?

③ Is $\operatorname{im} \varphi_* = \ker \psi_*$?

① No. $K \otimes_R -$ does not take injective maps to injective maps.
$\mathbb{Z}/6\mathbb{Z} \otimes_\mathbb{Z} \left(0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \right)$ was seen.

② Yes. $K \otimes_R N$ is generated by $\{ x \otimes z \mid x \in K, z \in N \}$
If ψ is surjective, $\forall z \in N$, $\exists y \in M$ s.t. $\psi(y) = z$.
Hence,

$$x \otimes z = x \otimes \psi(y) = \psi_*(x \otimes y).$$

Thus, $K \otimes_R -$ takes surjections to surjections.

③ We already know it is a complex. Thus, $\operatorname{im}(\varphi_*) \subset \ker(\psi_*)$.

(?) Let $\sum x_i \otimes y_i \in \ker(\psi_*)$. ← Doing this will almost never work.

We prove the natural map $\dfrac{K \otimes_R M}{\operatorname{im} \varphi_*} \xrightarrow[\pi]{} \dfrac{K \otimes_R M}{\ker \psi_*}$

is an isomorphism.

This would prove $\operatorname{im} \varphi_* = \ker \psi_*$.

this is the natural onto obtained because im ⊂ ker is known

To do this, we prove that the map

duced

$K \otimes M \xrightarrow{\bar{\psi}_* \pi} K \otimes$

map induced by quotient

$\overline{\psi}_* : \frac{K\otimes_R M}{\ker \psi_*} \longrightarrow K\otimes_R N$

$$\frac{K\otimes_R M}{\text{im }\varphi_*} \xrightarrow{\overline{\psi}_* \pi} K \otimes_R N \longrightarrow 0$$ is invertible.

Construct the inverse as follows

$\forall x \in K$, $z \in N$, choose $y \in M$ s.t. $\psi(y) = z$.

(Ex1.) Verify $x \otimes z \longmapsto x \otimes y + \text{im }\varphi_*$ is a well-defined R-linear map

This is an inverse of $\overline{\psi}_* \pi$.

Thus, we have shown : If $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is an s.e.s, then

$K\otimes_R L \xrightarrow{\varphi_*} K \otimes_R M \xrightarrow{\psi_*} N \otimes_R M \longrightarrow 0$ is exact

and injective maps need not go to injective maps.

We say that $K\otimes_R -$ is not an exact functor but a right exact functor. In fact, our proof did not need injectivity of φ.

(Right exact functor, exact functor)

This is the def^n of right exactness. → Thus, $L \to M \to N \to 0$ exact $\Rightarrow$ $K\otimes_R (L \to M \to N \to 0)$ exact.

Note the lack of $0 \to$ here.

Hom as a functor

Note that $K \otimes_R -$ and $- \otimes_R K$ are the same due to commutativity. However, as Hom is different.

Ex. Given R-modules M and N, is $\text{Hom}_R(M, N) \cong \text{Hom}_R(N, M)$?

No. Take $R = M = \mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z}$.

(1) Fix an R-module K.

Given an R-module M, $\text{Hom}_R(M, K)$ is an R-module.

Given an R-linear map $\varphi: M \to N$, we get a function

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, dashed] & N \arrow[d, dashed] \\ & K \end{tikzcd}$$

$$\begin{array}{ccc} \mathrm{Hom}(N, K) & \xrightarrow[\varphi_*]{\mathrm{Hom}(\varphi, K)} & \mathrm{Hom}(M, K) \\ \alpha & \longmapsto & \alpha \circ \varphi \end{array}$$

Note the reversal!

This association respects compositions (reverse), identity, and zero.

Thus, $\mathrm{Hom}_R(-, K)$ is a contravariant functor.

(Contravariant functor)

Since it preserves compositions and zeroes, it preserves complexes. That is, given an s.e.s

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0,$$ we get a complex

$$0 \to \mathrm{Hom}_R(N, K) \xrightarrow{\psi_*} \mathrm{Hom}_R(M, K) \xrightarrow{\varphi_*} \mathrm{Hom}_R(L, K) \to 0$$

At what place(s) is it exact?

(2) Fix K. Given M, $\mathrm{Hom}_R(K, M)$ is an R-module and given $\varphi : M \to N$, we get a map

$$\varphi_* = \mathrm{Hom}(K, \varphi) : \mathrm{Hom}(K, M) \longrightarrow \mathrm{Hom}(K, N).$$

(Ex2.) Verify that $\mathrm{Hom}_R(-, K)$ is a contravariant functor, and $\mathrm{Hom}_R(K, -)$ is a covariant functor. Both are left exact.

(Ex1.) First, we show $\Phi : K \times N \to \dfrac{K \otimes_R M}{\mathrm{im}\, \varphi_*}$

$(x, z) \mapsto x \otimes y + \mathrm{im}\, \varphi_*$ is well-defined

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

Note that if $\psi(y_1) = \psi(y_2)$, then $y_1 - y_2 \in \ker \psi = \mathrm{im}\, \varphi$.

Thus, $y_1 = y_2 + \varphi(x')$ for some $x' \in L$.

$\Rightarrow \quad x \otimes y_1 = x \otimes y_2 + x \otimes \varphi(x')$

$\Rightarrow \quad x \otimes y_1 = x \otimes y_2 + \varphi_*(x \otimes x')$

Thus, $x \otimes y_1 + \mathrm{im}\,\varphi_* = x \otimes y_2 + \mathrm{im}\,\varphi_*$.

Thus, Φ is a well-defined map. That it is bilinear is clear.

Thus we get a well-defined map

$$\tilde{\Phi} : K \times N \longrightarrow \frac{K \otimes_R M}{\mathrm{im}\,\varphi^*} \quad \text{defined by}$$

$$x \otimes z \longmapsto x \otimes y + \mathrm{im}\,\varphi^*$$

It suffices to show that it is the left inv of $\bar{\Psi}_*\pi$. ← already know it is onto

$$\frac{K \otimes_R M}{\mathrm{im}\,\varphi_*} \overset{\pi}{\twoheadrightarrow} \frac{K \otimes_R M}{\ker \psi_*} \xrightarrow[\bar{\psi}_*]{} K \otimes_R N \xrightarrow[\tilde{\Phi}]{} \frac{K \otimes_R M}{\mathrm{im}\,\varphi_*}$$

on generators: $x \otimes y + \mathrm{im}\,\varphi_* \longmapsto x \otimes y + \ker \psi_* \longmapsto x \otimes \psi(y) \longmapsto x \otimes y + \mathrm{im}\,\varphi_*$. ✓

(Ex2.) $L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathrm{Hom}(N, K) \xrightarrow{\psi_*} \mathrm{Hom}(M, K) \xrightarrow{\varphi_*} \mathrm{Hom}(L, K)$ exact

- ψ_* 1-1

$$\begin{tikzcd} N \arrow[r, "\alpha"] & K \\ M \arrow[u, "\psi"'] \arrow[ur, "\psi_*\alpha = \alpha \circ \psi"'] & \end{tikzcd}$$

Let $\alpha \in \mathrm{Hom}(N, K)$ s.t. $\psi_* \alpha = 0$.

Thus, $\alpha \circ \psi = 0$ map

$\Rightarrow (\alpha \circ \psi)(m) = 0 \quad \forall m \in M$

$\Rightarrow \alpha(\psi(m)) = 0 \quad \forall m \in M$

$\Rightarrow \alpha(n) = 0 \quad \forall n \in N \quad (\because \psi$ is onto$)$

$\Rightarrow \alpha = 0$ map $\quad \boxtimes$

- $\mathrm{im}\,\psi_* = \ker \varphi_*$

$$\begin{tikzcd} N \arrow[dr, dashed, "\alpha"] & \\ M \arrow[u, "\psi"] \arrow[r, "\beta"] & K \\ L \arrow[u, "\varphi"] \arrow[ur, "\beta \circ \varphi"'] & \end{tikzcd}$$

$(\subseteq)$ since complex.

$\in \mathrm{Hom}(M, K)$

(2) Let $\beta \in \ker \varphi_*$. Then, $\psi_* \beta = 0$ map

$\Leftrightarrow$ $\beta \circ \psi = 0$ map

$\Rightarrow$ $\beta \circ \varphi(l) = 0 \quad \forall l \in L$

$\beta \in \text{im}\,\psi_*$ $\Rightarrow \boxed{\beta = \alpha \circ \varphi}$

$\Rightarrow$ $\ker \beta \supset \text{im}\,\varphi = \ker \psi$

Thus, if $\beta(m_1) = \beta(m_2)$, then $\psi(m_1) = \psi(m_2)$.

Thus, by UMP of quotients, $\alpha(n) = \alpha(\psi(m))$
$= \beta(m)$

is well defined and R-linear.

Thus, $\beta = \alpha \circ \psi = \varphi_* \alpha \in \text{im}\,\psi_*$. ∎

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \quad \text{exact}$$

$$\Downarrow$$

$$0 \longrightarrow \text{Hom}(K, L) \xrightarrow{\varphi_*} \text{Hom}(K, M) \xrightarrow{\psi_*} \text{Hom}(K, N) \quad \text{exact}$$

- φ_* is 1-1

Let $\alpha \in \ker \psi_*$. $\psi_* \alpha = 0 \Rightarrow \varphi \circ \alpha = 0$

$\Rightarrow (\varphi \circ \alpha)(k) = 0 \quad \forall k \in K$

$\Rightarrow \alpha(k) = 0 \quad \forall k \in K \quad (\because \varphi \text{ is } 1\text{-}1)$

$\Rightarrow \alpha = 0$. ∎

- $\text{im}\,\varphi_* = \ker \psi_*$

$$\begin{tikzcd} & L \arrow[d, "\varphi"] \\ K \arrow[ur, dashed, "\alpha"] \arrow[r, "\beta"] \arrow[dr, "\psi\circ\beta"'] & M \arrow[d, "\psi"] \\ & N \end{tikzcd}$$

(⊆) clear.

(2) Let $\beta \in \ker \psi_*$. Then, $\psi \circ \beta = 0$

$\Rightarrow (\psi \circ \beta)(k) = 0 \quad \forall k$

$\Rightarrow \beta(k) \in \ker \psi = \text{im}\,\varphi \quad \forall k$

$\Rightarrow \forall k \; \exists l_k \in L$ s.t. $\varphi(l_k) = \beta(k)$.

$\because \varphi$ is 1-1, $\exists!\, l_k$

Moreover, $\alpha = (k \mapsto l_k)$ is R-linear.

Thus, $\alpha \in \text{Hom}(K, L)$ and

$\beta(k) = \varphi(l_k) = (\varphi \circ \alpha)(k) \quad \forall k$.

Thus, $\beta = \varphi \circ \alpha = \varphi_* \alpha \in \text{im}\,\varphi_*$. ∎

Lecture 7 (25-01-2021)

25 January 2021 10:35

Covariant Hom is left exact, i.e.,

$$\underset{\text{exact}}{0 \to L \to M \to N} \Rightarrow \underset{\text{exact}}{0 \to \mathrm{Hom}_R(K,L) \to \mathrm{Hom}_R(K,M) \to \mathrm{Hom}_R(K,N)}$$

Contravariant Hom is also left exact, i.e.,

$$\underset{\text{exact}}{L \to M \to N \to 0} \Rightarrow \underset{\text{exact}}{0 \to \mathrm{Hom}_R(L,K) \to \mathrm{Hom}_R(M,K) \to \mathrm{Hom}_R(N,K)}$$

(Note the assumption of $\to 0$ on LHS.)

Usually we will be more relaxed and start with an s.e.s. to begin with.

Q. Is $K \otimes_R -$ exact? No. $K = \mathbb{Z}/6\mathbb{Z}$ over $R = \mathbb{Z}$.

Is $K \otimes_R -$ exact for some K over some R? Yes, $K = R$ for any R.

(1) Over any R, can you find a K s.t. $K \otimes_R -$ is not exact.

(2) Can you find a class of examples of K s.t. ⓐ $K \otimes_R -$ is not exact?
ⓑ $K \otimes_R -$ is exact?

Q. Can ask and (try to) answer similar questions about both the Hom functors.

Def^n. A s.e.s. $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is split exact if $\exists$ a map $\chi : N \to M$ which is a splitting, i.e., $\psi \circ \chi = id_N$.

(Split exact sequences)

(map will always refer to the appropriate morphisms.)

Ex. ① $M = \varphi(L) \oplus \chi(N)$

② x is injective and hence $M \simeq L \oplus N$.

① Split exact sequence captures the notion of $\oplus$.
(s.e.s. captures submodule and quotient.)

If $M = L \oplus N$, then $0 \to L \to M \to N \to 0$ is split exact with the natural maps.

② Let $\mathcal{F}$ be a functor from R-modules to R-modules which is additive.

(Definition at end)

If

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

is split exact,

the what can one say about

$$0 \to \mathcal{F}(L) \xrightarrow{\varphi_*} \mathcal{F}(M) \xrightarrow{\psi_*} \mathcal{F}(N) \to 0 \ ?$$

→ The "splitness" is preserved. $\psi x = id_N \Rightarrow \psi_* x_* = id_{\mathcal{F}(N)}$.
In particular, ψ_* remains surjective.

Is φ_* injective?

$\varphi(L) \oplus x(N)$ // $\simeq$

Actually, splitting gives a map $\pi : M \to L$ as well s.t. $\pi \circ \varphi = id_L$

Thus, $\pi_* \circ \varphi_* = id_{\mathcal{F}(L)}$ and thus, φ_* is injective.

$$\begin{tikzcd} 0 \arrow[r, shift left] & L \arrow[l, shift left] \arrow[r, shift left, "\varphi"] & M \arrow[l, shift left, "\pi"] \arrow[r, shift left, "\psi"] & N \arrow[l, shift left, "x"] \arrow[r, shift left] & 0 \arrow[l, shift left] \end{tikzcd}$$

The below seq. is exact as well.

We also can say:

$$0 \to N \xrightarrow{x} M \xrightarrow{\pi} L \to 0$$ is split exact

with $\varphi : L \to M$ being the splitting map.

To conclude:

$$0 \longrightarrow \mathcal{F}(L) \xrightarrow{\varphi_*} \mathcal{F}(M) \xrightarrow{\psi_*} \mathcal{F}(N) \longrightarrow 0$$

is split exact. (Verify at middle point. Do we need additivity?)

Thus, if $E = (0 \to L \to M \to N \to 0)$ is split exact,
then, for any K: $K \otimes_R E$, $\mathrm{Hom}_R(K, E)$, $\mathrm{Hom}_R(E, K)$ are all split exact.

Def^n. (Additive functor)
A functor $\mathcal{F} : R\text{-Mod} \to R\text{-Mod}$ is called additive if

covariant

$$\mathcal{F} : \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(\mathcal{F}(M), \mathcal{F}(N))$$ is

a group homomorphism.

Free resolutions (II)

$\mathbb{Z}$ $\quad \underline{\underline{2\mathbb{Z}}}$ $\quad \underline{\underline{\mathbb{Z}}}$ $\quad \mathbb{Z}/\mathbb{Z} = 0$

$\mathbb{Z}/2\mathbb{Z} \to$

$\to M, L_1, L_2$

Q! Given an s.e.s. $0 \to L \to M \to N \to 0$ of R-modules, if we know free resolutions of two of the three, can construct a free resolution for the third?

(Given our experience, we might expect that a resolution of M gives for both. However, this is "hopeless".)

Example. Take N as any module, we know that $\exists$ free F s.t. $F \to N \to 0$.
Take $0 \to \ker \to F \to N \to 0$.
know for this!

As an example

$$0 \to I \to R \to R/I \to 0.$$

↳ this can't tell for both ends

Q2 Suppose M is f.g. and F a finite rank free module s.t. $M \simeq F/K$. Is K f.g.?

No. Take R to be non-noe., let $\overset{\trianglelefteq R}{I}$ be a non-f.g. ideal. Then, $F = R$, $M = R/I$, $K = I$ is a counterexample.

Q3. With same notation, give a condition of R which forces K to be f.g.

Ans. R is Noetherian. Then, $F = R^{\oplus n}$ ← Noetherian and hence, $\overset{\subseteq F}{K}$ is f.g.

In fact, we can say more

$$\begin{tikzcd}
 & & & & & 0 \arrow[dl] \\
 & & & & K_1 \arrow[dl] & \text{f.g. again} \arrow[l] \\
0 & M \arrow[l] & F_0 = R^{\oplus n_0} \arrow[l] & R^{\oplus n_1} \arrow[l] \arrow[dl, dashed] & R^{\oplus n_2} \arrow[l] \arrow[u, dashed] & \cdots \arrow[l] \\
 & & K_0 \arrow[u] & 0 \arrow[l] & & \\
 & \text{f.g.} \arrow[ur] & & & &
\end{tikzcd}$$

Thus, over a Noetherian ring R, a f.g. module M has a free resolution of the form

$$F_\bullet \ : \ \cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0 \quad \text{where}$$

$$F_i \simeq R^{\oplus n_i}.$$

Hence, fixing bases for F_i, we can write φ_i as matrices.

Def^n. (Presentation)

A matrix representation of φ_1 is called a presentation of M.

Q4. If M is f.g and $M \simeq F_1/K_1 \simeq F_2/K_2$, where $F_1 \simeq R^{\oplus n_1}$ and $F_2 \simeq R^{\oplus n_2}$, then

(a) is $n_1 = n_2$? No. Have seen already. Can always pad more Rs.

(b) Is it necessary that $K_1 \simeq K_2$? $\longrightarrow$ $n_1 = n_2 = 1$, we know

(c) How are f_1 and f_2 related?

Think about examples: $\mathbb{Z}$, $\mathbb{k}[x] \rightarrow$ quotients,

$\swarrow$

Think of $\curvearrowright$ as $\rightarrow$ $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$

rings or $\mathbb{Z}$-modules

$\cong$

$\swarrow$

Construct modules over it

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

$\dfrac{\mathbb{k}[x,y]}{I}$ where

$I = \langle x, y\rangle, \langle x^2, y^2\rangle, \langle x^2, xy\rangle, \langle x^2, xy, y^2\rangle, \langle x\rangle$

$\dfrac{\mathbb{k}[x,y,z]}{I}$; $I = \langle x, y\rangle, \langle x, y, z\rangle, \langle x^2, xy, y^2\rangle, \ldots$

Lecture 8 (26-01-2021)

26 January 2021 11:37

Optimality of free resolutions

Example Consider $M = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module.

$$\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

Then, $M = \langle (1,1) \rangle = \langle (1,0), (0,1) \rangle$.

(i)
$$\begin{tikzcd} 0 & M \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "6"'] & 0 \arrow[l] \\ & (1,1) & 1 \arrow[l, mapsto] & & \end{tikzcd}$$

(ii)
$$\begin{tikzcd} 0 & M \arrow[l] & \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \arrow[l] & \mathbb{Z} \oplus \mathbb{Z} \arrow[l, "\begin{bmatrix}2 & \\ & 3\end{bmatrix}"'] & 0 \arrow[l] \\ & (1,0) & e_1 \arrow[l, mapsto] & & \\ & (0,1) & e_2 \arrow[l, mapsto] & & \end{tikzcd}$$

Which is more optimal? How could we make the first step optimal?
Pick gen. set which is least in cardinality?

(Note that both the gen sets above are minimal, since that is w.r.t. inclusion.)

Does this guarantee optimality in second step?

Recall. If $(R, \mathfrak{m}, \Bbbk)$ is local and M a f.g. module, then every minimal generating set of M has the same cardinality, denoted $\mu(M)$, where $\mu(M) = \dim_{\Bbbk}(M/\mathfrak{m}M)$.
[Follows from Nakayama.]

(\mu(M))

We would also want the kernels to be f.g.
Thus, we work in the following setting:

$(R, \mathfrak{m}, \Bbbk)$ is Noetherian local, M f.g.

(In fact, some authors : local includes Noetherian and non-Noetherian local is quasi-local for them)

A way to ensure optimality : Let $\mu(M) = b_0$

Map $F_0 = R^{\oplus b_0}$ onto M.

$$\begin{tikzcd}
0 & M \arrow[l] & R^{\oplus b_0} \arrow[l] & K_0 \arrow[l] & 0 \arrow[l] \\
& \langle x_1, \dots, x_{b_0} \rangle \arrow[u, equal] & & & \\
& x_i & e_i \arrow[l, mapsto] & &
\end{tikzcd}$$

Put $b_1 := \mu(K_0)$ and map $F = R^{\oplus b_1}$ onto K_0 and continue.

$$\begin{tikzcd}
0 & R^{\oplus b_0} \arrow[l] & R^{\oplus b_1} \arrow[l, "\varphi_1"'] & R^{\oplus b_2} \arrow[l, "\varphi_2"'] & \cdots \arrow[l]
\end{tikzcd}$$

Here, $b_0 = \mu(M)$ and $b_i = \mu(\operatorname{im} \varphi_i)$ for $i \geq 1$.

This is called a minimal free resolution of M over R.

(Minimal free resolution)

Q. If $\langle y_1, \dots, y_{b_0} \rangle = M$ and K_0' is the kernel obtained by mapping $e_i \mapsto y_i$, how are K_0 and K_0' related?

Is $\mu(K_0) = \mu(K_0')$. → This guarantees b_1 is well-defined.

Doesn't guarantee anything for b_2, however.

Would like to see : $K_0 \overset{?}{\cong} K_0'$? If yes, then everything would go well ad infitum.

$(n = b_0)$

Note that $y_j \in \langle x_1, \dots, x_n \rangle$ $\forall j$ and

$x_i \in \langle y_1, \dots, y_n \rangle$ $\forall i$.

$$y_j = a_{j1} x_1 + \cdots + a_{jn} x_n \qquad ; \; j = 1, \ldots, n$$

That is, $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Similarly, $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

where $A, B \in M_n(R)$.

Note $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = BA \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Idea: Try to show A is invertible. → hope to show $K_0 \cong K_0'$ with this.

Note that: modulo $\mathfrak{m}$, $\overline{BA} = \overline{id}$ since $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is a basis and there is a unique way to express it in terms of itself.

Thus, $\det(\overline{BA}) \neq 0 \mod \mathfrak{m}$

$\Rightarrow \det BA \notin \mathfrak{m}$ $\quad \big) \; (R, \mathfrak{m})$ local

$\Rightarrow \det BA$ is a unit in R

$\Rightarrow BA$ is invertible in $M_n(R)$

$\Rightarrow A$ and B are invertible in $M_n(R)$.

Lecture 9 (28-01-2021)

28 January 2021 09:16

Setup: $(R, \mathfrak{m}, k)$ local Noetherian

Would like the following: ① A minimal free resolution of M over R (say $F_\cdot$) is truly minimal in the following sense: If $G_\cdot$ is a free resolution of M, then

$$\operatorname{rank}(F_i) \leq \operatorname{rank}(G_i) \quad \forall i.$$

② If $F_\cdot$ is of the form $\cdots F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$

and $G_\cdot$ is of the form $\cdots G_1 \xrightarrow{\psi_1} G_0 \to M \to 0$

We would like to relate $\ker \varphi_1$ and $\ker \psi_1$. Would like $\ker \varphi_1 \hookrightarrow \ker \psi_1$.

All relations here are also here

③ For the next step:

$$\begin{tikzcd} F_2 \arrow[r, "\varphi_2"] & F_1 \arrow[r] & \ker \varphi_1 \arrow[r] & 0 \\ G_2 \arrow[r, "\psi_2"'] & G_1 \arrow[r] & \ker \psi_1 \arrow[r] & 0 \end{tikzcd}$$

What now? We only expect $\ker \varphi_1 \hookrightarrow \ker \psi_1$.

How do rank F_1 and rank G_1 compare?

Is there a relation between $\ker \varphi_2$ and $\ker \psi_2$?

Note that we have defined "minimal" last time, least cardinality of generating set at each step. Want to know if it is truly minimal.

The following technical Lemma takes care of it:

Lemma (Splitting Lemma) Let M and N be f.g. R-modules, where $(R, \mathfrak{m}, k)$ is a local Noetherian ring. Let F and G

be free modules mapping onto M and $M \oplus N$, respectively. Further assume that rank $F = \mu(M)$. Then, F split off G (i.e., $\exists$ an R-module P s.t. $G \simeq F \oplus P$) in a "natural way."

Moreover, if $\varphi : F \to M$ and $\psi : G \to M \oplus N$ are the given maps, then $\ker \varphi$ splits off $\ker \psi$.

rank $(F) = \mu(M)$ ensures the minimality.

Proof. Consider the s.e.s. es

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] & M \oplus N \arrow[r] & 0, \\ 0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"] & M \arrow[r] & 0, \end{tikzcd}$$

where $L = \ker \psi$ and $K = \ker \varphi$.

Note that rank $(F) \leq$ rank (G) since F maps minimally onto M. (This trivially gives a splitting of F off G, by the way.)

but we want a "natural" one!

Let $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ be bases of F and G (over R), respectively.

Let $x_i = \varphi(e_i)$ and $y_j = \psi(f_j)$. Then, $\{x_1, \dots, x_n\}$ is a minimal gen set of M over R.

(That is, $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a k-basis of $M/\mathfrak{m}M$.)

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] & M \oplus N \arrow[r] \arrow[d, shift left, "\pi"] & 0 \\ 0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"] & M \arrow[r] \arrow[u, shift left, "i"] & 0 \end{tikzcd}$$

Under the above inclusions and projection, we can "write y_js in terms of the x_is" and vice-versa.

More precisely:

$$\pi(y_1) = a_{11} x_1 + \cdots + a_{1n} x_n$$
$$\vdots$$
$$\pi(y_m) = a_{m1} x_1 + \cdots + a_{mn} x_n$$

$$\begin{bmatrix} \pi(y_1) \\ \vdots \\ \pi(y_m) \end{bmatrix} = A_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Keep in mind that the

Keep in mind that the elements of the column are "vectors" themselves

$$i(x_1) = b_{11} y_1 + \cdots + b_{1m} y_m$$

$$\vdots$$

$$i(x_n) = b_{n1} y_1 + \cdots + b_{nm} y_m$$

$$\begin{bmatrix} i(x_1) \\ \vdots \\ i(x_n) \end{bmatrix} = B_{n\times m} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

So far, we have

$$\begin{tikzcd}
f_j \arrow[rrr, mapsto] & & & y_j \arrow[dd] \\
G \arrow[rr, "\psi"] \arrow[d, dashed] & & M \oplus N \arrow[d, shift left, "\pi"] & \\
F \arrow[rr, "\varphi"] & & M \arrow[u, shift left, "i"] &
\end{tikzcd}$$

$$\pi(y_j) = \sum_{k=1}^{n} a_{jk} x_k$$

Want a blue map to make square commute.

Define $G \to F$ by $f_i \mapsto \sum_{k=1}^{n} a_{jk} e_k$.

Since G and F are free modules, we can represent it as a matrix. It is A^T.

Similarly, we have $F \to G$ given as B^T.

$$\begin{tikzcd}
G \arrow[r, "\psi"] \arrow[d, shift right, "A^T"'] & M \oplus N \arrow[d, shift left, "\pi"] \\
F \arrow[r, "\varphi"'] \arrow[u, shift right, "B^T"'] & M \arrow[u, shift left, "i"]
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, two heads, "\psi"] & M \oplus N \\
F \arrow[u, dashed] \arrow[ur, dash, dashed] \arrow[r, two heads, "\varphi"'] & M \arrow[u]
\end{tikzcd}$$

$$BA \left\{ \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & & \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right.$$

$$= \begin{bmatrix} c_1 x_1 + \cdots + c_{1n} x_n \\ \vdots \end{bmatrix}_{n\times 1} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(The "inner" and "outer" squares commute.)

Note that $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = BA \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. $\left(\text{why? } \underbrace{\pi i(x_1) = x_1}, \ldots \right)$

Go modulo m:

$$\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \overline{BA} \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix},$$ i.e.,

$$(\bar{I} - \bar{BA}) \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = 0.$$

Since $\{\bar{x}_1, \ldots, \bar{x}_n\}$ is a basis, we have $\bar{I} = \bar{BA}$.

Thus, $\det(\bar{BA}) = \bar{1}$ and hence, $\det(BA) \in 1 + \mathfrak{m}$.

Thus, $\det(BA) \in \mathcal{U}(R)$.

Hence BA is invertible.

Thus, so is $(BA)^T = A^T B^T$. $\Big\}$ in $M_n(R)$

Let $x : F \to F$ be s.t. $x A^T B^T : F \to F$ is id_F.

Thus, $B^T : F \to G$ and $xA^T : G \to F$ are s.t.

$$(xA^T) B^T = \mathrm{id}_F.$$

$B^T x : F \to G$

$A^T : G \to F$

$G - B^T x(F) \oplus \ker A^T$

$$\begin{tikzcd} 0 \arrow[r] & \ker(xA^T) \arrow[r] & G \arrow[r, "xA^T"] & F \arrow[l, dashed, bend left, "B^T"] \arrow[r] & 0 \end{tikzcd}$$

Thus, the above s.e.s. splits and hence

$$G = B^T(F) \oplus \ker(xA^T).$$

$\Big\}$ shows the naturality!

Next, we show : $L = B^T(K) \oplus (\ker(xA^T) \cap L)$

(Try it!)

- $B^T(K) \subset L$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & G \arrow[r, "\psi"] \arrow[d, shift left, "A^T"] & M \oplus N \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r] & F \arrow[r, "\varphi"'] \arrow[u, shift left, "B^T"] \arrow[loop below, "x"] & M \arrow[r] & 0 \end{tikzcd}$$

<u>Proof.</u> Recall: $L = \ker \psi$ and $K = \ker \varphi$.

Let $x \in K = \ker \varphi$.

Then, $\varphi(x) = 0$ and thus, $i(\varphi(x)) = 0$.

But $i(\varphi(x)) = \psi(B^T(x))$.

$\therefore$ $B^T(x) \in \ker \psi = L$. ∎

Lecture 10 (01-02-2021)

01 February 2021 10:31

Recall:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] \arrow[d, shift left, "\alpha"] & M \oplus N \arrow[r] \arrow[d, shift right, "\pi"'] & 0 \\ 0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"] \arrow[u, shift left, "\beta"] & M \arrow[r] \arrow[u, shift right, "i"'] & 0 \end{tikzcd}$$

$(i\varphi = \psi\beta)$

Where β is multiplication by B^T and α by A^T.

(After we fixed an appropriate basis.)

We showed: $G = \beta(F) \oplus \ker(\chi\alpha)$

Claim. $\beta(K)$ is a direct summand of L.

Proof. We show that: $L \cap \beta(F) = \beta(K)$

$(\supseteq)$ Let $x \in K$. Then, $\varphi(x) = 0$ and

$$\begin{array}{c} i\varphi(x) = 0 \\ \| \\ \psi\beta(x) \end{array}$$

Thus, $\beta(x) \in \ker\psi = L$.

Thus, $\beta(K) \subseteq L \cap \beta(F)$.

$(\subseteq)$ Suppose $y \in L \cap \beta(F)$. Then, $y = \beta(x)$ for some $x \in F$.

We show $x \in K$, i.e., $\varphi(x) = 0$.

Note that
$$\begin{aligned} i\varphi(x) &= \psi\beta(x) \\ &= \psi(y) \quad (\because y \in L = \ker\psi) \\ &= 0 \end{aligned}$$

$\therefore$ $i\varphi(x) = 0$. Since i is 1-1, we get $\varphi(x) = 0$.

This finishes the proof. ⊠

———X———

Restating the lemma:

Lemma: Let M and N be R-modules, $\varphi : F \to M$, $\psi : G \to M \oplus N$ be onto, $\ker\varphi = K$, $\ker\psi = L$, $F = R^{\oplus n}$, $G = R^{\oplus m}$, and $n = \mu(M)$. Consider the s.e.s.es

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, hook] & G \arrow[r, "\psi"] & M \oplus N \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\varphi"'] \arrow[u, "\beta"] & M \arrow[r] \arrow[u, "i"'] & 0
\end{tikzcd}$$

Then, $\exists \beta : F \to G$ s.t.

① $\psi\beta = i\varphi$.

② $\exists \gamma : G \to F$ onto s.t. β is a splitting.

③ $\beta|_K : K \to L$ is a splitting.

(Part of ③ that $\varphi(K) \subset L$.)

Notation: $K | L$ that K is (isomorphic to) a direct summand of L.

Note:

$$\begin{tikzcd}
0 \arrow[r] & \overset{x\alpha}{\overset{\shortparallel}{\ker\gamma}} \arrow[r] & G \arrow[r, shift left, "\gamma"] & F \arrow[l, shift left, dashed, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \ker\gamma \cap L \arrow[r] & L \arrow[r, shift left, "\gamma"] & K \arrow[l, shift left, dashed, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

Some observations:

① Free modules have a lifting property:

$$\begin{tikzcd}
G \arrow[r, "\psi"] & M \oplus N \arrow[r] & 0 \\
F \arrow[u, dashed, "\beta"] \arrow[ur, "i\varphi"'] & &
\end{tikzcd}$$

constructed this → β

$\{e_1, \ldots, e_n\}$

Suppose $e_1 \overset{i\varphi}{\longmapsto} z_1 \in M \oplus N$.

Let $z_1' \in G$ be s.t. $\psi(z_1') = z_1$.

Then, we define $\beta(e_1) := z_1'$.

Can do this for every e_i.

Then, this defines a map $\beta : F \to G$ since F is free with basis $\{e_i\}_i$.

All we really used is: ① ψ is onto.

② F is free.

More generally, we have: (Lifting property)

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \arrow[r] & 0 \\ F \arrow[u, dashed, "\bar{x}"] \arrow[ur, "x"', "2"] & & \end{tikzcd}$$

Let φ be onto and F free.

Then, $\exists\, \bar{x} : F \to M$ s.t.

$$\varphi \bar{x} = x.$$

(rank $F < \infty$ not necessary.)

Def^n. This defines an R-module being projective.
(One that has the lifting property as above.)

(Projective modules)

Eg. Free modules are projective.

② Let β, β' be two lifts. How are they related?

$\beta - \beta'$ must be in ker:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & G \arrow[r] & M \oplus N \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r] & F \arrow[r] \arrow[ul, dashed, "\beta-\beta'"] \arrow[u, shift left, "\beta"'] \arrow[u, shift right=-3, "\beta'"] & M \arrow[r] & 0 \end{tikzcd}$$

Suppose we have the free resolutions:

$$\begin{tikzcd} G_1 \arrow[rrr] \arrow[dr] & & & G \\ & L \arrow[r] \arrow[urr, leftrightarrow] & 0 & \\ & & & F_0 \arrow[u, shift left] \arrow[u, shift right] \arrow[ull, "\beta-\beta'"] \end{tikzcd}$$

By the lifting property, $\exists$ a map $F_0 \to G_1$ as:

$$\begin{tikzcd}
G_1 \arrow[r] & G_0 \arrow[r] & {} \\
F_1 \arrow[r] & F_0 \arrow[r] \arrow[u, shift left] \arrow[u, shift right] \arrow[ul, dashed] & {}
\end{tikzcd}$$

This idea will define a "homotopy" between two maps of chain complexes

③ Definition of chain maps :

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & F_0 \arrow[r] \arrow[dd, dashed] \arrow[ddr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[dd] & 0 \\
& F_1 \arrow[u] \arrow[ur] \arrow[dd, dashed] & & & \\
0 \arrow[r] & L \arrow[r] & G_0 \arrow[r] & M \oplus N \arrow[r] & 0 \\
& G_1 \arrow[u] \arrow[ur] & & &
\end{tikzcd}$$

This idea helps to define maps between chain complexes.

Lecture 11 (02-02-2021)

02 February 2021 11:33

Consequences of the proof of the Splitting Lemma :

① if $N=0$, then $L = \beta(K) \oplus \ker(\chi\, \alpha)$.

② if $m=n$, then ($N=0$ and) $L = \beta(K)$.

In particular, $L \cong K$.

Consequence of the Splitting Lemma:

Thm. Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring, M a f.g. R-module and $F_\cdot \to M$ be a minimal R-free resolution of M. If $G_\cdot \to M$ is any R-free resolution of M, then $\forall i \geq 0$, $\exists$ injective maps $\beta_i : F_i \to G_i$ satisfying:

(1)

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, equal, "\mathrm{id}_M"'] \arrow[dr, phantom, "\circlearrowright"] & F_0 \arrow[l, "\varphi_0"'] \arrow[d, "\beta_0"] \arrow[dr, phantom, "\circlearrowright"] & F_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta_1"] & \cdots \arrow[l] \\
0 & M \arrow[l] & G_0 \arrow[l, "\psi_0"] & G_1 \arrow[l, "\psi_1"] & \cdots \arrow[l]
\end{tikzcd}$$

$\psi_i \beta_i = \beta_{i-1} \varphi_i$ or $\psi \beta = \beta \varphi$ or each square commutes

② $\beta_i(F_i) \mid G_i$, i.e., $G_i = \beta_i(F_i) \oplus \cdots$

In particular, $\operatorname{rank}(F_i) \leq \operatorname{rank}(G_i)$. (Since β_i is 1-1.)

Proof. We use induction on i $(=: n)$ to show that

$\exists \beta_n : F_n \to G_n$ satisfying ① and ②

and ③ $\ker \psi_n = \beta(\ker(\varphi_n)) \oplus \cdots$

The base case $n=0$ is the splitting lemma.

By induction, assume that $\forall i \leq n$, we have

$\beta_i : F_i \to G_i$

satisfying ①, ②, and ③.

$$\begin{tikzcd}
0 \arrow[r] & L_n \arrow[r] & G_n \arrow[r, "\psi_n"] \arrow[dr, phantom, "\circlearrowright"] & G_{n-1} \arrow[r] & \cdots \\
0 \arrow[r] & K_n \arrow[r] & F_n \arrow[r, "\varphi_n"'] \arrow[u, "\beta_n"] & F_{n-1} \arrow[r] \arrow[u, "\beta_{n-1}"'] & \cdots
\end{tikzcd}$$

We know that $L_n = \beta(K_n) \oplus \cdots$

The splitting lemma applied to

$$\begin{tikzcd}
0 \arrow[r] & \ker \psi_{n+1} \arrow[r] & G_{n+1} \arrow[r] & L_n \arrow[r] & 0 \\
0 \arrow[r] & \ker \varphi_{n+1} \arrow[r] & F_{n+1} \arrow[r] & K_n \arrow[r] \arrow[u, "\beta_n"'] & 0
\end{tikzcd}$$

gives β_{n+1} satisfying ①, ② and ③.

Remark. The compatibility of β with φ and ψ shows that every free resolution of M contains a minimal free resolution.
(Can think of β_is as inclusions $F_i \hookrightarrow G_i$ and ψ_is restrict to maps $F_i \to F_{i-1}$ where it becomes φ_i.)

Next consequence:

Thm If F and G are two minimal resolutions of M over R, then rank (F_i) = rank (G_i).

Remark. In fact, the two resolutions are "isomorphic".
(We have chain maps which are isomorphisms. ← definition pending)

Defn Let $F_\bullet$ be a minimal free resolution (m.f.r.) of a f.g. module M over a Noetherian local ring R.
Then,

① the i^{th} Betti number of M over R, denoted $\beta_i^R(M) = \text{rank}_R(F_i)$.

② $\ker \varphi_i = \text{im}\, \varphi_{i+1}$ is called the $(i+1)^{st}$ syzygy module of M over R, denoted $\Omega^R_{i+1}(M)$.

(Note the shift of index, $K_0 = \Omega^R_1(M)$.)

The above is well-defined, by the discussion above.

Note

① Thus, $F_\cdot$ breaks into short exact sequences (with $\Omega_0(M) = M$):

$$0 \longrightarrow \Omega_{i+1}(M) \longrightarrow R^{\oplus \beta_i} \longrightarrow \Omega_i(M) \to 0.$$

$\therefore$ $\Omega_{i+1}(M)$ is the first syzygy of $\Omega_i(M)$ since β_i chosen minimaly

② $\beta_i = \mu(\Omega_i(M))$.

Remark The above also makes sense in the category of graded modules over graded rings.

(There's a notion of graded local, graded NAK, graded free resolutions, graded free resolution)

Testing minimality:

Q. What does it mean for $\{x_1, \dots, x_n\}$ to be a minimal generating set of M?

(Notation: Let $\varphi : F = \bigoplus_{i=1}^{n} Re_i \twoheadrightarrow M$ with $e_i \mapsto x_i$. Assume it actually is a gen. set.)

$\{x_1, \dots, x_n\}$ is a minimal gen. set of M over R

$\Leftrightarrow$ $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a K-basis of $M/\mathfrak{m}M$

Lecture 12 (04-02-2021)

04 February 2021 09:31

Q. What does it mean for $\{x_1, \dots, x_n\}$ to be a minimal generating set of M?
(Notation: Let $\varphi : F = \bigoplus_{i=1}^{n} Re_i \twoheadrightarrow M$ with $e_i \mapsto x_i$. Assume it actually is a gen. set.)

$\{x_1, \dots, x_n\}$ is a minimal gen. set of M over R

$\Leftrightarrow$ $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a K-basis of $M/\mathfrak{m}M$ $\Leftrightarrow$ rank $F = \mu(M)$

observe :

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\
F/\mathfrak{m}F \arrow[r, dashed, "\bar{\varphi}"'] & M/\mathfrak{m}F
\end{tikzcd}$$

$\varphi : F \twoheadrightarrow M$ induces an onto map $\bar{\varphi} : F/\mathfrak{m}F \longrightarrow M/\mathfrak{m}M$ of k-vector spaces. (Can verify manually or observe this as tensor $\otimes_R k$.)

Now: $\{x_1, \dots, x_n\}$ is a minimal gen set of M $\Leftrightarrow$ $\bar{\varphi}$ is an iso.

$(\Rightarrow)$ Can use right-exactness to show $\bar{\varphi}$ is onto.
However, $\dim_k (F/\mathfrak{m}F) = \dim_k (M/\mathfrak{m}F) = n < \infty$. $\therefore \bar{\varphi}$ iso. ☒

$(\Leftarrow)$ If $S = \{x_1, \dots, x_n\}$ is not minimal, then $\exists A \subsetneq S$ minimal.
But then $\bar{A}$ is a basis with $< n$ elements.
Contradiction since $\bar{F}$ has dim. n. ☒

Q. What does this say about $\ker \varphi$?

Ans. $\ker \varphi \subset \mathfrak{m} F$.

Proof. Let $y \in \ker \varphi$. Write $y = \sum a_i e_i$.

Claim: $a_i \in \mathfrak{m}$ $\forall i$. (This would prove $\ker \varphi \subset \mathfrak{m}F$.)

Proof. $\sum a_i x_i = 0$ in M over R

$\Rightarrow \sum \bar{a}_i \bar{x}_i = 0$ in $M/\mathfrak{m}M$ over $R/\mathfrak{m}$

$\Rightarrow \bar{a}_i = 0$ $\forall i$ in $R/\mathfrak{m}$

$\Rightarrow\ a_i \in \mathfrak{m}\ \ \forall i$ in R

Alternate proof by diagram:

$$\begin{tikzcd}
 & & y \arrow[r, mapsto] & y \arrow[r, mapsto] & 0 \\
 & 0 \arrow[r] & \ker \varphi \arrow[r] & F \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\
 & & & F/\mathfrak{m}F \arrow[r, "\bar{\varphi}"'] & M/\mathfrak{m}M
\end{tikzcd}$$

$\therefore\ y \longmapsto 0 \in M/\mathfrak{m}M$

$\therefore y \longmapsto 0 \in F/\mathfrak{m}F$

$\therefore\ y \in \mathfrak{m}F.$

The above is again equivalent. ⇝ do element wise or use the diagram

Thus, we have: $(R, \mathfrak{m}, k)$ local Noetherian, M f.g.,
$M = \langle x_1, \dots, x_n \rangle$.

$F = Re_1 \oplus \cdots \oplus Re_n$, $\varphi : F \to M$ where $e_i \mapsto x_i$.

Then TFAE

(1) $\{x_1, \dots, x_n\}$ is a minimal gen. set of M.

(2) $\{\bar{x}_1, \dots, \bar{x}_n\}$ is a basis of $M/\mathfrak{m}M$ (over k)

(3) rank $F = \mu(M)$

(4) $\bar{\varphi} : F/\mathfrak{m}F \to M/\mathfrak{m}M$ is an iso.

(5) $\ker \varphi \subset \mathfrak{m} F$

Consider $F = F_0$ and F_1 a free module mapping onto $\ker\varphi$.
Then, im $F_1 \subset \mathfrak{m} F_0$. Equivalently, if im $F_1 \subset \mathfrak{m} F_0$, then $\ker \varphi \subset \mathfrak{m} F_0$ and hence, $\{x_1, \dots, x_n\}$ is a min. gen. set of M.

Thus, we have the following: With M and R as above, let $F_\bullet \to M$ be a free resolution of M over R.
Then, $F_\bullet$ is minimal $\iff$ im $\varphi_i \subset \mathfrak{m} F_{i-1}\ \ \forall i \geq 1$

$$\begin{tikzcd}
 & & & \ker \varphi_{i-1} \arrow[dr] & & \\
\cdots \arrow[r] & F_{i+1} \arrow[r] & F_i \arrow[ur] \arrow[rr, "\varphi_i"] & & F_{i-1} \arrow[r, "\varphi_{i-1}"] & \cdots
\end{tikzcd}$$

im φ_i = ker $\varphi_{i-1} \subset \mathfrak{m}F_{i-1}$ ⇝ since F_{i-1} maps minimally onto ker φ_{i-2}.

$\iff$ (writing φ_i as a matrix) the entries of φ_i are in $\mathfrak{m}$.

Q. Let $G_\cdot \longrightarrow M$ be a free resolution of M over R. (R local Noetherian, M f.g.). If $\mathrm{rank}(G_i) = n_i$, what can we say about the Betti numbers of M?

Q. If $G_\cdot$ is minimal after one stage, then can we write it as $(F_\cdot) \oplus N$?

Can we drop local-ness?

Lecture 13 (08-02-2021)

08 February 2021 10:36

Note: Over a Noetherian ring, our convention has and will be that the free modules in a free resolution have finite ranks.

Remarks $(R, \mathfrak{m}, \Bbbk)$ is Noetherian local, M is a f.g. R-module

① Bounds on the Betti numbers of M. How to get?

- Upper bound: Construct any free resolution of M over R.
  By def^n, we will have $\beta_i^R(M) \le \mathrm{rank}(F_i)$.

Obs. $\beta_0^R(M) = \mu(M) = \dim_{\Bbbk}(M/\mathfrak{m}M)$.

- Lower bound: Find a complex of free modules which can be "put inside" a minimal free resolution of M.
  (Easier said than done.)

(Open) Conjecture (Buchsbaum – Eisenbud – Horrocks) Given M, $\exists c$ (an invariant of M) such that $\forall i$ $\beta_i^R(M) \ge \binom{c}{i}$.

Not stated precisely

(Why binomial co-efficients? See Koszul complexes.)

② Poincaré series: $P_M^R(t) = \sum_{t \geq 0} \beta_i^R(M) t^i$

(polynomial iff minimal resolution is finite length

For $R = \frac{\Bbbk[x]}{\langle x \rangle}$ and $M = \Bbbk$, we had seen (in tut.) $P_M^R(t) = \sum_{t \ge 0} t^i = \frac{1}{1-t}$.

Q. Can ask: Is $P_M^R(t) \in \mathbb{Z}(t)$?

↳ rational polynomial

was asked by Serre

Serre proved: For $M = \Bbbk \leftarrow R/\mathfrak{m}$ the Poincare series is term-wise bounded above by a rational function (which comes from a certain Koszul complex) and asked the above Q for $M = \Bbbk$.

Example ① $R = \dfrac{\Bbbk[x,y]}{\langle x,y\rangle^2}$, $M = \Bbbk$. Q. Is $P_{\Bbbk}^{R}(t)$ rational?

↪ Noe. local

② $R = \dfrac{\Bbbk[x,y]}{\langle x^2, xy\rangle}$ $M = \Bbbk$. ↗

Anick proved that : $P_{\Bbbk}^{R}(t)$ is not necessarily rational.

Bøgvad used "idealisation" to show that this fails even for nice rings (Gorenstein rings).
(See A. Kustein's write-up " Georgia Southern University talk".)

③ Suppose S is an R-algebra (via φ) and $F_\bullet \to M \to 0$ is a free resolution of M over R.

ⓐ Recall $S \otimes_R F_\bullet$ is indeed a complex of free S-modules where

$$S \otimes_R F_1 \to S \otimes_R F_\bullet \to S \otimes_R M \to 0 \quad \text{is exact.}$$

We have already seen an example where $S \otimes_R F_\bullet$ is not free.

Q. When will it give a free resolution?

A. Tensoring with S should preserve exactness.

Defn. (Flat modules) An R module K is flat if $K \otimes_R -$ is exact.

Eg. If $A \subset R$ is an m.c.s., take $K = R_A$.

Remark $K \otimes_R -$ being exact says that s.e.s es go to s.e.ses.
However, if $C_\bullet$ is any exact complex of R-modules, then above implies $K \otimes_R C_\bullet$ is exact.
(Reason: Every long exact sequence (l.e.s.) breaks up into

s.e.ses.)

Thus, if S is an R-algebra which is flat as an R-module, then $S \otimes_R F_\cdot \to S \otimes_R M \to 0$ is an S-free resolution of $S \otimes_R M$.

ⓑ What about minimality?

Suppose $F_\cdot \to M$ is a minimal R-free resolution.

Q. Does $S \otimes_R F$ give a minimal S-free resolution of $S \otimes_R M$?

To talk about that, we assume that S is Noetherian and local.

Let $(S, \mathfrak{n}, l)$ be local, $\varphi : R \to S$ a ring map, and $F_\cdot \to M \to 0$ a min'l R-free resolution of M. Is $S \otimes_R F_\cdot \to S \otimes_R M \to 0$ a min'l free resolution of $S \otimes_R M$ over S?

Note that: ① Just flatness is not enough. (Find an example.)

(Take $R = \mathbb{Q}[x]$ and $S =$ field of fracs. $= \mathbb{Q}(x)$.)

② If $\varphi(\mathfrak{m}) \subset \mathfrak{n}$, then the entries of the matrices in $S \otimes_R F_\cdot$ lie in $\mathfrak{n}$ and hence, $S \otimes_R F_\cdot \to S \otimes_R M \to 0$ is a min'l free resolution.

Thus, we need that $\varphi : R \to S$ is a flat, local map.

Def^n. (Flat, local map) A ring map $\varphi : R \to S$ is flat if S is a flat R-module (via φ) and local if $(S, \mathfrak{n})$ is local with $\varphi(\mathfrak{m}) \subset \mathfrak{n}$.

Example. If $A \subset R$ is an m.c.s., then $S = R_A$ is a flat R-algebra, local if $A = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$ but may not satisfy $\varphi_A : R \to R_A$ is local.

Ex. Suppose $\varphi : R \to S$ is flat.
Suppose $F_\cdot \to M$ is an R-free resolution. Suppose $S \otimes_R F_\cdot \to S \otimes_R M$ is a min'l S-free resolution. Then, $F_\cdot \to M$ was minimal.

Example There is the notion of "completion in the m-adic topology," denoted $\hat{R}$ and the natural map $\varphi : R \to \hat{R}$ is flat, local.

Aside Standard reduction in many Problems: WLOG, assume R is a complete local domain.

$$R \xrightarrow{\text{localise}} (R, m) \xrightarrow{\text{complete}} (\hat{R}, m\hat{R}) \xrightarrow[\text{suitable prime}]{\text{go mod}} \text{complete local domain?}$$

Complexes and Homology

Def^n ① Given complexes $C_\cdot$ and $D_\cdot$, a map of complexes $\alpha_\cdot : C_\cdot \to D_\cdot$ is a collection of R-linear maps $\alpha_i : C_i \to D_i$ such that each sequence below commutes. Also called a chain map.

$$\begin{tikzcd}
\cdots \arrow[r] & C_i \arrow[r, "\partial^c"] \arrow[d, "\alpha_i"'] \arrow[dr, phantom, "\circlearrowright"] & C_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_i \arrow[r, "\partial^D"'] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

$(\alpha \partial^c = \partial^d \alpha)$

② $C_\cdot$ and $D_\cdot$ are isomorphic if $\exists \alpha_\cdot : C_\cdot \to D_\cdot$ s.t. $\alpha_i : C_i \to D_i$ is an isomorphism for all i.

③ Given $\alpha_\cdot, \beta_\cdot : C_\cdot \to D_\cdot$ chain maps, we say $\alpha_\cdot$ is homotopic to $\beta_\cdot$ $\exists$ maps $\delta_i : C_i \to D_{i+1}$ s.t.

$$\alpha_i - \beta_i = \delta_{i-1} \partial_i^c + \partial_{i+1}^D \delta_i.$$

$$\begin{tikzcd}
 & \cdots \arrow[r] & C_i \arrow[r, "\partial^c"] \arrow[dl, "\delta_i"'] \arrow[d, shift right, "\alpha_i"'] \arrow[d, shift left, "\beta_i"] & C_{i-1} \arrow[r] \arrow[dl, "\delta_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r] & D_i \arrow[r] & \cdots &
\end{tikzcd}$$

∂^D

(Chain maps, homotopic maps)

Observations:

① ⓐ A chain map $\alpha_\cdot : C_\cdot \to D_\cdot$ induces a map

$$H_\cdot(\alpha) : H_\cdot(C_\cdot) \to H_\cdot(D_\cdot), \quad \text{i.e.,}$$

we get maps

$$H_\cdot(\alpha_i) : H_i(C_\cdot) \longrightarrow H_i(D).$$

This follows since $\alpha_i(\ker(\partial_i^C)) \subset \ker(\partial_i^D)$ and $\alpha_i(\operatorname{im}(\partial_{i+1}^C)) \subset \operatorname{im}(\partial_{i+1}^D)$.

$$\begin{tikzcd}
 & \ker(\partial_i^C) \arrow[r, "\alpha"] \arrow[d] & \ker(\partial_i^D) \arrow[d] & \\
\frac{\ker(\partial_i^c)}{\operatorname{im}(\partial_{i+1}^D)} = & H_i(C) \arrow[r, dashed] & H_i(D) & = \frac{\ker(\partial_i^c)}{\operatorname{im}(\partial_{i+1}^D)}
\end{tikzcd}$$

ⓑ If $\alpha_\cdot = \mathrm{id}$, then $H_\cdot(\alpha_\cdot) = \mathrm{id}$.

ⓒ If $C_\cdot \xrightarrow{\alpha_\cdot} C_\cdot' \xrightarrow{\beta_\cdot} C_\cdot''$ are chain maps, then $C_\cdot \xrightarrow{\beta_\cdot \circ \alpha_\cdot} C_\cdot''$ is a chain map and

$$\begin{tikzcd}
H_\cdot(C_\cdot) \arrow[r, "H_\cdot(\alpha_\cdot)"] \arrow[dr, bend right, "H_\cdot(\beta_\cdot)\circ H_\cdot(\alpha_\cdot)"'] & H_\cdot(C_\cdot') \arrow[d, "H_\cdot(\beta_\cdot)"] \\
\arrow[ur, phantom, "\circlearrowright"] & H_\cdot(C_\cdot'')
\end{tikzcd}$$

Commutes.

② If $\alpha_\cdot$ and $\beta_\cdot$ are homotopic, then $H_\cdot(\alpha_\cdot) = H_\cdot(\beta_\cdot)$, i.e., homotopic chain maps induce the same maps on homology.

Lecture 14 (09-02-2021)

09 February 2021 11:35

Remark $H_\bullet$ is a functor from the category of chain complexes to chain complexes. We think of $H_\bullet(C_\bullet)$ as a chain complex with zero differential (i.e, all maps are the zero maps).

$$\cdots \to H_n(C_\bullet) \xrightarrow{0} H_{n-1}(C_\bullet) \to \cdots$$

Putting 0 maps ensures that homology is again $H_\bullet(C_\bullet)$.

Prop^n If $\alpha_\bullet$ and $\beta_\bullet$ are homotopic, then $H_\bullet(\alpha_\bullet) = H_\bullet(\beta_\bullet)$.

Proof. Denote $H_i(\alpha_\bullet)$ as $\alpha_{i*} : H_i(C_\bullet) \longrightarrow H_i(D_\bullet)$.

Claim $\forall i : \alpha_{i*} = \beta_{i*}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial^C"] & C_i \arrow[r, "\partial^C"] \arrow[dl, "\gamma_i"'] \arrow[d, shift right, "\alpha_i"'] \arrow[d, shift left, "\beta_i"] & C_{i-1} \arrow[r] \arrow[dl, "\gamma_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r, "\partial^D"'] & D_i \arrow[r, "\partial^D"'] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\alpha_i - \beta_i = \gamma_{i-1}\partial^C + \partial^D \gamma_i$$

Notation: $Z_i(C) = \ker \partial_i^C$, $B_i(C) = \operatorname{im} \partial_{i+1}^C$, $H_i(C) = Z_i(C)/B_i(C)$.

Let $x \in Z_i$. Consider $\bar{x} \in H_i(C_\bullet)$. Want: $\alpha_{i*}(\bar{x}) = \beta_{i*}(\bar{x})$.
That is, want to show

$$\alpha_i(x) - \beta_i(x) \in B_i(D).$$

However,

$$\begin{aligned}(\alpha_i - \beta_i)(x) &= \gamma_{i-1}\overbrace{\partial^C(x)}^{=0 \text{ since } x \in Z_i \subseteq \ker \partial^C} + \partial^D \gamma_i(x) \\ &= \partial^D \gamma_i(x) \in \operatorname{im} \partial_{i+1}^D = B_i(D). \quad \blacksquare\end{aligned}$$

③ Let $C_\cdot$ and $D_\cdot$ be complexes. We say that $C_\cdot$ is homotopic to $D_\cdot$ (denoted $C_\cdot \simeq D_\cdot$) if
$\exists$ chain maps $\alpha_\cdot : C_\cdot \to D_\cdot$, $\beta_\cdot : D_\cdot \to C_\cdot$ such that
$$\alpha_\cdot \beta_\cdot \simeq id_{D_\cdot} \quad \text{and} \quad \beta_\cdot \alpha_\cdot \simeq id_{C_\cdot}.$$

(homotopic complexes)

If $C_\cdot \simeq D_\cdot$ then $H_\cdot(C_\cdot) = H_\cdot(D_\cdot)$.

Q. If $C_\cdot$ and $D_\cdot$ are homotopic, are they isomorphic?

No.

Comment. Let $F_\cdot$: … be a free resolution of M.

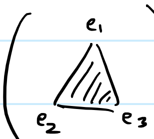
Think of M as a complex

$$\begin{tikzcd}
F_\cdot : \cdots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r, "\varphi_1"] \arrow[d] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\
M_\cdot : \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$M = \operatorname{coker} \varphi_1$.

(See "derived categories".)

Koszul complex

Boundary (

) $= e_1e_2 + e_2e_3 + e_3e_1$ (formal sum)

$= e_1 \, e_2 + e_2e_3 - e_1e_3$

Boundary $(e_1 \, e_2 \, e_3)$

$$\partial(e_1 \wedge e_2 \wedge e_3) = \underset{\substack{\uparrow \\ \text{excluding } e_1}}{e_2 \wedge e_3} - \underset{\substack{\uparrow \\ \text{excl. } e_2}}{e_1 \wedge e_3} + \underset{\substack{\uparrow \\ \text{exc. } e_3}}{e_1 \wedge e_2}$$

$$\partial(e_2 \wedge e_3) = e_3 - e_2$$
$$\partial(e_1 \wedge e_3) = e_3 - e_1$$
$$\partial(e_1 \wedge e_2) = e_2 - e_1$$

$$\therefore \partial^2(e_1 \wedge e_2 \wedge e_3) = (e_3 - e_2) - (e_3 - e_1) + (e_2 - e_1)$$
$$= 0$$

$$0 \to \mathbb{Z}(e_1\wedge e_2\wedge e_3) \xrightarrow{\partial} \begin{matrix}\mathbb{Z}(e_2\wedge e_3)\\ \oplus \\ \mathbb{Z}(e_1\wedge e_3)\\ \oplus \\ \mathbb{Z}(e_1\wedge e_2)\end{matrix} \xrightarrow{\partial} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \to 0$$

A different boundary map:

Let $\varphi : \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \to \mathbb{Z}$ be a map.

Define

$$\partial(e_1\wedge e_2\wedge e_3) = \varphi(e_1)\, e_2\wedge e_3 - \varphi(e_2)\, e_1\wedge e_3 + \varphi(e_3)\, e_1\wedge e_2.$$

Koszul complex on $a_1, \ldots, a_n \in R$. Define $M = Re_1 \oplus \cdots \oplus Re_n$ and $\varphi : M \to R$ by $e_i \mapsto a_i$.

Write ΛM as:

$$0 \to \Lambda^n M \xrightarrow{\partial} \Lambda^{n-1} M \to \cdots \to \Lambda^2 M \xrightarrow{\partial} \Lambda^1 M \xrightarrow{\varphi} R \to 0$$

where

$$\partial(e_{i_1}\wedge\cdots\wedge e_{i_k}) = \sum_{1\le j\le k} (-1)^{j-1}\varphi(e_{i_j})\, e_{i_1}\wedge\cdots\wedge \underset{\substack{\uparrow\\ \text{drop}}}{\widehat{e_{i_j}}}\wedge\cdots\wedge e_{i_k}$$

Verify that the above is a complex.

Assignment. ① Explicitly write down the modules in $K_\bullet(\underline{a})$.
$\underline{a} = a_1, \ldots, a_n$.

② Identify $H_j(K_\bullet(\underline{a}))$ at least when $j = 0, 1, n$.

③ For $n = 1, 2, 3$ try many examples.

④ Is it a free resolution of $H_0(K_\bullet(\underline{a}))$?

[When is it ?]

↳ when exalt

Lecture 15 (11-02-2021)

11 February 2021 09:40

(Direct sums and tensor products of complexes)

Q. We had discussed "chain complexes" as a category.
Can you think of direct sums and tensor products (of two complexes)?

ⓐ Can verify direct sum works in natural way. Define

$$\begin{tikzcd} (C_\bullet \oplus C'_\bullet)_i = C_i \oplus C'_i \arrow[d, "\partial = (\partial^C_i \, , \, \partial^{C'}_i)"] \\ (C_\bullet \oplus C'_\bullet)_{i-1} = C_{i-1} \oplus C'_{i-1} \end{tikzcd}$$

and verify $\partial^2 = 0$.

ⓑ

$$\begin{tikzcd} (C \otimes C')_i \arrow[d] & = & C_i \otimes C'_i \arrow[d, "\partial = \partial^c \otimes \partial^{c'}"] \\ (C \otimes C')_{i-1} & = & C_{i-1} \otimes C'_{i-1} \end{tikzcd}$$

This does define a chain complex but we won't consider this as the tensor product! (Informal: Note that $\partial^2 = 0$ above is because we get $\partial^2 = (\partial^c)^2 \otimes (\partial^{c'})^2 = 0 \otimes 0 = 0$. However just one component is enough for $\otimes$ to be 0. This is "overkill".)

Example : $C_\bullet$: $\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ R-complex

$C'_\bullet$: $0 \to S \to 0$ S : R-alg.

The above product gives : $0 \to S \otimes C_0 \to 0$

Would want something like :

$$\to S \otimes C_n \to \cdots \to S \otimes C_1 \to S \otimes C_0 \to 0.$$

Q. Define the complex $(C_\bullet \otimes C'_\bullet)$. Identify a "good" differential.

$$\bigoplus_{i=0}^{n} C_i \otimes C'_{n-i} \ ?$$

Snake lemma

Given two short exact sequences of R-modules with compatible maps between them

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha"'] & M_1 \arrow[r, "\psi_1"] \arrow[d, "\beta"'] & N_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & L_2 \arrow[r, "\varphi_2"'] & M_2 \arrow[r, "\psi_2"'] & N_2 \arrow[r] & 0,
\end{tikzcd}$$

we get an exact sequence

$$0 \to \ker\alpha \xrightarrow{\varphi_1} \ker\beta \xrightarrow{\psi_1} \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \xrightarrow{\bar\varphi_2} \operatorname{coker}\beta \xrightarrow{\bar\psi_2} \operatorname{coker}\gamma \to 0.$$

($\varphi_1, \psi_1, \bar\varphi_2, \bar\psi_2$ above are the induced maps)

δ is called the connecting homomorphism.

[Tool: If $0 \to K \to 0$ is exact, then $K = 0$.]

Consequences:

① If two out of α, β, γ are isomorphisms, then so is the third.

② β injective $\Rightarrow$ α injective

β surjective $\Rightarrow$ γ surjective

③ β is omorphism $\Rightarrow$ α injective, γ surjective AND $\ker\gamma \overset{\delta}{\cong} \operatorname{coker}\alpha$

Proofs. φ is 1-1 $\Leftrightarrow$ $\ker\varphi = 0$

φ is onto $\Leftrightarrow$ $\operatorname{coker}\varphi = 0$

$0 \to A \to B \to 0$ is exact $\Leftrightarrow$ $A \cong B$

Q. If N_1, N_2 are submodules of M such that $M/N_1 \cong M/N_2$, is $N_1 \cong N_2$?

Look at $0 \to N_1 \to M \to M/N_1 \to 0$

Look at

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, "id"'] & M/N_1 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N_2 \arrow[r] & M \arrow[r] & M/N_2 \arrow[r] & 0
\end{tikzcd}$$

"Proof" Using the lemma, $N_1 \cong N_2$.

Attempt at Tensor product:

Consider $D_\cdot$ defined by

$$D_i = \bigoplus_{j=0}^{i} C_j \otimes C'_{i-j}$$

① $D_i \xrightarrow{\partial} D_{i-1}$

$$\partial(\underset{\underset{C_j \otimes C'_{j-1}}{\Cap}}{C_j \otimes C'_{i-j}}) = \underset{\underset{C_{j-1} \otimes C'_{i-j}}{\Cap}}{(\partial^c C_j) \otimes C'_{i-j}} + \underset{\underset{C_j \otimes C'_{i-j-1}}{\Cap}}{C_j \otimes \partial^{c'} C'_{i-j}} \quad \in \quad D_{i-1}$$

$$\partial^2(C_j \otimes C'_{i-j}) = \partial^c C_j \otimes \partial^{c'} C'_{i-j} + \partial^c C_j \otimes \partial^{c'} C'_{i-j} \rightarrow \text{NOT ZERO!}$$

② $D_i \xrightarrow{\partial} D_{i-1}$

$$\partial(\underset{\underset{C_j \otimes C'_{j-1}}{\Cap}}{C_j \otimes C'_{i-j}}) = \underset{\underset{C_{j-1} \otimes C'_{i-j}}{\Cap}}{(\partial^c C_j) \otimes C'_{i-j}} - \underset{\underset{C_j \otimes C'_{i-j-1}}{\Cap}}{C_j \otimes \partial^{c'} C'_{i-j}} \quad \in \quad D_{i-1}$$

$$\partial^2(C_j \otimes C'_{i-j}) = -\partial^c C_j \otimes \partial^{c'} C'_{i-j} - \partial^c C_j \otimes \partial^{c'} C'_{i-j} \rightarrow \text{NOT ZERO!}$$

③ $D_i \xrightarrow{\partial} D_{i-1}$

$$\partial(\underset{\underset{C_j \otimes C'_{j-1}}{\Cap}}{C_j \otimes C'_{i-j}}) = \underset{\underset{C_{j-1} \otimes C'_{i-j}}{\Cap}}{(\partial^c C_j) \otimes C'_{i-j}} + (-1)^i \underset{\underset{C_j \otimes C'_{i-j-1}}{\Cap}}{C_j \otimes \partial^{c'} C'_{i-j}} \quad \in \quad D_{i-1}$$

$$\partial^2 (C_j \otimes C'_{i-j}) = (-1)^{i-1} \partial^c C_j \otimes \partial^{c'} C'_{i-j} + (-1)^i \partial^c C_j \otimes \partial^{c'} C'_{i-j} = 0.$$

③ works?

Lecture 16 (16-02-2021)

16 February 2021 11:35

Snake Lemma (a more general version)

Thm. Let

$$\begin{tikzcd}
 & {} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\varphi_{n+1}"] \arrow[d, "\alpha_{n+1}"'] & B_{n+1} \arrow[r, "\psi_{n+1}"] \arrow[d, "\beta_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\gamma_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\varphi_n"] \arrow[d, "\alpha_n"'] & B_n \arrow[r, "\psi_n"] \arrow[d, "\beta_n"] & C_n \arrow[r] \arrow[d, "\gamma_n"] \arrow[dll, dashed, "\exists \delta_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\varphi_{n-1}"] \arrow[d, dashed, "\alpha_{n-1}"'] & B_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d, dashed, "\beta_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, dashed, "\gamma_{n-1}"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

$\downarrow$ differentials

$\rightarrow$ rows exact

all squares commute

be given. (δ_n is not actually a map $C_n \to A_{n-1}$. It will be on homology!)

Then, we have an exact sequence on homology:

$$\cdots \to H_{n+1}(C_\cdot) \xrightarrow{\delta_{n+1}} H_n(A_\cdot) \xrightarrow{\varphi_*} H_n(B_\cdot) \xrightarrow{\psi_*} H_n(C_\cdot) \xrightarrow{\delta_n} H_{n-1}(A_\cdot) \to \cdots$$

δ is called the connecting homomorphism. (We already know $\exists \varphi_*, \psi_*$.)

That is, a short exact sequence of complexes induces a long exact sequence on homology.

Proof. Step 1. Construction of δ_n.

Want: $\delta_n : H_n(C_\cdot) \longrightarrow H_{n-1}(A_\cdot)$

Let $z \in \ker \gamma_n$. (Then, $\bar{z} \in H_n(C_\cdot)$.)

Want to define: $\delta(\bar{z}) \in H_{n-1}(A)$.

That is, we want $x \in \ker \alpha$ s.t. $\bar{x} \in H_{n-1}(A)$ can be defined as $\delta(\bar{z})$.

ψ is onto

$\exists y$ s.t. $\psi(y) = z$

$$\begin{tikzcd}
y \arrow[r, mapsto] & z \\
B_{n-1} \arrow[r] & C_{n-1}
\end{tikzcd}$$

$\exists y$ s.t. $\psi(y) = z$

$$\begin{tikzcd}
 & & y \arrow[r, mapsto] \arrow[dd, bend right, mapsto] & z & \\
 & & B_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[d, "\gamma"] & {} \arrow[d, mapsto] \\
A_{n-1} \arrow[r] & \beta(y) \arrow[rr, bend right, mapsto] & B_{n-1} \arrow[r] & C_{n-1} & 0 \\
x \arrow[ur, bend right, mapsto] & & & 0 &
\end{tikzcd}$$

$\psi(\beta(y)) = \gamma(\psi(y)) = 0$

$\therefore \quad \beta(y) \in \ker \psi = \operatorname{im} \varphi. \qquad \therefore \exists! x \in A_{n-1}$ s.t. $\varphi(x) = \beta(y).$

φ is one-one

Want : $\bar{z} \mapsto \bar{x}.$ Does $\bar{x}$ even make sense?

Check :
① $x \in \ker \alpha$
② $\bar{x}$ should not depend on (choice of) y
③ same comment for z also.

↪ can then define $\delta(\bar{z}) = \bar{x}$

① $\varphi_{n-2}(\alpha(x)) = \beta(\varphi_{n-1}(x)) = \beta(\beta(y)) = 0$
and hence, $\alpha(x) = 0$ since φ_{n-2} is 1-1.

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[ddd, mapsto] & \beta(y) \arrow[ddd, mapsto] \\
A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[d] \\
A_{n-2} \arrow[r] & B_{n-2} \\
\alpha(x) \arrow[r, mapsto] & \beta^2(y) = 0
\end{tikzcd}$$

② and ③: Given $z, z' \in \ker \gamma$ with $y, y' \in B_n$ s.t. $\psi(y) = z$, $\psi(y') = z'$ and $x, x' \in A_{n-1}$ s.t. $\varphi(x) = \beta(y)$ and $\varphi(x') = \beta(y')$. We show that if $\bar{z} = \overline{z'}$, then $\bar{x} = \overline{x'}$.

Proof. $z - z' = \gamma(z'')$ for some $z'' \in B_{n+1}$.
$\exists y'' \in C_{n+1}$ s.t. $\psi(y'') = z''$.

Now, $\psi(y-y') = z-z' = \gamma(z'') = \gamma(\psi(y''))$.

$= \psi(\beta(y''))$

$$\begin{tikzcd}
n+1 & \bullet & \overset{y''}{\bullet} \arrow[r] \arrow[d] & \overset{z''}{\bullet} \arrow[d] \\
n & \overset{x''}{\bullet} \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \\
n-1 & \bullet \arrow[r] & \bullet & \bullet
\end{tikzcd}$$

$\Rightarrow y-y' - \beta(y'') \in \ker\psi = \operatorname{im}\varphi$

$\Rightarrow y-y' - \beta(y'') = \varphi(x'')$ for $x'' \in A_n$.

$\Rightarrow \beta(y) - \beta(y') - \cancelto{0}{\beta^2(y'')} = \beta\varphi(x'')$

φ is 1-1 $\therefore \alpha(x'') = x-x'$

$\Rightarrow \varphi(x-x') = \beta\varphi(x'') = \varphi\alpha(x'')$

$\Rightarrow x-x' \in \operatorname{im}\alpha$, as desired. ∎

Thus, we can now define $\delta : H_n(C_\cdot) \longrightarrow H_{n-1}(A_\cdot)$ as follows:

→ Take $z \in \ker\gamma$.

→ $\exists\, y \in B_n$ s.t. $\psi(y) = z$.

→ $\exists\, x \in A_{n-1}$ s.t. $\varphi(x) = \beta(y)$.

→ Define $\delta(\bar{z}) = \bar{x}$.

This is well-defined.

To be completed. Need to verify that the long sequence is exact. Assignment!

Koszul on $(a_1) \hookrightarrow K(a_1,a_2) \hookrightarrow K(a_1,a_2,a_3) \cdots$

Lecture 17 (18-02-2021)

18 February 2021 09:02

One application of Snake Lemma is the following:

Let $0 \to L \to M \to N \to 0$ be a s.e.s. of R-modules and $F. \to L$ and $G. \to N$ be free resolutions.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \\
 & F. \arrow[u] & F. \oplus G. \arrow[u] & G. \arrow[u] &
\end{tikzcd}$$

We will prove that $F. \oplus G. \to M$ is a free resolution. (Horse-shoe lemma)

Rewrite this as

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & F_{n+1} \arrow[r] \arrow[d] & F_{n+1} \oplus G_{n+1} \arrow[r] \arrow[d] & G_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_n \arrow[r] \arrow[d] & F_n \oplus G_n \arrow[r] \arrow[d] & G_n \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

(*)

Beware: The maps are not always the usual ones!

1. In the proof of horseshoe lemma, it would be enough to prove $F. \oplus G. \to M$ is a complex. The left and right are exact, which will give exactness of middle. Thus, it would become a resolution.

2. Let $\mathcal{F}$ be an additive functor on R-modules. Apply $\mathcal{F}$ on (*). Then,

$$0 \to \mathcal{F}(F.) \to \mathcal{F}(F. \oplus G.) \to \mathcal{F}(G.) \to 0$$

is a s.e.s. (in fact, split exact) since the rows were split exact.

↪ part of the horse-shoe lemma

Then we can use Snake lemma again.

Lecture 18 (22-02-2021)

22 February 2021 10:33

Projective modules : A module which has the lifting property of free modules.

Def^n. We say that R-module P is projective if given R-linear maps $\varphi : M \twoheadrightarrow N$ and $\psi : P \to N$ with φ onto, then $\exists\ \tilde{\psi} : P \to M$ which lifts ψ, i.e.,

$$\begin{tikzcd}
M \arrow[r, "\varphi"] & N \arrow[r] & 0 \\
P \arrow[u, dashed, "\tilde{\psi}"] \arrow[ur, "\psi"'] \arrow[ur, phantom, bend left, "\circlearrowleft"] & &
\end{tikzcd}$$

$$\varphi \circ \tilde{\psi} = \psi.$$

Examples. (1) Every free module is projective.

Q. Is every projective module free?

Note that saying P is projective is equivalent to saying that $\mathrm{Hom}_R(P, -)$ is exact.

Reason: Projective $\Leftrightarrow$ $\underset{\varphi_*}{\underset{\parallel}{\mathrm{Hom}_R(P, -)}} : \mathrm{Hom}_R(P,M) \longrightarrow \mathrm{Hom}_R(P,N)$ is onto!

where $\varphi : M \to N$ is onto R-linear

And we already knew $\mathrm{Hom}_R(P, -)$ is always left exact.

Aside. Some definitions:

① An R module K is called flat if the functor $K \otimes_R -$ is exact.

② An R module K is called injective if $\mathrm{Hom}_R(-, K)$ is exact.

Obs. Let P be projective, F a free R-module mapping onto P (via φ).

$$\begin{tikzcd} F \arrow[r, "\varphi"] & P \arrow[r] & 0 \\ & P \arrow[u, "id"'] \arrow[ul, dashed, "\psi"] & \end{tikzcd}$$

$\exists \psi : P \to F$ s.t. $\varphi\psi = id_P$, i.e., $P \mid F$.

(P splits off F or P is a direct summand of F.)

In fact, $F = P \oplus \ker \varphi$.

① P is a direct summand of a free module.

② We never used F was free. Thus, every onto map splits. In other words:

A s.e.s. of the form $0 \to L \to M \to P \to 0$ splits.

Q. Are the converses true?

①' Suppose $\exists$ a free module F and R-module Q s.t. $F = P \oplus Q$.

Is P projective?

Given $\varphi : M \to N$ onto and a map $\psi : P \to N$, does there exist $\tilde{\psi} : P \to M$ s.t. $\varphi\tilde{\psi} = \psi$?

First, we extend to $F \to N$ by composing ψ with $\pi : F \to P$. Then, we can use lifting of F to get a map $F \to M$. We also have a natural $P \hookrightarrow F$. Verify it works.

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \arrow[r] & 0 \\ F \arrow[u, dashed, "\tilde{\psi}"] \arrow[r, shift left, "\pi"] & P \arrow[u, "\psi"'] \arrow[l, shift left, "i"] & \end{tikzcd}$$

In fact, the above shows that direct summands of projective modules are projective. It also shows Q is projective.

Also, the above shows that not all projective modules are free. Consider any direct summand of a free module which is not free. (E.g. $R = \mathbb{Z}/6\mathbb{Z} = F$. $F = \langle \bar{2} \rangle \oplus \langle \bar{3} \rangle$.)

not free but proj.

① + ①' : P is projective $\Leftrightarrow$ P is a direct summand of a free module

Q. Are direct sums of projective R-modules also proj.?

②' If every s.e.s. of the form $0 \to N \to M \to P \to 0$ splits, is P projective?

Yes! Take M to be a free-module mapping onto P. Apply ①'.

We saw $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ are projective over $\mathbb{Z}/6\mathbb{Z}$ but not free.

More generally: If I, J are non-zero ideals s.t. $R = I \oplus J$, then I and J are projective but not free.

$\text{ann } I \supset J$
$\text{ann } J \supset I$

Q Suppose R is indecomposable as a module over itself. Is every projective R-module free? No! Find example(s)!

Remarks: ① Direct sum of projective modules are projective.

Let $\{P_i\}_{i \in I}$ be a family of proj. modules. Then,

$$F_i = P_i \oplus Q_i.$$ Then,

$$\bigoplus F_i = \left(\bigoplus P_i\right) \oplus \left(\bigoplus Q_i\right).$$

Thus, $\bigoplus P_i$ is proj., since $\bigoplus F_i$ is free.

② If P_1, P_2 are projective, then so is $P_1 \otimes_R P_2$.

$F_1 = P_1 \oplus Q_2$, $F_2 = P_2 \oplus Q_2$, then

$$\underset{\substack{\downarrow \\ \text{free}}}{F_1 \otimes F_2} = \underset{\substack{\downarrow \\ \text{direct summand}}}{(P_1 \otimes P_2)} \oplus (\quad) \oplus (\quad) \oplus (\quad)$$

③ **Q.** If P_1, P_2 are projective, is $\mathrm{Hom}_R(P_1, P_2)$?

④ In general, quotients and submodules of projectives need not be projective. Find examples.

⑤ If $A \subset R$ is a m.c.s., P a proj. R-module, then P_A is a projective R_A-module.
(Localisation)

Same as before. Localising and $\oplus$ commutes.
Localising free R-mod. gives free R_A-mod

⑥ If $I \subset R$ is an ideal, P is a projective R-module, then P/IP is a projective R/I-module.
(Quotient)

⑦ If $R \xrightarrow{\varphi} S$ is a ring map, P a projective R-modules, then $S \otimes_R P$ is a projective S-module
(Base change)

$$F = P \oplus Q$$
$$S \otimes_R F = (S \otimes_R P) \oplus (S \otimes_R Q)$$

$\hookrightarrow$ free S $\qquad \hookrightarrow \therefore$ this is projective

Proof works for ⑤ and ⑥ also!

Q. If $Q \subset P$ and $P, Q \to$ projective, is $Q|P$?
No. Take $R = P = \mathbb{Z}$ and $Q = 2\mathbb{Z}$.

Q. If P_p is a projective R_p-module for all $p \in \mathrm{Spec}\, R$, is P a projective R-module?

Ans. Let $0 \to L \to M \to N \to 0$ be a s.e.s. of R-modules.
Let P be s.t. $P_{\mathfrak{p}}$ is proj. $\forall\, \mathfrak{p} \in \operatorname{Spec} R$.
(over $R_{\mathfrak{p}}$)

Consider

$$0 \to \operatorname{Hom}_R(P, L) \xrightarrow{\psi^*} \operatorname{Hom}_R(P, M) \xrightarrow{\psi^*} \operatorname{Hom}_R(P, N) \to K \to 0$$

where $K = \operatorname{coker} \psi^*$.

Since $P_{\mathfrak{p}}$ is proj. over $R_{\mathfrak{p}}$, and

This is not true in general. $\longrightarrow$ $(\operatorname{Hom}_R(P, M))_{\mathfrak{p}} \simeq \operatorname{Hom}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}, M_{\mathfrak{p}})$, we get $K_{\mathfrak{p}} = 0$ $\forall \mathfrak{p}$.

$\therefore K = 0$.

Then, $\operatorname{Hom}_R(P, -)$ is exact. $\therefore$ P is projective.
(Incomplete.)

⑧ (Projective module in local rings)
Let P be a f.g. projective R-module on a local ring $(R, \mathfrak{m}, k)$. Then, P is free. (Consequence of NAK)

Proof. Let F be a free-module of rank $\mu(P)$. Then F maps onto P (minimally) which gives us $F = P \oplus K$. (P projective $\Downarrow$ $F \xrightarrow{\varphi} P \to 0$ splits.)

Here $K = \ker \varphi$ where $0 \to K \to F \xrightarrow{\varphi} P \to 0$.

Then,

$$F/\mathfrak{m}F \simeq P/\mathfrak{m}P \oplus K/\mathfrak{m}K.$$

Both $F/\mathfrak{m}F$ and $P/\mathfrak{m}P$ have the same dim over $\Bbbk$.
Thus, $K/\mathfrak{m}K = 0$ or $\mathfrak{m}K = K$.

K is f.g. since $K \simeq F/P$.

By NAK, $K = 0$. ⊠

In fact, by Kaplansky, every projective module over $(R, \mathfrak{m}, \Bbbk)$ is free.

is free.

Next: (Schanuel's Lemma) Let P_1, P_2 be projective R-modules, K_1, K_2 and M R-modules s.t. we have the s.e.s.es

$$0 \to K_1 \to P_1 \to M \to 0 \quad \text{and}$$
$$0 \to K_2 \to P_2 \to M \to 0.$$

Then, $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Similar to free resolutions, one can define "projective resolutions"

Projective dimension of M as: $pd_R(M) =$ min length of proj. res'l of M, if it exists.

Lecture 19 (23-02-2021)

23 February 2021 11:36

Def^n. (Finitely presented) An R-module M is finitely presented if ∃ free R-modules F_0 and F_1 of finite rank and an R-linear $\varphi : F_1 \to F_0$ s.t. $M \cong \operatorname{coker} \varphi$.

$$\begin{tikzcd} (0 \arrow[r] & \varphi(F_1) \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[dl] & M \arrow[r] & 0) \\ & \text{f.g.} & & & \end{tikzcd}$$

φ can be thought of as a matrix, called a "presentation matrix" of M.

Remark If R is Noetherian, then M is f.g. ⇔ M is finitely presented.

Ex. If M is finitely presented, N is an R-module, then for all $p \in \operatorname{Spec} R$:

$$[\operatorname{Hom}_R(M, N)]_p \cong_{R_p} \operatorname{Hom}_{R_p}(M_p, N_p).$$

(With this assumption, the proof from yesterday goes through.)

((Proof of : P_p proj. $\forall\, p^{\in \operatorname{Spec}} \Rightarrow P$ proj.

If we assume P is f.pres. then we are done.))

Q. If every localisation is free, then is the module free?

No. Take any f.g. proj. module which is not free. The localisation is f.g. + proj over local ring. Thus, it is free.

Thm (Schanuel's Lemma) Let P_1, P_2 be projective R-modules, K_1, K_2 and M R-modules s.t. we have the s.e.s.es

$$0 \to K_1 \to P_1 \to M \to 0 \quad \text{and}$$

$$0 \to K_2 \to P_2 \to M \to 0.$$

Then, $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Proof. Given:

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "\varphi_1"] & P_1 \arrow[r, "\psi_1"] & M \arrow[r] & 0 \\
0 \arrow[r] & K_2 \arrow[r, "\varphi_2"'] & P_2 \arrow[r, "\psi_2"'] \arrow[u, dashed, "\alpha"] & M \arrow[r] \arrow[u, equal, "id"'] & 0
\end{tikzcd}$$

Idea: Get an onto map $K_1 \oplus P_2 \twoheadrightarrow P_1$ and hope the kernel is K_2.

Lift $id_M : M \to M$ to $\alpha : P_2 \to P_1$ using projectivity of P_2.

Thus, $\psi_1 \alpha = \psi_2$. We already have $\psi_1 : K_1 \to P_1$.

Thus, using the red maps, we get

$$\varphi : K_1 \oplus P_2 \to P_1$$ defined by

$$\varphi(x, y) = \varphi_1(x) + \alpha(y).$$

Is φ onto? Let $z \in P_1$. $\exists y \in P_2$ s.t. $\psi_2(y) = \psi_1(z)$.
(in M)

But $\psi_2 = \psi_1 \alpha$.

$\therefore$ $\psi_1 \alpha(y) = \psi_1(z)$ or $\alpha(y) - z \in \underset{\operatorname{im}\varphi_1.}{\underset{\shortparallel}{\ker \psi_1}}$

$\therefore$ $\alpha(y) - z = -\varphi_1(x)$ for some $x \in K_1$.

$\Rightarrow$ $z = \varphi_1(x) + \alpha(y) = \varphi(x, y)$.

φ is onto. $\boxtimes$

Now, let $(x, y) \in \ker \varphi$. Then, $\varphi_1(x) = -\alpha(y)$. ① apply ψ_1

$0 = -\underset{\psi_2(y)}{\underset{\shortparallel}{\psi_1\alpha(y)}}$

$\therefore$ $y \in \ker \psi_2 = \operatorname{im} \varphi_2$.

Thus, $y = \varphi_2(x')$ for some $x' \in K_2$.

Put in ①: $\varphi_1(x) = -\alpha\varphi_2(x')$

In fact, α restricts to map $K_2 \to K_1$. We get:

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "\varphi_1"] & P_1 \arrow[r, "\psi_1"] & M \arrow[r] & 0 \\
0 \arrow[r] & K_2 \arrow[r, "\varphi_2"'] \arrow[u, "\alpha"] & P_2 \arrow[r, "\psi_2"'] \arrow[u, "\alpha"'] & M \arrow[r] \arrow[u, equal, "id"'] & 0
\end{tikzcd}$$

$\varphi_1(x) = -\alpha\varphi_2(x') = -\varphi_1\alpha(x')$ and thus, injectivity gives

$$\alpha(x') = -x.$$

Thus, $(x, y) = (-\alpha(x'), \varphi_2(x'))$.

Clearly, if $x' \in K_2$, then $(-\alpha(x'), \varphi_2(x')) \in \ker \varphi$.

Thus, the map

$$\begin{aligned} \psi : K_2 &\longrightarrow \ker \varphi \\ x' &\longmapsto (-\alpha(x'), \varphi_2(x')) \end{aligned}$$

is well-defined, onto, and one-one (since φ_2 is 1-1).

$\therefore K_2 \cong \ker \varphi$.

We have $0 \to \ker \varphi \to K_1 \oplus P_2 \longrightarrow P_1 \longrightarrow 0$

Since P_1 is projective, the s.e.s. splits and we have

$$K_1 \oplus P_2 \cong P_1 \oplus \ker \varphi \cong P_1 \oplus K_2. \qquad ∎$$

Q. ① What information does Schanuel's lemma give about a finitely presented module?

② If every R-module M has a free-res'l of the form

$$0 \to R^n \to R^m \longrightarrow M \longrightarrow 0$$

what can you conclude about R?

Next: ① (Lifting lemma) Let $P_\bullet \to M$ and $Q_\bullet \to N$ be projective resolutions and $f : M \longrightarrow N$ be R-linear.

Then, $\exists$ a chain map $\alpha_\bullet : P_\bullet \to Q_\bullet$ lifting f. $\alpha_\bullet$ is unique up to homotopy.

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[l, "\varphi_0"'] \arrow[d, "\alpha_0"] \arrow[dr, phantom, "\circlearrowleft"] & P_1 \arrow[l, "\varphi_1"'] \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \arrow[l, "\varphi_2"'] \arrow[d, "\alpha_2"] & \cdots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l, "\psi_0"] & P_1 \arrow[l, "\psi_1"] & P_2 \arrow[l, "\psi_2"] & \cdots \arrow[l]
\end{tikzcd}$$

defn of "lifts f"

commutes by defn of chain map

② (Horseshoe Lemma) Gives proj. res'ls $P_\cdot \to L$, $Q_\cdot \to N$ and a s.e.c.

$$0 \to L \to M \to N \to 0,$$

∃ a proj. res'l $P_\cdot'$ of M s.t. $(P_\cdot')_i = P_i \oplus Q_i$.

(Try to find an appropriate map $P_1 \oplus Q_1 \to P_0 \oplus Q_0$..)

Lecture 20 (04-03-2021)

04 March 2021 09:30

Thm. Lifting Lemma: Let $P_\cdot \xrightarrow{\alpha_\cdot} M \to 0$, $Q_\cdot \xrightarrow{\beta_\cdot} N \to 0$ be projective resolutions and $f : M \to N$ be R-linear. Then, f can be lifted to a chain map $\gamma_\cdot : P_\cdot \to Q_\cdot$, which is unique up to homotopy.

Proof. Want α_i s.t. each square commutes

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & P_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & P_0 \arrow[r, "\alpha_0"] \arrow[d, "\gamma_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\beta_2"'] & Q_1 \arrow[r, "\beta_1"'] & Q_0 \arrow[r, "\beta_0"'] & N \arrow[r] & 0
\end{tikzcd}$$

- γ_0 : we have $P_0 \xrightarrow{f\alpha_0} N$ and $Q \overset{\beta_0}{\twoheadrightarrow} N$. This gives γ_0 s.t. the rightmost square $\circlearrowleft$.

- γ_1 : Let $M_1 = \underset{\text{im } \alpha_1}{\overset{\|}{\ker \alpha_0}}$ and $N_1 = \underset{\text{im } \beta_1}{\overset{\|}{\ker \beta_0}}$ and consider

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] & P_0 \arrow[r] \arrow[d, "\gamma_0"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
P_3 \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "\gamma_1 \leftarrow \text{want}"] & M_1 \arrow[r] & 0 \\
Q_3 \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & N_1 \arrow[r] & 0
\end{tikzcd} \qquad (1)$$

(Now, we want a map $M_1 \to N_1$. Then we can get a γ_1.)

Easy to check that $\gamma_0(M_1) \subset N_1$. Thus, γ_0 restricts to a map $\gamma_0|_{M_1} : M_1 \to N_1$.

Now, as earlier, we lift $\gamma_0|_{M_1}$ to γ_1 in (1).
(Note that $Q_1 \xrightarrow{\beta_1} N_1$ is onto. $Q_1 \xrightarrow{\beta_1} Q_0$ need not have been.)

Thus, we get γ_1 as

$$\begin{tikzcd}
P_1 \arrow[r, "\alpha_1"] & M_1
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, "\gamma_1"'] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[d, "\gamma_0"] \\ Q_1 \arrow[r, "\beta_1"'] & N_1 \end{tikzcd}$$

The above γ_1 also makes the following commute:

$$\begin{tikzcd} P_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"'] \arrow[dr, phantom, "\circlearrowright"] & P_0 \arrow[d, "\gamma_0"] \\ Q_1 \arrow[r, "\beta_1"'] & Q_0 \end{tikzcd}$$

<u>Ex.</u> (a) Complete the above proof using induction.
(b) What is the "best statement" of this proof?

Now, we prove uniqueness, up to homotopy.
Suppose $\gamma_0, \delta_0 : P_0 \to Q_0$ are two liftings of f.
(That is, $f\alpha_0 = \beta_0\gamma_0 = \beta_0\delta_0$ and $\gamma_{i-1}\alpha_i = \beta_i\gamma_i$ AND
$\delta_{i-1}\alpha_i = \beta_i\delta_i \quad \forall\, i \geq 1$.)

<u>Claim:</u> γ_0 and δ_0 are homotopic, i.e.

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r, "\alpha_1"] \arrow[d, shift right, "\gamma_1"'] \arrow[d, shift left, "\delta_1"] \arrow[dl, dashed, "\sigma_1"'] & P_0 \arrow[r, "\alpha_0"] \arrow[d, shift right, "\gamma_0"'] \arrow[d, shift left, "\delta_0"] \arrow[dl, dashed, "\sigma_0"] & M \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\sigma_{-1}"] & 0 \\ \cdots \arrow[r] & Q_2 \arrow[r, "\beta_2"'] & Q_1 \arrow[r, "\beta_1"'] & Q_0 \arrow[r, "\beta_0"'] & N \arrow[r] & 0 \end{tikzcd}$$

- σ_0: Note that $\beta_0 \circ (\gamma_0 - \delta_0) = 0$. Thus, $\operatorname{im}(\gamma_0 - \delta_0) \subset \operatorname{im} \beta_1$.

Moreover β_1 maps Q_1 onto $\operatorname{im} \beta_1$.

Thus, projectivity of P_0 gives us σ_0 lifting $\gamma_0 - \delta_0$.

$$\begin{tikzcd} & P_0 \arrow[dl, "\sigma_0"'] \arrow[d, "\gamma_0 - \delta_0"] \\ Q_0 \arrow[r, "\beta_1"'] \arrow[ur, phantom, "\circlearrowright"] & Q \end{tikzcd}$$ commutes

Take $\sigma_{-1} = 0$ map. Thus,

$$\gamma_0 - \delta_0 = \beta_1 \sigma_0 + \sigma_{-1} \alpha_0.$$

- σ_1 : Want $\sigma_1 : P_1 \to Q_2$ s.t.

$$\beta_2 \sigma_1 + \sigma_0 \alpha_1 = \gamma_1 - \delta_1.$$

Or

$$\beta_2 \sigma_1 = \underbrace{\gamma_1 - \delta_1 - \sigma_0 \alpha_1}_{\text{Thus try to lift this!}}$$

Note that $\beta_1 \circ (\gamma_1 - \delta_1 - \sigma_0 \alpha_1)$

$$= \gamma_0 \alpha_1 - \delta_0 \alpha_1 - \beta_1 \sigma_0 \alpha_1$$

$$= (\gamma_0 - \delta_0 - \beta_1 \sigma_0) \alpha_1 = \sigma_{-1} \alpha_0 \alpha_1 = 0.$$

Thus, again $\operatorname{im}(\gamma_1 - \delta_1 - \sigma_0 \alpha_1) \subset \operatorname{im} \beta_2$ which gives us a lift $P_1 \to Q_2$.

Complete using induction. ☒

Two main applications:

Take $M = N$. (i) $f = id_M$, (ii) $f = \mu_a$.

Then, we get maps $P_\cdot \xrightarrow{\alpha_\cdot} Q_\cdot$ and $Q_\cdot \xrightarrow{\beta_\cdot} P_\cdot$ which are lifts. $\alpha_\cdot \beta_\cdot$ and $id_{Q_\cdot}$ both are lifts.

Thus, they are homotopic. Similarly, $\beta_\cdot \alpha_\cdot \simeq id_{P_\cdot}$

$\therefore \quad P \simeq Q.$ (Homotopic)

An interesting consequence: If $\mathcal{F}$ is an additive functor of R-modules, then $\mathcal{F}(P_\cdot)$ and $\mathcal{F}(Q_\cdot)$ are also homotopic. In particular, the homologies are isomorphic even if $\mathcal{F}(P_\cdot)$ and $\mathcal{F}(Q_\cdot)$ are not exact anymore.

This allows one to define "right derived functors" of a left exact additive functor, et cetera.

Lecture 21 (08-03-2021)

08 March 2021 10:40

Horseshoe lemma. Let $F_\cdot$ and $G_\cdot$ be projective resolutions of L and N. If

$$0 \to L \to M \to N \to 0$$ is a s.e.s.

of R-modules, then there exists a resolution of M where the i^{th} projective module is $F_i \oplus G_i$.

Proof. Consider the diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & F_2 \arrow[d, "\alpha_2"'] & & G_2 \arrow[d, "\gamma_2"] & \\
 & F_1 \arrow[d, "\alpha_1"'] & & G_1 \arrow[d, "\gamma_1"] & \\
 & F_0 \arrow[r, dashed] \arrow[d, "\alpha_0"'] & F_0 \oplus G_0 \arrow[r, dashed] \arrow[d, dashed] & G_0 \arrow[d, "\gamma_0"] \arrow[ld, dashed, "\delta_0"'] & \\
0 \arrow[r] & L \arrow[r, "\varphi"'] \arrow[d] & M \arrow[r, "\psi"'] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

(The horizontal maps are usual.)

Claim 1. $\exists$ an onto map $F_0 \oplus G_0 \longrightarrow M$ making the diagram commute.

Lift $\gamma_0 : G_0 \to N$ to $\delta_0 : G_0 \longrightarrow M$.

For $(x, z) \in F_0 \oplus G_0$ define

$$\beta_0(x, z) = \varphi\alpha_0(x) + \delta_0(z).$$ (Clearly R-linear.)

Q. Is β_0 surjective?

Answer. Yes!

Let $y \in M$.

$\exists z \in G_0$ s.t. $\gamma_0(z) = \psi(y)$. (γ_0 is onto!)

Let $y' = \delta_0 z$.

Then, $\psi(y - y') = \psi y - \psi\delta_0 z$
$= \gamma_0(z) - \gamma_0(z) = 0$.

$\therefore$ $y - y' \in \ker \psi = \operatorname{im} \varphi$.

$\Rightarrow$ $y - y' = \varphi(x')$ for some $x' \in L$.

Since α_0 is onto, $\exists x \in F_0$ s.t. $\alpha_0 x = x'$.

$\therefore$ $y = \varphi(x') + y' = \varphi\alpha_0(x) + \delta_0(z) = \beta_0(x, z)$. $\boxtimes$

Note: Could have also used first form of Snake Lemma for this Q.

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_1 \oplus G_1 \arrow[r] \arrow[d, dashed] & G_1 \arrow[r] \arrow[d, "\gamma_1"] & 0 \\
0 \arrow[r] & F_0 \arrow[r] \arrow[d] & F_0 \oplus G_0 \arrow[r] \arrow[d, "\beta_0"'] & G_0 \arrow[r] \arrow[d] \arrow[dl, "\delta_0"] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Claim 2. $\exists \beta_1 : F_1 \oplus G_1 \to F_0 \oplus G_0$ making the squares commute s.t.
$\operatorname{im} \beta_1 = \ker \beta_0$.

Attempt 1. We know $\exists \delta_1 : G_1 \to F_0 \oplus G_0$
In fact, we have the lift
$\delta_1 = (0, \gamma_1)$

But
$\beta_0\beta_1(x_1, z_1) = \varphi\alpha_0\alpha_1(\cancelto{0}{x_1}) + \delta_0\gamma_1(z_1)$.
However, we only know $\delta_0\gamma_1(z_1) \in \ker \psi = \operatorname{im} \varphi$, not that it is 0.

Attempt 2. Observe that $\delta_0\gamma_1(G_1) \subset \varphi(L)$ and
$\varphi\alpha_0 : F_0 \to \varphi(L)$ is onto.
Use projectivity of G_1 to get a lift

$$\begin{tikzcd}
 & G_1 \arrow[dl, dashed, "\delta_1"'] \arrow[d, "\delta_0\gamma_1"] & \\
F_0 \arrow[r, "\varphi\alpha_0"'] & \varphi(L) \arrow[r] & 0 \\
 & \cap &
\end{tikzcd}$$

$\delta_1 : G_1 \longrightarrow F_0$

Now, define

M $\quad \beta_1(x_1, z_1) = (\alpha_1(x_1) - \delta_1(z_1), \gamma_1(z_1)).$

Now,
$$\begin{aligned}\beta_0\beta_1(x_1, z_1) &= \varphi\alpha_0(\alpha_1(x_1) - \delta_1(z_1)) + \delta_0(\gamma_1(z_1))\\ &= 0 - \varphi\alpha_0\delta_1(z_1) + \delta_0\gamma_1(z_1)\\ &= 0\end{aligned}$$

(In fact, we chose the sign $-$ precisely so that $\beta_0\circ\beta_1 = 0$.)

Now, we show $\operatorname{im}\beta_1 \supset \ker\beta_0$.

Let $(x, z) \in F_0 \oplus G_0$ be s.t. $\varphi\alpha_0(x) + \delta_0(z) = 0$.

Claim. $\exists\, (x_1, z_1) \in F_1 \oplus G_1$ s.t. $\beta(x_1, z_1) = (x, z)$.

Ex. (1) Prove $\ker\beta_0 \subset \operatorname{im}\beta_1$.

(2) Complete the proof of Horseshoe Lemma by induction.

Tor and Ext modules

Defⁿ. Let K be a fixed R-module, M an R-module with projective resolution $F_\cdot$.

$$F_\cdot : \quad 0 \leftarrow F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \qquad \text{with} \qquad H_0(F_\cdot) = M.$$

Then, $F_\cdot \otimes K : \quad 0 \leftarrow F_0 \otimes K \xleftarrow{\varphi_1\otimes id} F_1 \otimes K \xleftarrow{\varphi_2\otimes id} F_2 \otimes K \leftarrow \cdots$

may not be exact.

Define $\operatorname{Tor}_i^R(M, K) := H_i(F_\cdot \otimes K)$.

Q. (1) Is it independent of the resolution chosen?

(Note that we didn't write $\operatorname{Tor}_i^R(M, K, F_\cdot)$. We often suppress notations in homological alg.)

Obs. (0) $\mathrm{Tor}_0^R(M, K) \cong M \otimes K$ since $- \otimes K$ is right exact.

(1) If K is flat, we see that $\mathrm{Tor}_i^R(M, K) = 0$ $\forall i \geq 1$

(2) (Answering Q) Suppose $F_\cdot \to M$ and $G_\cdot \to M$ are proj resolutions. Then, we had seen that $F_\cdot$ and $G_\cdot$ are homotopic. Then, so are $F_\cdot \otimes K$ and $G_\cdot \otimes K$.

$(- \otimes K$ is additive.)

Thus, $H_i(F_\cdot \otimes K) = H_i(G \otimes K)$ $\forall i$.

Therefore, there is no abuse of notation!

That is, $\mathrm{Tor}_i^R(M, K)$ depends only on M and K.

(More precisely, $- \otimes K$.)

Q. (2) If $\mathrm{Tor}_i^R(M, K) = 0$ $\forall i \geq 1$ $\forall M$, then is K flat?

(3) For a fixed M, what are the K s.t. $\mathrm{Tor}_i^R(M, K) = 0$ $\forall i \geq 1$.

(fix n) $\forall i \geq n$.

(4) (Functoriality of Tor) Given $f : M \longrightarrow N$ R linear, $\exists$ an R-linear

$$f_i : \mathrm{Tor}_i^R(M, K) \longrightarrow \mathrm{Tor}_i^R(N, K)$$

constructed from f such that $(f \circ g)_i = f_i \circ g_i$ $\forall i \geq 0$,

$(\mathrm{id}_M)_i = \mathrm{id}_{\mathrm{Tor}_i^R(M,K)}$ $\forall i \geq 0$,

$0_i = 0$ $\forall i \geq 0$,

$(M = N$ and $a \in R)$ $(\mu_a)_i = \mu_a$ $\forall i \geq 0$.

$\mu_a : M \to M$ $\quad \hookrightarrow \mu_a : \mathrm{Tor}_i^R(M, K) \to \mathrm{Tor}_i^R(M, K)$

- Construction of f_i:

Given proj. res'ls $F_\cdot \longrightarrow M$ and $G_\cdot \longrightarrow N$.

By the lifting lemma, we may lift $f : M \to N$ to a chain map $\gamma_\cdot : F_\cdot \to G_\cdot$.

This induces a chain map $\gamma_\cdot \otimes K : F_\cdot \otimes K \to G_\cdot \otimes K$.

This, in turn, induces a map on the homologies, which were the Tors.

the Tors.

By lifting lemma, we also know that $g_{\cdot}$ is unique up to homotopy. Thus, f_i is well-defined.

Lecture 22 (09-03-2021)

09 March 2021 11:36

(5) If M is projective, then $\mathrm{Tor}_i^R(M, K) = 0$ $\forall i \geq 1$, $\forall K$.

Proof Choose the resolution

$$0 \to M \to M \to 0 \quad \text{for } M.$$

Thus, if first co-ordinate is proj. or second is flat, we get 0.

(6) Let P be projective and L be such that

$$0 \to L \to P \to M \to 0 \quad \text{is a s.e.s.}$$

Then, $\mathrm{Tor}_i^R(M, K) \cong ?$

Let $Q_\bullet \to L \to 0$ be a proj. res'l of L/R.

Then, $Q_\bullet \to P \to M$ is a proj res'l of M/R.

$$\begin{tikzcd}
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r, "/" marking] & Q_0 \arrow[r] & L \arrow[r] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r, "/" marking] & Q_0 \arrow[r] & P \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then, $\mathrm{Tor}_i^R(M, K) \cong \mathrm{Tor}_{i-1}^R(L, K)$ for $i \geq 2$.

For $i = 0$, $\mathrm{Tor}_i^R(M, K) \cong M \otimes K$.

For $i = 1$, $\mathrm{Tor}_1^R(M, K) \cong ?$

(7) Let $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ be exact. What can one say about $\mathrm{Tor}_i^R(L, K)$, $\mathrm{Tor}_i^R(M, K)$, $\mathrm{Tor}_i^R(N, K)$.

Let $F_\bullet$ and $G_\bullet$ be projective resolutions of L and N, resp. Then, we construct a resolution $H_\bullet$ of M, where $H_i = F_i \oplus G_i$.

(by Horseshoe Lemma)

Tensor this with K to get:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \arrow[d] & \\
0 \arrow[r] & F_{i+1}\otimes K \arrow[r] \arrow[d] & F_{i+1}\otimes K \oplus G_{i+1}\otimes K \arrow[r] \arrow[d] & G_{i+1}\otimes K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_i\otimes K \arrow[r] \arrow[d] & F_i\otimes K \oplus G_i\otimes K \arrow[r] \arrow[d] & G_i\otimes K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{i-1}\otimes K \arrow[r] \arrow[d] & F_{i-1}\otimes K \oplus G_{i-1}\otimes K \arrow[r] \arrow[d] & G_{i-1}\otimes K \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

(Note rows were (and are) <u>split</u> exact.)

Apply Snake lemma to get a l.e.s. on homology:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_{i+1}^R(N,K) \arrow[r] & \mathrm{Tor}_i^R(L,K) \arrow[r] & \mathrm{Tor}_i^R(M,K) \arrow[r] & \mathrm{Tor}_i^R(N,K) \arrow[d] \\
\mathrm{Tor}_1^R(M,K) \arrow[d, bend right] & \cdots \arrow[l] & & \cdots & \mathrm{Tor}_{i-1}^R(L,K) \arrow[l] \\
\mathrm{Tor}_1^R(N,K) \arrow[r] & L\otimes K \arrow[r] & M\otimes K \arrow[r] & N\otimes K \arrow[r] & 0.
\end{tikzcd}$$

natural maps

Consequences:

ⓐ $0 \to L \to P \to M \to 0$ s.e.s. P projective

$$\Rightarrow \mathrm{Tor}_i^R(M,K) \cong \begin{cases} \mathrm{Tor}_{i-1}^R(L,K) & ; \quad i \geq 2 \\ \ker(L\otimes K \to P\otimes K) & ; \quad i=1 \\ M\otimes K & ; \quad i=0. \end{cases}$$

(Since $\mathrm{Tor}_i^R(P,K)=0 \quad \forall\, i \geq 1$.)

ⓑ If two out of L, M, N have the property that $\mathrm{Tor}_i^R(-,K)$ is 0 for $i \gg 0$, then so does the third.

ⓒ Suppose $\mathrm{Tor}_1^R(N, K) = 0$ $\forall N$. Then, K is flat.

Proof. Consider $0 \to L \to M \to N \to 0$. we have the les

$$\cdots \to \underset{0}{\mathrm{Tor}_1^R(N, K)} \to L \otimes K \to M \otimes K \to N \otimes K \to 0.$$

natural

$\therefore$ $0 \to L \otimes K \to M \otimes K \to N \otimes K \to 0$ is exact

Conclusion. Let K be an R-module.

TFAE :

① K is flat.

② $\mathrm{Tor}_i^R(M, K) = 0$ $\forall M$, $\forall i \geq 1$

③ $\mathrm{Tor}_1^R(M, K) = 0$ $\forall M$

?④ Fix $n \in \mathbb{N}$. $\mathrm{Tor}_n^R(M, K) = 0$ $\forall M$.

Q. $\mathrm{Tor}_i^R(M, N) \overset{?}{\cong} \mathrm{Tor}_i^R(N, M)$?

If yes, then projectives are flat.

Ex. Let $(R, \mathfrak{m}, \Bbbk)$ be Noetherian local, M a f.g. R-module.

Then,

$$\beta_i^R(M) = \dim_{\Bbbk}\left(\mathrm{Tor}_i^R(M, \Bbbk)\right)$$

Lecture 23 (11-03-2021)

11 March 2021 09:36

General comment.

If $\mathcal{F}$ is a covariant, right-exact, additive functor on R-modules, then given an R-module M, we construct the "left derived functors" of $\mathcal{F}$ as follows:

(left derived functors)

- Take a projective resolution $P_\cdot \to M \to 0$ and define

$$\mathcal{L}^i\mathcal{F}(M) := H_i(\mathcal{F}(P_\cdot)) \quad \forall\, i \geq 0$$

(One functor for each i.)

Project 3. 20th Mar

(a) Discuss properties. (Including what it does to maps as well as functoriality.)

Q. Does the "flatness" go through?

(b) What happens if $\mathcal{F}$ is contravariant left-exact?

Example: $\mathrm{Hom}_R(-, K)$ ⟵ What we get is $\mathrm{Ext}_R^i(-, k)$.

Back to Tor:

⑧ Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring, M a f.g. R-module and $F_\cdot \to M \to 0$ a min'l free resolution of M.

Recall that the maps $\varphi_\cdot$ in $F_\cdot$ have entries in $\mathfrak{m}$. (The matrices.)

Thus, the maps in $F_\cdot \otimes \Bbbk$ are 0. ($\because \Bbbk = R/\mathfrak{m}$)

Thus, $H_i(F_\cdot \otimes \Bbbk) \cong F_i \otimes \Bbbk$.

Thus, $\mathrm{Tor}_i^R(M, \Bbbk) \cong \Bbbk^{\beta_i(M)}$ or $\dim_{\Bbbk}(\mathrm{Tor}_i^R(M, \Bbbk)) = \beta_i(M)$.

Consequence: (a) $\mathrm{pdim}_R(M) = \max\{i : \mathrm{Tor}_i^R(M, \Bbbk) \neq 0\}$.

(b) [Two out of three] Given a s.e.s. $0 \to L \to M \to N \to 0$,

if two out of three have finite projective dimension, then so does the third.
(Consequence of (a) + l.e.s. of Tors.)

Ex.) Get bounds on $\mathrm{pdim}_R(-)$ of the third.

Aside. Q. In the Horseshoe lemma, if the res'l of L and N are min'l, is the constructed res'l of M also min'l?

No. Take M s.t. $\mathrm{pdim}_R(M) < \max(\mathrm{pdim}_R(L), \mathrm{pdim}_R(N))$.

Also, we have $\beta_i^R(M) \leq \beta_i^R(L) + \beta_i^R(N)$.

(And $\mathrm{pdim}_R(M) \leq \max(\mathrm{pdim}_R(L), \mathrm{pdim}_R(N))$.)

(c) Assume symmetry of Tor: In particular, assume $\mathrm{Tor}_i^R(M, \mathbb{k}) \cong \mathrm{Tor}_i^R(\mathbb{k}, M)$.

Then, $\mathrm{pdim}_R(M) \leq \mathrm{pdim}_R(\mathbb{k})$.

In particular, if $\mathrm{pdim}_R(\mathbb{k}) < \infty$, then so is $\mathrm{pdim}_R(M)$ for all f.g. M.

Proof. Compute $\mathrm{Tor}_i^R(M, \mathbb{k})$ using a projective (free) resolution of $\mathbb{k}$ and use $\beta_i^R(M) = \dim_{\mathbb{k}}(\mathrm{Tor}_i^R(M, \mathbb{k}))$.

Lecture 24 (15-03-2021)

15 March 2021 10:36

(Regular local ring)

Def^n. Let (R, m, k) be a Noetherian local ring. We say that R is a regular local ring (RLR) if $\text{pdim}_R(k) < \infty$.

- If R is an RLR, then $\forall$ f.g. module M, then $\text{pdim}_R(M) < \infty$.

Ex. If R is an RLR, then so is R_p $\forall$ $p \in \text{Spec } R$.

Q. If R is an RLR, and $a \in m$ is a non-zero-div, when is $\bar{R} = R/aR$ an RLR?

Back to Tor:

⑨ Let R be a ring, N, M be R-modules.
Then, $\forall i \geq 0$, we have $\text{Tor}_i^R(M, K) \cong \text{Tor}_i^R(K, M)$.
(For $i=0$, we already know.)

Strategy: Prove it for free modules and use a free presentation of K (or M).

Proof. Step I. Let M be free.
If we show that M is flat, then both sides are 0 $\forall K$ $\forall i \geq 1$.
Now, a free module is flat since it is a direct sum of copies of R. Tensor product distributes over $\oplus$ and R is flat as an R-module with $R \otimes N \cong N$ $\forall N$.

Thus, $\text{Tor}_i^R(M, K) \cong \text{Tor}_i^R(K, M)$ $\forall K$ $\forall i \geq 0$.
($i=0$ is $K \otimes M$.)

Step II. Let $0 \to L \to F \to M \to 0$ be a s.e.s. where F

is free and M is an arbitrary R-module

Let $\cdots \to G_2 \to G_1 \to G_0 \to K \to 0$ be a free resolution of K.

Tensor the first with G_i to get

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & G_2 \otimes L \arrow[r] \arrow[d] & G_2 \otimes F \arrow[r] \arrow[d] & G_2 \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_1 \otimes L \arrow[r] \arrow[d] & G_1 \otimes F \arrow[r] \arrow[d] & G_1 \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_0 \otimes L \arrow[r] \arrow[d] & G_0 \otimes F \arrow[r] \arrow[d] & G_0 \otimes M \arrow[r] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

exact since G_i are free and hence flat

((Columns are complexes.))

By Snake lemma, we get

since F flat

$$\cancelto{0}{\mathrm{Tor}_1^R(K, F)} \to \mathrm{Tor}_1(K, M) \to K \otimes L \to K \otimes F \to K \otimes M \to 0.$$

Thus, $\mathrm{Tor}_1(K,M) = \ker(K \otimes L \to K \otimes F) \cong \mathrm{Tor}_1(M,K)$,

by consequence 7(a).

Furthermore, since $\mathrm{Tor}_i^R(K, F) = 0 \ \forall \ i \geq 1$, we get

$\mathrm{Tor}_i^R(K, M) \cong \mathrm{Tor}_{i-1}^R(K, L) \ \forall \ i \geq 2$.

Since we know $\mathrm{Tor}_i^R(M, K) \cong \mathrm{Tor}_{i-1}^R(L, K) \ \forall \ i \geq 2$,

we are done by induction. ∎

⑩ Let $a \in R$ be a non-zero-div. This gives a natural s.e.s

$$0 \to R \xrightarrow{\cdot a} R \to \underset{\overset{\shortparallel}{R/aR}}{\bar{R}} \to 0$$ which is

a free res'l. Given an R-module M, what is $\mathrm{Tor}_i^R(M, \bar{R})$?

$$\mathrm{Tor}_i^R(M, \bar{R}) \cong \begin{cases} 0 ; & i \geq 2 \\ M/aM ; & i = 0 \end{cases}$$

$$\mathrm{Tor}_i^R(M, \bar{R}) \cong \begin{cases} 0 & ; \ i \geq 2 \\ M/aM & ; \ i = 0 \\ \underset{\substack{\| \\ \ker \mu_a}}{\mathrm{ann}_M(a)} & ; \ i = 1 \end{cases}$$

Special case: ⓐ if a is also a non-zero-divisor on M, i.e., $\mathrm{ann}_M(a) = 0$, then $\mathrm{Tor}_i^R(M, \bar{R}) = 0 \ \forall \ i \geq 1$.

Consequence: Let a be a n.z.d on R and M and $F_\cdot \longrightarrow M$ be an R-free resolution of M. Then, $0 = \mathrm{Tor}_i^R(M, \bar{R}) \cong H_i(F_\cdot \otimes R) \quad \forall \ i \geq 1$.

Thus, $H_i(F_\cdot \otimes R) = \begin{cases} \bar{M} & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases}$

$(\bar{M} = M/aM.)$

That is, $F_\cdot \otimes \bar{R}$ is a free-res'l of $\bar{M}$ over $\bar{R}$.

In particular, if (R, m, k) is Noe. local, then

$$\beta_i^R(M) = \beta_i^{\bar{R}}(\bar{M}) \quad \forall \ i \geq 1, \text{ and}$$

$$\mathrm{pdim}_R(M) = \mathrm{pdim}_{\bar{R}}(\bar{M}).$$

Lecture 25 (16-03-2021)

16 March 2021 11:36

Recall. Here, assume that $a \in R$ is not a unit.

10. ⓐ If a is a nzd on R and M, and if $F_\cdot \longrightarrow M$ is an R-free resolution, then $F_\cdot \otimes_R \bar{R}$ is a free res'l of $\bar{M}$.
In particular, $\operatorname{pdim}_R(\bar{M}) < \operatorname{pdim}_R(M)$.

ⓑ Furthermore, if $(R, \mathfrak{m}, \Bbbk)$ is Noetherian local, then

(i) $\beta_i^R(M) = \beta_i^{\bar{R}}(\bar{M})$,

(ii) $\operatorname{pdim}_R(M) = \operatorname{pdim}_{\bar{R}}(\bar{M})$,

Reason: If $F_\cdot \to M$ is min'l $/R$, then $F_\cdot \otimes_R \bar{R} \to \bar{M}$ is min'l $/\bar{R}$.

(iii) in particular,
$$\operatorname{pdim}_{\bar{R}}(\bar{M}) < \infty \iff \operatorname{pdim}_R(M).$$

⑪ Suppose a is a nzd on M. Then, what can we say about $\operatorname{pdim}_R(\bar{M})$?

We have a natural s.e.s.

$$0 \longrightarrow M \xrightarrow{a} M \longrightarrow \bar{M} \longrightarrow 0.$$

Given an R-module K, tensoring the s.e.s. with K gives a l.e.s. of Tors.

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Tor}_{j+1}^R(\bar{M}, K) \arrow[r] & \operatorname{Tor}_j^R(M, K) \arrow[r, "\mu_a"] & \operatorname{Tor}_j^R(M, K) \arrow[r] & \operatorname{Tor}_j^R(\bar{M}, K) \arrow[d] \\
 & & & \cdots & \operatorname{Tor}_{j-1}^R(\bar{M}, K) \arrow[l] \\
\arrow[r, hook] & \operatorname{Tor}_1^R(\bar{M}, K) \arrow[r] & M \otimes K \arrow[r, "\mu_a"] & M \otimes K \arrow[r] & \bar{M} \otimes K \arrow[r] & 0.
\end{tikzcd}$$

(a) If $\operatorname{Tor}_j^R(M, K) = 0$ for $j >> 0$, then same is true for $\bar{M}$.
In fact, if $\operatorname{Tor}_j^R(M, K) = 0 \ \forall\, j > t$, then $\operatorname{Tor}_j^R(M, K)$

$\forall j > t+1.$

Q. Suppose $\mathrm{Tor}_j^R(\bar{M}, K) = 0 \ \forall j >> 0$, then is $\mathrm{Tor}_j^R(M,K) \ \forall j >> 0$?

(b) Suppose R is Noetherian, M, K are f.g. and $a \in \mathrm{Jac}(R)$, then the above Q has a positive answer.

Proof. First observe that, with the given hypothesis, we can choose $F_\cdot \to M \to 0$ to be a free res'l where $\mathrm{rank}(F_i) < \infty$ $\forall i$. Hence, $F_i \otimes K$ is f.g $\forall i$.

Thus, each $\mathrm{Tor}_i^R(M, K)$ is f.g. (A subquotient.)

Now, we have

$$\mathrm{Tor}_{i+1}^R(\bar{M}, K) = 0 \longrightarrow \mathrm{Tor}_i^R(M, K) \xrightarrow{\mu_a} \mathrm{Tor}_i^R(M, K) \longrightarrow 0 = \mathrm{Tor}_i^R(M,K).$$

Use NAK to get $\mathrm{Tor}_i^R(M, K) = 0$ for $i >> 0$.

Thus, if $\mathrm{Tor}_i^R(\bar{M}, K) = 0 \ \forall i \geq t$, then $\mathrm{Tor}_i^R(M,K) = 0$ $\forall i \geq t$.

(c) Special case of (b): $(R, \mathfrak{m}, \mathbb{k})$ is Noetherian local, $a \in \mathfrak{m}$ is a nzd on $M \leftarrow$ f.g., $K = \mathbb{k}$.

Recall: $\mathrm{Tor}_j^R(M, \mathbb{k})$ is a $\mathbb{k}$ vector space. Hence,

$\mu_a : \mathrm{Tor}_j^R(M, \mathbb{k}) \to \mathrm{Tor}_j^R(M, \mathbb{k})$ is the 0-map.

(Since $a \in \mathfrak{m}$.)

Thus, $\forall j$: $\mathrm{Tor}_j(M, \mathbb{k}) \cong \mathrm{Tor}_{j+1}^R(\bar{M}, \mathbb{k})$.

In particular, $\mathrm{pdim}_R(M) < \infty \Leftrightarrow \mathrm{pdim}_R(\bar{M}) < \infty$.

In this case, $\mathrm{pdim}_R(M) + 1 = \mathrm{pdim}_R(\bar{M})$.

General comments on associated primes:

Assume R is Noetherian. Then, $\mathfrak{p} \in \mathrm{Spec}\, R$ is associated

to an R-module M if $R/\mathfrak{p} \hookrightarrow M$ as an R-module.
This naturally gives an s.e.s

$$0 \to R/\mathfrak{p} \hookrightarrow M \to N \to 0$$

which is useful in homological computations.

(Recall: $M \neq 0 \iff \mathrm{Ass}(M) \neq \emptyset$.)

Moreover, every f.g. R-module M has a prime filtration, i.e.,

$\exists$ a chain of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$, where
$M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Spec}(R)$.

Any series as above gives a series of s.e.s es:

$$\begin{gathered} 0 \to M_1 \hookrightarrow M_2 \to M_2/M_1 \to 0 \\ \vdots \\ 0 \to M_{n-1} \hookrightarrow M \to M/M_{n-1} \to 0 \end{gathered}$$

Observe: $M_1 \cong R/\mathfrak{p}_1$ and $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ $\forall\, i \geq 2$.

Consequence: By the "boot strapping" process, prove that if R is Noetherian, K is an R-mod, then $\forall$ f.g. M
$\mathrm{Tor}_i^R(M, K) = 0$ if $\mathrm{Tor}_i^R(R/\mathfrak{p}, K) = 0$
$\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$.

Recall: If R Noe, M f.g., then
① $\mathrm{Ass}_R(M)$ is a finite set,
② $\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$ = set of zero-div. of M.

Finding a nzd on M is precisely an element in $R \setminus \bigcup \mathfrak{p}$.

$\cup$ $\bigcup$ $p \in \mathrm{Ass}(M)$

Q. If $(R, \mathfrak{m}, k)$ is Noeth. local, M f.g., when does $\exists\, a \in \mathfrak{m}$ which is a n.z.d. on M & R?

$\mathfrak{m} \notin \mathrm{Ass}(M) \cup \mathrm{Ass}(R)$. (Prime avoidance.)

Lecture 26 (18-03-2021)

18 March 2021 09:34

Q. Let $(R, \mathfrak{m}, k)$ be Noetherian local, $I \subsetneq R$.
When does $\exists$ a minimal generator which is a n.z.d.?

- If r is such an element, then $r \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$ and $r \in I \setminus \mathfrak{m}I$.

Thus, $r \notin \left(\bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p} \right) \cup (\mathfrak{m}I)$. Can use the general prime avoidance.

Thus, existence is guaranteed if $I \not\subseteq \mathfrak{p}$ $\forall \mathfrak{p} \in \mathrm{Ass}(R)$ and $I \not\subseteq \mathfrak{m}I$.

Ex. Let $(R, \mathfrak{m}, k)$ be Noe. local, $I \subsetneq R$. If I contains a nzd, then it has a min'l gen. which is a nzd.
(Prime avoidance + NAK.)

Regular sequences and depth

Defn Let R be a ring, M an R-module. We say that a sequence $\underline{a} = (a_1, \ldots, a_n)$ is an M-regular sequence (or M-sequence) if

① a_1 is a n.z.d. on M.

② a_i is a n.z.d. on $\dfrac{M}{\langle a_1, \ldots, a_{i-1} \rangle M}$ $\forall i \geqslant 2$ and

③ $\langle \underline{a} \rangle M \neq M$.
↳ in particular, no a_i is a unit.

- If $a \in R$, then $\underline{a} = (a)$ is a regular sequence, then $\mu_a : M \to M$ is one-one but not onto.
If $M = R$, then a is a n.z.d. and non-unit.
(Converse as well.)

Def^n Let $I \subset R$ be an ideal and M an R-module. The length of a maximal M-sequence on I, of maximum length, is called depth(I, M). (depth)

Q. Is this finite? Are there some maximal sequences which are finite?

Notation ① If $M = R$, depth(I, R) is often denoted grade(I).
② If $(R, \mathfrak{m}, k)$ is local, then depth$(M) :=$ depth$(\mathfrak{m}, M)$.

Q. Let $(R, \mathfrak{m}, k)$ be Noetherian local, M f.g. When is depth$(M) = 0$?

Obs. $\text{depth}(M) = 0 \Rightarrow \mathfrak{m} \subset Z(M) = \bigcup\limits_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$

$|\text{Ass}(M)| < \infty$, use prime avoidance

$$\Downarrow$$

$\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(M)$ $\Leftarrow$ $\mathfrak{m} \subset \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(M)$

$$\Downarrow$$

$\mathfrak{m} \in \text{Ass}(M)$.

The converse is also true. $\mathfrak{m} \in \text{Ass}(M) \Rightarrow \mathfrak{m} \subset Z(M)$
$\Downarrow$
no reg. sequence. ☒

Thus, $\text{depth}(M) = 0 \Leftrightarrow \mathfrak{m} \in \text{Ass}_R(M) \Leftrightarrow k \hookrightarrow M$ as an R-module
$\Leftrightarrow \exists x \in M$ s.t. $(0 :_R x) = \mathfrak{m}$.
$\Leftrightarrow \text{ann}_M(\mathfrak{m}) = 0 :_M \mathfrak{m} \neq 0$

Def^n Given an R-module M, define soc(M) (called the socle of M) as $\text{ann}_M(\mathfrak{m}) = 0 :_M \mathfrak{m} \cong \text{Hom}_R(k, M)$.
($(R, \mathfrak{m}, k)$ local above)

Thus, $\text{depth}(M) = 0 \Leftrightarrow \mathfrak{m} \in \text{Ass}_R(M) \Leftrightarrow \text{soc}(M) \neq 0$

Ex. If (R, m, k) is local, M an Artinian R-module, then $\operatorname{soc}(M) \neq 0$ (and thus, $\operatorname{depth}(M) = 0$).

Q. Let $a \in m$ be M-regular. What can you say about depth of M/aM.

Take a reg. sequence in M/aM. Put a at beginning. We get an M-sequence.

Thus if $\operatorname{depth}(M/aM) < \infty$, then $\operatorname{depth}(M) \geq \operatorname{depth}(M/aM) + 1$.

Auslander-Buchsbaum Formula: Let (R, m, k) be Noetherian local, M f.g. s.t. $\operatorname{pdim}_R(M) < \infty$.

Then, $\operatorname{pdim}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R)$.

What can we say about $\overset{\text{reg. local ring}}{\text{RLR}}$ R and depth.

Def^n. A Noetherian ring R is regular if R_p is a RLR $\forall p \in \operatorname{Spec}(R)$.

Remind project on Monday.

Lecture 27 (22-03-2021)

22 March 2021 10:30

Ex. ① A permutation of a regular sequence need not be regular. Find examples.

② If R is Noetherian, and M an R-module, is depth$(I, M) < \infty$?
(Will local help?)

③ Let R be Noetherian local. Is depth$(I, R) \geq 1$, then

ⓐ depth$(R) \geq 1$,

ⓑ I contains a n.z.d. and hence, $\operatorname{ann}_R(I) = 0$.

ⓒ I has a min'l generator which is a n.z.d. by prime avoidance (modified).

④ Let R be Noetherian local, depth$(I, R) = r$.
Then, does $\exists$ a min'l gen. set $a_1, \ldots, a_n$ of I s.t. $(a_1, \ldots, a_r)$ is I-regular?

Tensor Product of Complexes

Let $(C_\cdot, \partial_\cdot)$ and $(D_\cdot, \delta_\cdot)$ be complexes of R-modules.
We define the complex $(C_\cdot \otimes D_\cdot)$ as the complex with

$$(C_\cdot \otimes D_\cdot)_n = \bigoplus_{i+j=n} C_i \otimes D_j$$

with differential defined as :

for $x_i \in C_i$, $y \in D_j$, set $x \otimes y \longmapsto \underset{\underset{C_{i-1} \otimes D_j}{\cap}}{\partial x \otimes y} + (-1)^i \underset{\underset{C_i \otimes D_{j-1}}{\cap}}{x \otimes \delta y}$

Ex. Verify that $C_\cdot \otimes D_\cdot$ is a complex.

$$\begin{tikzcd}
 & & & & \vdots \arrow[d] \\
\cdots \arrow[r] & C_{i+1} \arrow[r] & C_i \arrow[r] & C_{i-1} \arrow[r] & \cdots \quad \otimes D_{j+1} \arrow[d] \\
\cdots \arrow[r] & C_{i+1} \arrow[r] & C_i \arrow[r] & C_{i-1} \arrow[r] & \cdots \quad \otimes D_j \arrow[d] \\
\cdots \arrow[r] & C_{i+1} \arrow[r] & C_i \arrow[r] & C_{i-1} \arrow[r] & \cdots \quad \otimes D_{j-1} \arrow[d] \\
 & & & & \vdots
\end{tikzcd}$$

The above gives

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{i+1} \otimes D_{j+1} \arrow[r] \arrow[d] & C_i \otimes D_{j+1} \arrow[r] \arrow[d] & C_{i-1} \otimes D_{j+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{i+1} \otimes D_j \arrow[r] \arrow[d] & C_i \otimes D_j \arrow[r] \arrow[d] & C_{i-1} \otimes D_j \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{i+1} \otimes D_{j-1} \arrow[r] \arrow[d] & C_i \otimes D_{j-1} \arrow[r] \arrow[d] & C_{i-1} \otimes D_{j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Diagonal gives the modules in $C \otimes D_\bullet$, called the total complex of the given double complex. Working through the double complex also gives a proof of symmetry of Tor.

Q. If $F_\bullet \to M \to 0$, $G_\bullet \to N \to 0$ are free resolutions, what is $H_0(F_\bullet \otimes G_\bullet)$. Does $F_\bullet \otimes G_\bullet$ give a free resolution of H_0? If not, identify conditions when it is.

Example ① Let $(C_\bullet, \delta)$ be a complex and $D_\bullet : 0 \to \underset{D_0, \text{ convention}}{M} \to 0$ where M is an R-module.

$$\begin{tikzcd}
\text{Then, } (C_\bullet \otimes D_\bullet)_n \arrow[d] & = & C_n \otimes M \arrow[d, "\delta \otimes id"] & x \otimes y \arrow[d, mapsto] \\
 & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
{} \arrow[d] & {} \arrow[d, "\delta \otimes 1a"] & {} \arrow[d] \\
(C_\cdot \otimes D_\cdot)_{n-1} = & C_{n-1} \otimes M & \delta x \otimes y
\end{tikzcd}$$

Q. What is $H_n(C_\cdot \otimes D_\cdot)$? Not exactly $H_n(C_\cdot) \otimes M$ since $\delta x \otimes y = 0 \not\Rightarrow \delta x = 0$.

For example, if $C_\cdot$ is a resolution, then we get Tors, not necessarily 0.

Remark Note that if we took $D_\cdot \otimes C_\cdot$, the maps would have a sign.

---

② Let $(C_\cdot, \delta_\cdot)$ be a complex and

$$\begin{tikzcd}
D_\cdot = 0 \arrow[r] & Re \arrow[r, "\cdot a"] \arrow[d, equal] & R \arrow[r] \arrow[d, equal] & 0. \\
& D_1 & D_0 &
\end{tikzcd}$$

Then,

$$\begin{tikzcd}
(C_\cdot \otimes D_\cdot)_n = & C_n \otimes R & \oplus & C_{n-1} \otimes Re \\
& x \otimes 1 \arrow[d, mapsto] & & y \otimes e \arrow[d, mapsto] \\
& \delta x \otimes 1 + 0 & & \delta y \otimes e + (-1)^{n-1} y \otimes a \\
(C_\cdot \otimes D_\cdot)_n = & C_{n-1} \otimes R & \oplus & C_{n-2} \otimes Re
\end{tikzcd}$$

In general, identify image of $(x \otimes b, y \otimes ce)$. $\leftrightarrow (bx, cy)$

$$(x \otimes b, y \otimes ce) = (\delta x \otimes b + (-1)^{n-1} y \otimes a, \delta y \otimes ce)$$

or

$$(b\delta x + (-1)^n cay, c\delta y).$$

Note that the second co-ord. is just δ.

Notation. Shift of a complex.

Given $(C_\cdot, \delta_\cdot)$, define $\Sigma^{-1} C_\cdot$ to be the complex with modules $(\Sigma^{-1} C_\cdot)_n = C_{n-1}$

with differential $(\Sigma^{-1} \delta_\cdot)_n = \delta_{n-1}$.

$$\begin{tikzcd}
& & i+1 & i & i-1 & \\
C_\cdot : & \cdots \arrow[r] & C_{i+1} \arrow[r, "\delta_{i+1}"] & C_i \arrow[r, "\delta_i"] & C_{i-1} \arrow[r, "\delta_{i-1}"] & \cdots \\
\Sigma^{-1} C_\cdot : & \cdots \arrow[r] & C_i \arrow[r, "\delta_i"'] & C_{i-1} \arrow[r, "\delta_{i-1}"'] & C_{i-2} \arrow[r, "\delta_{i-2}"'] & \cdots
\end{tikzcd}$$

δ_i δ_{i-1} δ_{i-2}

More generally, $(\Sigma^j C_\bullet)$ is defined.

Back to example: $(C_\bullet \otimes D_\bullet)_n = C_n \otimes R \oplus C_{n-1} \otimes Re$

$\cong$

$C_n \oplus C_{n-1}$

Thus, we have $0 \to C_\bullet \to C_\bullet \otimes D_\bullet \to \Sigma^{-1} C_\bullet \to 0$

(The maps on coker $\Sigma^{-1} C_\bullet$ are actually $\Sigma^{-1} \delta_\bullet$)

The above is a s.e.s of complexes which gives a l.e.s. on homologies as

$$\cdots \to H_{n+1}(C_\bullet \otimes D_\bullet) \to H_{n+1}(\Sigma^{-1} C_\bullet) \to H_n(C_\bullet) \to H_n(C_\bullet \otimes D_\bullet) \to H_n(\Sigma^{-1} C_\bullet) \to \cdots$$

What is $H_i(\Sigma^{-1} C_\bullet)$? What is the connecting homomorphism?

$\|$

$H_{i-1}(C_\bullet)$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_n \arrow[r] \arrow[d, "\delta"'] & C_n \oplus C_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & C_{n-1} \oplus C_{n-2} \arrow[r] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

middle map : $(x, y) \mapsto (\delta x + (-1)^{n-1} ay, \delta y)$

Verify the connecting homom. is $\pm \mu_a$.

Thus, the l.e.s. is

$$\cdots \to H_{n+1}(C_\bullet \otimes D_\bullet) \to H_n(C_\bullet) \xrightarrow{\pm a} H_n(C_\bullet) \to H_n(C_\bullet \otimes D_\bullet) \to H_{n-1}(C_\bullet) \xrightarrow{\pm a} \cdots$$

Q. What can you conclude?

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$$\cdots \to H_{n+1}(C_\bullet \otimes D_\bullet) \to H_n(C_\bullet) \xrightarrow{\pm a} H_n(C_\bullet) \to H_n(C_\bullet \otimes D_\bullet) \to H_{n-1}(C_\bullet) \xrightarrow{\pm a} \cdots$$

Conclusions:

① If $a \in \mathcal{U}(R)$, then $H_n(C_\bullet \otimes D_\bullet) = 0$.
(In fact, we only need each $\pm\mu_a$ to be an isomorphism on $H_n(C_\bullet)$.)

② If $H_i(C_\bullet) = 0 \ \forall i \geq n$, then $H_i(C_\bullet \otimes D_\bullet) = 0 \ \forall i \geq n+1$.
Moreover, $H_n(C_\bullet \otimes D_\bullet) = \ker(\pm\mu_a : H_{n-1}(C_\bullet) \to H_{n-1}(C_\bullet))$.
In fact if $H_n(C_\bullet) = H_{n-1}(C_\bullet) = 0$, then $H_n(C_\bullet \otimes D_\bullet) = 0$.

③ $H_i(C_\bullet \otimes D_\bullet) = 0 = 0 \ \forall i \geq n \Rightarrow \mu_a : H_i(C_\bullet) \to H_i(C_\bullet)$ is an iso. for all $i \geq n$.

④ The converse of ② is not true. Find examples

⑤ If each C_i is Noetherian and $a \in \mathrm{Jac}(R)$, then by NAK, if $H_n(C_\bullet \otimes D_\bullet) = 0$, then $H_n(C_\bullet) = 0$. (Since μ_a becomes on to.)

This acts as a converse to ②.

Special case of example ②:

$C_\bullet : 0 \to Re_1 \xrightarrow{\cdot a} R \to 0$; $D_\bullet : 0 \to Re_2 \xrightarrow{\cdot a} R \to 0$.

What is $C_\bullet \otimes D_\bullet$?

$$0 \to Re_1 \otimes Re_2 \to Re_1 \oplus Re_2 \to R \to 0.$$

$e_1 \otimes e_2 \longmapsto a_1 e_2 - a_2 e_1$

$e_1 \longmapsto a_1$

$e_2 \longmapsto a_2$

One can now explicitly construct an isomorphism (of complexes) with $K_\bullet(a_1, a_2)$.
↳ Koszul on two elements

More generally, by induction, one can show that if $a_1, \ldots, a_n \in R$, then $K_\bullet(a_1) \otimes \cdots \otimes K_\bullet(a_n) \cong K_\bullet(a_1, \ldots, a_n)$.

[Show $\underset{\overset{\shortparallel}{C_\bullet}}{K_\bullet(a_1, \ldots, a_{n-1})} \otimes \underset{\overset{\shortparallel}{D_\bullet}}{K_\bullet(a_n)} \cong K_\bullet(a_1, \ldots, a_n)$.]

By the observations in example ②, we can relate Koszul homology of $(a_1, \ldots, a_n)$ with Koszul homology of $(a_1, \ldots, a_{n-1})$ and $H a_n$.

Def^n Let $a_1, \ldots, a_n \in R$, M an R-module. We define the Koszul complex on $\underline{a} = (a_1, \ldots, a_n)$ w.r.t. M as

$$K_\bullet(\underline{a}, M) := K_\bullet(\underline{a}) \otimes_R M.$$

Special case: If S is an R-algebra via φ; $a_1, \ldots, a_n \in R$, then

$$K(\underline{a}, R) \otimes_R S \cong K(\varphi(\underline{a}), S).$$

Based on the observations following Example ②, we have the following result:

Thm. Let $a_1, \ldots, a_n \in R$, M an R-module. Consider the following statements:

① $a_1, \ldots, a_n$ is M-regular.

② $H_i(K_\bullet(a_1, \ldots, a_n, M)) = 0 \quad \forall i \geq 1$

③ $H_1(\underline{a}, M) = 0$.

Then, ① $\Rightarrow$ ② $\Rightarrow$ ③. Moreover, if M is Noetherian and $a_1, \ldots, a_n \in \text{Jac}(R)$, then ③ $\Rightarrow$ ①.

Proof. Note that ① $\Leftrightarrow$ ② $\Leftrightarrow$ ③ for $n = 1$.

Observe that $K(a_1, \ldots, a_n, M) \cong K(a_1, \ldots, a_{n-1}, M) \otimes K(a_n)$ and

$a_1, \ldots, a_n$ is M-regular $\Leftrightarrow$ $a_1, \ldots, a_{n-1}$ is M-regular and a_n is a n.z.d. on $M/\langle a_1, \ldots, a_{n-1} \rangle M$.

These observations allow us to use induction.

- Assume ①. Then $a_1, \ldots, a_{n-1}$ is M-reg $\Rightarrow$ $H_i(a_1, \ldots, a_{n-1}, M) = 0 \quad \forall i \geq 1$

$$\Downarrow$$

$$H_i(a_1, \ldots, a_n, M) = 0 \quad \forall i \geq 2$$

Moreover, $H_1(a_1,\dots,a_n,M) = \ker(\mu_{a_n}: H_0(a_1,\dots,a_{n-1},M) \to H_0(a_1,\dots,a_{n-1},M))$

Since a_n is a n.z.d. on $H_0(a_1,\dots,a_{n-1},M) \cong \frac{M}{\langle a_1,\dots,a_{n-1}\rangle M}$, $H_1 = 0$.

Thus, ① ⇒ ②.

- ② ⇒ ③ obvious.

Assume M is Noetherian and $\underline{a} \subset \operatorname{Jac}(R)$. Use induction for ③ ⇒ ①.

$H_1(\underline{a}, M) = 0 \Rightarrow \mu_{a_n}$ is injective on $H_0(a_1,\dots,a_{n-1},M) \cong \frac{M}{\langle a_1,\dots,a_{n-1}\rangle M}$.

By obs. ⑤, $H_1(a_1,\dots,a_{n-1},M) = 0$.

By induction, $a_1,\dots,a_{n-1}$ is M-reg and a_n is a n.z.d. on .

∴ $a_1,\dots,a_n$ is M-reg.

Consequences:

① If $a_1,\dots,a_n$ is R-regular, then $K_\bullet(a_1,\dots,a_n)$ is a free resolution of $H_0 = R/\langle a_1,\dots,a_n\rangle$.

In particular, $H_i K_\bullet(\underline{a}, M) \cong \operatorname{Tor}_i^R(R/\langle \underline{a}\rangle, M)$.

② If $(R, \mathfrak{m}, k)$ is Noetherian local and $\mathfrak{m}$ is generated by a regular sequence, then $\operatorname{pdim}_R(\mathbb{k}) < \infty$ and hence, R is an RLR.
Moreover, if $\mu(\mathfrak{m}) = n$, then $\operatorname{pdim}_R(\mathbb{k}) = n \geq \operatorname{pdim}_R(M)$ $\forall$ f.g. M.

E.g. $\mathbb{k}[[x_1,\dots,x_n]]$ is an RLR. What is $\beta_i^R(k)$? $\binom{n}{i}$.

⓪ If $\underline{a} \subset \operatorname{Jac}(R)$, then any permutation of $\underline{a}$ is regular.

Lecture 29 (25-03-2021)

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Observations.

① Let $a_1, \dots, a_n \in R$ and $I = \langle a_1, \dots, a_n \rangle$ (and $\underline{a} = a_1, \dots, a_n$).

Recall : $H_0(\underline{a}, M) = M/IM$ and $H_n(\underline{a}, M) = \operatorname{ann}_M(I)$.

$$\operatorname{ann}_M(I) = (0 :_M I)$$

$0 \to R \to R^{\oplus n}$

$\otimes M \qquad \otimes M$

$1 \mapsto M \to M^n$

In both cases, the homology is annihilated by I.

In fact, we now show that $\forall i$, $I \cdot H_i(\underline{a}, M) = 0$, i.e., I annihilates every homology.

Proof. Let $S := R[T_1, \dots, T_n]$. Then, R is an S-algebra via $\varphi : S \to R$ is defined by $T_i \mapsto a_i$ (and id_R on R).

Let M be an R-module. Then, M is an S-module via φ.

$$K_\cdot^R(\underline{a}; M) \cong K_\cdot^S(\underline{T}; M)$$

as S-complexes.

Note $T_1, \dots, T_n$ is a regular sequence (in S) and hence,

$$H_i^S(\underline{T}; M) \cong \operatorname{Tor}_i^S(S/\langle \underline{T} \rangle, M)$$

Fact : If M, N are S-modules, then $\forall i$
$$\operatorname{ann}_S(M) + \operatorname{ann}_S(N) \subset \operatorname{ann}_S(\operatorname{Tor}_i^S(M, N)). \qquad \text{(Ex.)}$$

Thus, $\operatorname{ann}_S(H_i^S(\underline{T}; M)) \supset \operatorname{ann}_S(S/\langle \underline{T} \rangle) + \operatorname{ann}_S(M)$

$$\operatorname{ann}_S(S/\langle \underline{T} \rangle) = \langle T_1, \dots, T_n \rangle, \qquad \operatorname{ann}_S(M) \supseteq \operatorname{ann}_S(R) = \ker(\varphi)$$

Thus, $\operatorname{ann}_S(H_i^S(\underline{T}; M)) \supseteq \langle T_1, \dots, T_n \rangle + \langle T_1 - a_1, \dots, T_n - a_n \rangle$

$\supseteq \langle a_1, \dots, a_n \rangle = I$

ideal gen. in R

This forces $\mathrm{ann}_R(H_i^S(I, M)) \supset I$. But $H_i^S(I, M) \cong H_i^R(\underline{a}, M)$. $\boxtimes$

② Let $(R, \mathfrak{m}, \Bbbk)$ be local. Let $\{a_1, \dots, a_n\}$ be a min'l gen. set of I. If $I = \langle b_1, \dots, b_n \rangle$, then is $H_i(\underline{a}) \cong H_i(\underline{b})$?

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = A \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{tikzcd}
{} \arrow[r] & \bigoplus_{i=1}^{n} Re_i \arrow[r] \arrow[d, "A^T"'] & R \arrow[r] \arrow[d, "id_R"] & R/\langle \underline{a} \rangle \arrow[r] & 0 \\
{} \arrow[r] & \bigoplus_{i=1}^{n} Rf_i \arrow[r] & R \arrow[r] & R/\langle \underline{b} \rangle \arrow[r] & 0
\end{tikzcd}$$

$e_i \longmapsto a_i$

$f_i \longmapsto b_i$

As in the proof of the splitting lemma, A^T is invertible.

$$\bigoplus Re_i = K_1 \xrightarrow{\varphi} \bigoplus Rf_i = K_1'.$$

Focus now on:

Given φ, we construct $\bigwedge^i \varphi : \bigwedge^i K_1 \longrightarrow \bigwedge^i K_1'$

$$\begin{array}{ccc} \bigwedge^i K_1 & \longrightarrow & \bigwedge^i K_1' \\ \| & & \| \\ K_i & & K_i' \end{array}$$

modules in Koszul complex

Suppose φ is given by $A = [a_{ij}]$.

Then, $e_1 \wedge e_2 \longmapsto (a_{11} f_1 + a_{21} f_2 + \cdots + a_{n1} f_n) \wedge (a_{12} f_1 + a_{22} f_2 + \cdots + a_{n2} f_n)$

$$= \sum_{1 \le i < j \le n} (a_{i1} a_{j2} - a_{i2} a_{j1})(f_i \wedge f_j)$$

2x2 minor formed by rows i&j, columns i and j.

By induction, use the $k \times k$ minor of A to define $\bigwedge^k \varphi : \bigwedge^k (K_1) \longrightarrow \bigwedge^k (K_1')$

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \longmapsto \sum_{j_1 < \cdots < j_k} a_{(\underline{i}, \underline{j})} (f_{j_1} \wedge \cdots \wedge f_{j_k}).$$

(And this gives a chain map.)

In particular, the map $\Lambda^n \varphi : \Lambda^n (K_1) \to \Lambda^n (K_1')$ is $\cdot \det(A)$.

If φ is an isomorphism, one would like to say that $\Lambda^i \varphi$ is an isomorphism $\forall i$. (And that it gives an isomorphism of complexes.)

Idea: φ is an iso $\Rightarrow \exists a_{ij} \notin m$, i.e., a_{ij} is a unit.

WLOG, $a_{11} \notin m$.

We show the isomorphism by changing basis elements one by one.

$\{e_1, \dots, e_n\} \to \{f_1, e_2, \dots, e_n\} \to \cdots \{f_1, \dots, f_n\}$.

Let $\varphi(e_1) := a_1 e_1 + \cdots + a_n e_n$, $a_i \notin m$

$\varphi(e_i) := e_i$ $\forall i \geq 2$

$\Lambda^k \varphi(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $1 < i_1 < \cdots < i_k \leq n$.

$\Lambda^k \varphi(e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = a_1 (e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})$ + other terms

Can choose basis s.t. matrix of $\Lambda^k \varphi$ is upper triangular, with units on diagonal. Thus, $\Lambda^k \varphi$ is invertible.

This will show $K_\bullet(\underline{a}; R) \cong K_\bullet(\underline{b}; R)$. ▨

Remark Can be extended:

① If $a_1, \dots, a_n \in R$. $b_1 = r_1 a_1 + \cdots + r_n a_n$, where $r_1 \in U(R)$, then

$K_\bullet(a_1, \dots, a_n) \cong K_\bullet(b_1, a_2, \dots, a_n)$.

② If $a_{n+1} \in \langle a_1, \dots, a_n \rangle$, then

$$K_\bullet(a_1, \dots, a_{n+1}) \cong K_\bullet(a_1, \dots, a_n, 0).$$

③ If $\langle a_1, \dots, a_n \rangle = \langle b_1, \dots, b_n \rangle$ (not nec. min'l), then $K_\bullet(\underline{a}) \cong K_\bullet(b)$

Q What can we do in the non-local case?

(Notes by Sean Sather-Wigstaff on Koszul complexes.)

Lecture 30 (30-03-2021)

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Let $C_\cdot = K_\cdot(a_1, \ldots, a_{n-1}; M)$, $D_\cdot = K_\cdot(a_n; R)$.

Recall that we have a l.e.s. of Koszul homology.

$$\begin{tikzcd}
{} \arrow[r] & H_{k+1}(C_\cdot \otimes D_\cdot) \arrow[r] & H_k(C_\cdot) \arrow[r, "\cdot(\pm a_n)"] & H_k(C_\cdot) \arrow[r] & H_k(C_\cdot \otimes D_\cdot) \arrow[d] \\
 & & \cdots & H_{k-1}(C_\cdot) \arrow[l] & H_{k-1}(C_\cdot) \arrow[l, "\cdot(\pm a_n)"']
\end{tikzcd}$$

Here, $C_\cdot \otimes D_\cdot = K_\cdot(\underline{a}; M)$.

Consider the following statements:

① $a_{n-r+1}, \ldots, a_n$ is M-regular.

② $H_i(\underline{a}; M) = 0 \quad \forall\, i \geq n-r+1$

③ $H_{n-r+1}(\underline{a}; M) = 0$.

(Last r regular

$\Downarrow$

Last r homologies are 0.)

Then, ① $\Rightarrow$ ② $\Rightarrow$ ③.

"Proof" Induct as before on length of sequence.

If ① is true, $a_{n-r+1}, \ldots, a_{n-1}$ is M-regular and hence

$H_i(C_\cdot) = 0 \quad \forall\, i \geq (n-1) - (r-1) + 1 = n-r+1.$

Conclude ②. ∎

Note (Rigidity of Koszul).

Let M be Noe. and $a_1, \ldots, a_n \in \operatorname{Jac}(R)$.

If $H_j(\underline{a}; M) = 0$ for some $j \in \mathbb{N}$, then $H_i(\underline{a}; M) = 0 \ \forall\, i \geq j$.

(Consequence of ③ $\Rightarrow$ ①.)

Depth sensitivity of Koszul complex

Let $(R, \mathfrak{m}, \Bbbk)$ be a Noetherian local ring and M a f.g. R-module. If $I = \langle \underline{a} \rangle$, then

$$\operatorname{depth}(I; M) = \min\{ j \mid H_j(\underline{a}, M) \neq 0 \}.$$

Consequences: ① $\operatorname{depth}(I, M) < \infty$ and is $\leq \mu(I)$.

② Every maximal M-regular sequence in I has the same length.

③ (Depth lemma) Let $0 \to L \to M \to N \to 0$ be a s.e.s. of f.g. R-modules.

How do the depths compare?

Idea: $0 \to L \to M \to N \to 0$ gives a s.e.s. of complexes.

$$0 \to K(\underline{a}; L) \to K(\underline{a}; M) \to K_\bullet(\underline{a}; N) \to 0.$$

Lecture 31 (01-04-2021)

01 April 2021 09:40

① Rigidity of Koszul

② Let R be Noetherian, M f.g., $a_1, \dots, a_r, b_1, \dots, b_s \in R$ be s.t. $a_1, \dots, a_r$ is M-regular. Then,

$$H_j(\underline{a}, \underline{b}; M) = \begin{cases} 0 & ; \quad j > s \\ \mathrm{ann}_{M/\langle a\rangle M}(\langle \underline{b} \rangle) & ; \quad j = s \end{cases}$$

Proof Induction on s.

Base case: $s = 1$. We have $H_i(\underline{a}; M) = 0$ for $i \geq 1$.

Hence, the l.e.s. gives $H_i(\underline{a}, b_1; M) = 0$ for $i \geq 2$

and $H_1(\underline{a}, b_1; M) = \ker\left(H_0(\underline{a}; M) \xrightarrow{\cdot b_1} H_0(\underline{a}; M)\right) = \mathrm{ann}_{M/\langle a\rangle M}(b_1)$.

Complete the proof.

Assume $IM \neq M$.

3. Let $a_1, \dots, a_r = \underline{a} \in \langle \underline{b} = b_1, \dots, b_s \rangle = I$ be a max'l M-regular sequence in I.

Then,

(*)
$$H_i(\underline{a}, \underline{b}; M) = \begin{cases} 0 & ; \quad i > s \\ \mathrm{ann}_{M/\langle a\rangle M}(\underline{b}) & ; \quad i = s \end{cases}$$

Note that $\mathrm{ann}_{M/\langle a\rangle M}(\underline{b}) \neq 0$. Indeed, I consists of zero divisors of $M/\underline{a}M$ and hence is contained in some $\mathfrak{p} \in \mathrm{Ass}_R(M/\langle \underline{a}\rangle M)$ by prime avoidance. Now, $\mathfrak{p} = \mathrm{ann}_R(\bar{a})$ for some $0 \neq \bar{a} \in M/\langle a\rangle M$ and $\underline{b} \cdot \bar{a} = 0$ in M/aM.

Thus, $0 \neq \bar{a} \in \mathrm{ann}_{M/\langle a\rangle M}(\underline{b})$.

On the other hand, recall that $K_\cdot(\underline{a}, \underline{b}; M) = K_\cdot(\underset{\swarrow}{\underline{0}}, \underline{b}; M)$

r-many 0s

$\cong$

$K(\underline{b};M)\otimes K_\cdot(\underline{0})$

$$\therefore \quad H_\cdot(\underline{a},\underline{b};M) \cong H_\cdot(\underline{b};M)\otimes \Lambda R^r$$

(Think of homologies as complex with zero maps.)

homologies of $K_\cdot(\underline{0})$ is just the koszul itself

Let $n = r+s$.

$$H_n(\underline{a},\underline{b};M) = H_s(\underline{b},M)\otimes\Lambda^r R^r$$

since $H_j(\underline{b},M)=0=\Lambda^j R^r$ if $j>s$ or $j>r$.

$$H_{n-1}(\underline{a},\underline{b};M)\cong H_s(\underline{b};M)\otimes\Lambda^{r-1}R^r \oplus H_{s-1}(\underline{b};M)\oplus\Lambda^r R^r$$

$\vdots$

$$H_{\underbrace{n-r+1}_{=s+1}}(\underline{a},\underline{b};M)\cong H_s(\underline{b};M)\otimes\Lambda^1R^r\oplus\cdots\oplus H_{s-r+1}(\underline{b};M)\otimes\Lambda^r R^n$$

$$H_s(\underline{a},\underline{b};M)\cong H_s(\underline{b};M)\otimes\Lambda^0R^r\oplus\cdots\oplus H_{s-r+1}(\underline{b};M)\otimes\Lambda^{r-1}R^r \oplus H_{s-r}(\underline{b};M)\otimes\Lambda^rR^r.$$

Comparing with (*) gives
$$H_j(\underline{b};M)\begin{cases}=0 & ;\ j>s-r\\ \neq 0 & =s-r\end{cases}$$

Also shows $s\not<r$.

Use the $\Lambda^i R^r$ are free and compare from top.

Thus, $r=\min\{j : H_{s-j}(\underline{b};M)\neq 0\}$.

independent of $\underline{a}$

Consequences:

① All maximal M-regular sequences have the same length and

$$\operatorname{depth}(I;M)=\min\{j \mid H_{s-j}(\underline{b},M)\neq 0\},$$

where $I=\langle\underline{b}\rangle$.

② If $I=\langle b_1,\dots,b_s\rangle$, then $\operatorname{depth}(I;M)\le s$.

③ Let $a \in I$ be M-regular.
Then, $\operatorname{depth}(I, M/aM) = \operatorname{depth}(I, M) - 1$.

Idea: $a_1, \dots, a_r \in I$ is a max'l M/aM-reg. seq. in I
$\iff a, a_1, \dots, a_r \in I$ is a max'l M-reg. seq. in I.

Eg. Let $(R, \mathfrak{m}, k)$ be Noe. local, $\operatorname{depth}(R) = r$, $a_1, \dots, a_r \in \mathfrak{m}$ a max'l R-reg seq. Then

$$\operatorname{depth}\left(R/\langle a_1, \dots, a_i \rangle \right) = r - i.$$

In particular, $\operatorname{depth}\left(R/\langle \underline{a} \rangle \right) = 0$.

④ (Depth lemma)
Let $0 \to L \to M \to N \to 0$ be a s.e.s. of f.g. R-modules and $I \subset R$ an ideal. What can we say about depths?

Idea: If $I = \langle \underline{b} \rangle$, we have a s.e.s. of complexes

$$0 \to K_\bullet(\underline{b}; L) \to K_\bullet(\underline{b}; M) \to K_\bullet(b; N) \to 0.$$

Lecture 32 (05-04-2021)

05 April 2021 10:40

① (Depth lemma)

Let $0 \to L \to M \to N \to 0$ be a s.e.s. of f.g. R-modules and $I \subset R$ an ideal.

Then,

$$\operatorname{depth}(I, M) \geq \min\{\operatorname{depth}(I, L), \operatorname{depth}(I, N)\}$$

$$\begin{tikzcd} H_{n-(j-1)}(\underline{b}, N) \arrow[r] & H_{n-j}(\underline{b}, L) \arrow[r] & H_{n-j}(\underline{b}, M) \arrow[r] & H_{n-j}(\underline{b}, N) \arrow[d] \\ & & & H_{n-(j+1)}(\underline{b}, L). \end{tikzcd}$$

$$\operatorname{depth}(I, N) \geq \min\{\operatorname{depth}(I, M), \operatorname{depth}(I, L) - 1\}$$

$$\operatorname{depth}(I, L) \geq \min\{\operatorname{depth}(I, M), \operatorname{depth}(I, N) + 1\}$$

An observation about the Koszul complex (mainly H_1)

Let $\underline{a} = a_1, \ldots, a_r \in R$ and $\to K_2 \xrightarrow{\varphi_2} K_1 \xrightarrow{\varphi_1} K_0 \to 0$ be $K.(\underline{a})$

What is $\ker(\varphi_1)$? $\ker(\varphi_1) = \left\{ \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in R^r : [a_1 \cdots a_r] \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} = 0 \right\}$

$\shortparallel$

set of relations on $(a_1, \ldots, a_r)$

$$\operatorname{im}(\varphi_2) = \langle \underbrace{a_j e_i - a_i e_j}_{\shortparallel} \mid 1 \leq i < j \leq r \rangle$$

Koszul relations

$H_1(\underline{a}) = 0 \iff$ Koszul relations generate $\ker(\varphi_1)$

(Hence, being a regular sequence is the closest to linear independence in the ring.)

Suppose $H_1(\underline{a}) = 0$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in \ker(\varphi)$ Then, $\begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in \ker(\varphi_2)$

$$\Rightarrow \exists\, c_{ij} \in R \text{ s.t. } \begin{bmatrix} b_1 \\ \vdots \\ b_2 \end{bmatrix} = \sum_{1 \le i < j \le r} c_{ij}\,(a_j e_i - a_i e_j)$$

$$= C \begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix},$$

where $C \in M_r(R)$ is skew-sym

Thm. (Auslander-Buchsbaum Formula) Let $(R, \mathfrak{m}, k)$ be a Noe local ring, M a f g R module. If $\operatorname{pdim}_R(M) < \infty$, then

$$\operatorname{pdim}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R)$$

[Consequence: In this setup, $\operatorname{depth}(M) \le \operatorname{depth}(R)$

Q. Is this true if $\operatorname{pdim}_R(M) = \infty$?]

Proof. We prove this by induction on $\operatorname{depth}(R)$.

Suppose $\operatorname{depth}(R) = 0$ Then, $\mathfrak{m} \in \operatorname{Ass}(R)$ and hence, $\exists\, a \in R \setminus \{0\}$ s.t. $\mathfrak{m} = \operatorname{ann}_R(a)$

Let $F_\cdot : 0 \to F_n \to \quad \to F_0 \to 0$ be a min'l free resolution of M over R ($\operatorname{pdim}_R(M) = n$.)

Either $n = 0$ or $n \geq 1$ and $\varphi_n(F_n) \subset \mathfrak{m} F_{n-1}$

If $n \geq 1$, Then $\varphi_n \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \neq 0$. Thus, φ is not injective $\rightarrow\leftarrow$

Hence, $n = 0$ which forces M to be free, i.e., $M = R^{\oplus r}$ for some r

Hence, $\operatorname{depth}(M) = 0$

(Here $\mathfrak{m} \in \operatorname{Ass}_R(M)$ lets us conclude.)

[Ex $\operatorname{depth}(M \oplus N) \le \min\{\operatorname{depth}(M), \operatorname{depth}(N)\}$]

(Here $\mathfrak{m} \in \mathrm{Ass}_R(M)$ lets us conclude.)

[<u>Ex</u> $\mathrm{depth}(M \oplus N) \leq \min\{\mathrm{depth}(M), \mathrm{depth}(N)\}$.
$(R, \mathfrak{m}, k)$ local, M, N f.g.]

Thus, we are done if $\mathrm{depth}(R) = 0$

Assume $\mathrm{depth}(R) > 0$ We finish the case of $\mathrm{depth}(M) \neq 0$ as follows

Suppose $\mathrm{depth}(M) > 0$ By prime avoidance, $\exists$ a n.z.d a on R and M since $\mathfrak{m} \notin \mathrm{Ass}(R) \cup \mathrm{Ass}(M)$

Now, first observe that $\mathrm{depth}(M/aM) = \mathrm{depth}(M) - 1$ and
$\mathrm{depth}(M/aR) = \mathrm{depth}(R) - 1$

<u>Recall</u>: $\mathrm{pdim}_R(M) = \mathrm{pdim}_{R/aR}(M/aM)$ (Use $0 \to M \xrightarrow{\cdot a} M \to M/aM \to 0$)

Thus, by induction, this case is done

Now we look at the case $\mathrm{depth}(R) > 0$ and $\mathrm{depth}(M) > 0$
Note that $\mathfrak{m} \in \mathrm{Ass}_R(M)$ and hence, $\exists\, x \in M \setminus \{0\}$ s.t
$$\mathfrak{m} = \mathrm{ann}_R(x)$$

Let $\mathrm{pdim}_R(M) = p$ and $\mathrm{depth}(R) = r$ and $a_1, \ldots, a_r \in \mathfrak{m}$ be a maximal regular sequence (on R) Then, if $\bar{R} = R/\langle \underline{a} \rangle$, we have $\mathrm{depth}(\bar{R}) = 0$

Idea of proof: Compute $\mathrm{Tor}_i^R(M, \bar{R})$ in two different ways.

Since $\underline{a}$ is regular, we see that $K_\bullet(\underline{a})$ resolves $\bar{R} = R/\langle \underline{a} \rangle$ over R.

Hence, $\mathrm{Tor}_i^R(M, \bar{R}) \cong H_i(\underline{a}, M)$

Thus, $H_i(\underline{a}, M) = \begin{cases} 0 & \text{for } i > r, \end{cases}$

Thus, $H_i(\underline{a}, M) = \begin{cases} 0 & \text{for } i > r, \\ \operatorname{ann}_M(\underline{a}) & \text{for } i = r \end{cases}$

Since $x \in \operatorname{ann}_M(\underline{a})$, we see that $\operatorname{Tor}_r^R(M, \bar{R}) \neq 0$ and $\operatorname{Tor}_i^R(M, \bar{R}) = 0 \quad \forall\, i > r.$

OTOH, since $\operatorname{pdim}_R(M) = p$, we see that $\operatorname{Tor}_i^R(M, \bar{R}) = 0 \ \forall\, i > p$. Hence, $p \geq r$. To show equality, we prove $\operatorname{Tor}_p^R(M, \bar{R}) \neq 0$

Note that $\operatorname{Tor}_p^R(M, \Bbbk) \neq 0$ since $\operatorname{pdim}_R(M) = p$.

Now, since $\operatorname{depth}(\bar{R}) = 0$, $\mathfrak{m} \in \operatorname{Ass}(\bar{R}) \Rightarrow k \hookrightarrow \bar{R}$. Let C be the co-kernel. Note that

$\operatorname{Tor}_{p+1}^R(M, C) = 0$ since $\operatorname{pdim}_R(M) = p$

Hence, the l.e.s of homology after tensoring

$0 \to k \to \bar{R} \to C \to 0$ with M yields

$$0 \to \underset{\substack{\nparallel \\ 0}}{\operatorname{Tor}_p^R(M, k)} \to \operatorname{Tor}_p^R(M, \bar{R}) \to$$

Since $\operatorname{Tor}_p^R(M, k) \neq 0$, we get $\operatorname{Tor}_p^R(M, \bar{R}) \neq 0$, completing the proof ∎

Obs. Let $(R, \mathfrak{m}, \Bbbk)$ be Noe. local, M a f.g R-module. Recall that $\operatorname{pdim}_R(M) = \sup\{i \cdot \operatorname{Tor}_i^R(M, k) \neq 0\}$.

$$\left(\dim_k(\operatorname{Tor}_i^R(M, \Bbbk)) = \beta_i^R(M).\right)$$

In the proof above, we showed that $\operatorname{pdim}_R(M) = \sup\{i \ \operatorname{Tor}_i^R(M, \bar{R}) = 0\}$

Q Can you generalise this? Can we replace $\bar{R}$ with something else?

Ex. $\operatorname{pdim}_R(M) = \sup\{i : \operatorname{Tor}_i^R(M, N) \neq 0\}$ for any R-module N with $\operatorname{depth}(N) = 0$

Lecture 33 (06-04-2021)

06 April 2021 11:37

Essence of the splitting lemma

① Let $(R, \mathfrak{m}, \Bbbk)$ be Noe local; F, G be free R-modules of finite rank and $\varphi : F \to G$ be R-linear.
If $\tilde{\varphi} : F/\mathfrak{m}F \to G/\mathfrak{m}G$ is an isomorphism, then so is φ

② Let R, F, G, φ be as above.
If $\tilde{\varphi}$ $F/\mathfrak{m}F \to G/\mathfrak{m}G$ is one-one, then φ induces a splitting $\psi : G \to F$. (In particular, φ is one-one.)

Proof. Let $\{e_1, , e_n\}$ be a basis of F. By hypothesis, $\{\tilde{\varphi}(\bar{e}_1), . , \tilde{\varphi}(e_n)\}$ is lin indep. Extend to a basis $\{\tilde{\varphi}(\bar{e}_i), \bar{f}_1, .., \bar{f}_m\}$
Then, $G = \langle \varphi(e_1), , \varphi(e_n), f_1, , f_m \rangle$ where $\operatorname{rank}(G) = n+m$

Now, we claim that $\{\varphi(e_i), f_j\}$ is a basis of G over R, and this completes the proof. [Define ψ by $\varphi(e_i) \mapsto e_i$ and $f_j \mapsto 0$]

But that is simply true in general ∎

Thm (Serre)

Let $(R, \mathfrak{m}, \Bbbk)$ be Noe local. If $\mu(\mathfrak{m}) = s$, then
$$\beta_i^R(\Bbbk) \geq \binom{s}{i} \qquad \forall\, i$$

Proof Let $\mathfrak{m} = \langle a_1, . . , a_s \rangle$, $K_\bullet = K_\bullet(\underline{a})$ and $F_\bullet$ be a min'l free resolution of K over R.
Key point $K_\bullet$ splits off $F_\bullet$

Remaining proof Presentation later

This says that $\Bbbk$ satisfies the Buchsbaum-Eisenbud-Horrocks conjecture.

Q Let $(R, \mathfrak{m}, \Bbbk)$ be a Noe. local ring with $\operatorname{depth}(R) = d$

TFAE

(i) $\operatorname{pdim}_R(\Bbbk) < \infty$,

(ii) $\operatorname{pdim}_R(M) < \infty \quad \forall$ f g M,

(iii) $\mathfrak{m}$ is generated by a regular sequence,

(iv) $\mathfrak{m}$ is generated by d elements,

(v) $\operatorname{pdim}_R(\Bbbk) = d$.

(Auslander-Buchsbaum-Serre Thm)

Graded objects (and arrows)

Def^n ⓐ Let R be a ring We say that R is $\mathbb{N}_0$-graded (or just graded) if $\exists$ subgroups R_i of R such that

(i) $R = \bigoplus_{i \geq 0} R_i$ as (abelian) groups,

(ii) $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{N}_0$

ⓑ An element $a \in R_i \setminus \{0\}$ is said to be homogeneous of degree i

ⓒ An R-module M is graded if ① $M = \bigoplus M_i$ as groups,
② $R_i M_j \subset M_{i+j} \quad \forall\, i \in \mathbb{N}_0, \forall j \in \mathbb{Z}$.

ⓓ Let M be a graded R-module. We say that the submodule $N \subset M$ is graded (or homogeneous) if it generated by homogeneous elements

Observations

(1) $1 \in R_0$ $R_0 \subset R$ is a subring

(2) Every M_i is an R_0- module.

(3) $N \subset M$ graded $\Leftrightarrow$ $N = \bigoplus_{i \in \mathbb{Z}} (N_i \cap M_j)$

$$\Updownarrow$$

$\forall x \in N$ if $x = x_0 + \quad + x_d$ is a homogeneous decomposition,

then $x_i \in N$ $\forall i$

Lecture 34 (08-04-2021)

08 April 2021 09:26

ⓔ Let $N \subset M$ be a graded submodule. Then, M/N is also graded with $(M/N)_i = M_i/N_i$

Observe ① Let $\pi : M \to M/N$ be the natural projection. Then, $\pi(M_i) \subset \underset{\underset{(M/N)_i}{\parallel}}{M_i/N_i} \ \forall i$

We say that π is a "graded map" of "degree 0"

② If $I \subset R$ is homogeneous, then R/I is a graded ring with $(R/I)_j = R_j/I_j$

ⓕ Let $\varphi \ \overset{\text{graded}}{M \longrightarrow N}$ be R-linear. We say φ is graded of degree i if $\varphi(M_j) \subset N_{i+j} \ \forall j$

Q ① Are compositions of graded maps graded?
② Are there non-graded R-linear maps?
③ Are ker(φ) and im(φ) graded?

ⓖ Let M, N be graded. What about $M \otimes N$?

It is graded with

$$(M \otimes N)_k = \bigoplus_{i+j=k} (M_i \otimes N_j)$$

Is $\bigoplus_{i \in \mathbb{Z}} (M_i \otimes N_i)$ a graded module?

ⓗ $M \oplus N$ is graded as $\bigoplus_{i \in \mathbb{Z}} (M_i \oplus N_i)$

Examples. ① $R = \mathbb{k}[x_1, \;, x_d]$ is graded with

$R_0 = \mathbb{k}$,

$R_n = \mathbb{k}\langle x^{a_1} \cdot \; x^{a_d} \cdot \; \Sigma a_i = n \rangle$ for $n \geq 1$

$\mathbb{k}$ - vec. space spanned by all monomials of degree n (in d-variables)

Q What is $\dim_{\mathbb{k}}(R_n)$? $\binom{n+d-1}{d-1}$

The above is called the standard grading

Note that there is always the trivial grading with $R_0 = R$ and $R_n = 0 \ \forall n \geq 1$

② Let $R = k[x, y]$ Is $x^2 - y^3$ homogeneous?

No (In standard grading)

Can we make it?

Standard ways to make it homogeneous:

① Homogenization Embed $R \hookrightarrow k[x, y, z]$ $f = x^2 - y^3$

$f_h = x^2 z - y^3$

② Weighted variables Set $\deg(x) = 3$, $\deg(y) = 2$

Then, f is homogeneous of degree 6