Lecture 1 (03-01-2022)

03 January 2022 13:58

Texts : · Real and Complex Analysis – Rudin
· Complex Analysis – Lang

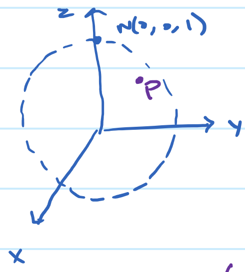
Topics : Review of basic $\mathbb{C}$ analysis, Harmonic functions, ...,
Maximum modulus theorem, ...,
Runge's, Mittag-Leffler, Weierstrass theorems,
Riemann mapping theorem,
Analytic continuation,
Little / Big Picard's theorem,
(If time) Introduction to Several Complex variables.

Evaluation (tentative!!!) : · Presentation 10-15 %
· Assignments 10-15 %
· Midsem, Endsem
· 1/2 Quizzes maybe

Riemann sphere / The Extended Complex plane.

$$\hat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}.$$

The stereographic projection is a function $\phi : S^2 \longrightarrow \hat{\mathbb{C}}$. ($S^2 \subseteq \mathbb{R}^3$)



$P \in S^3$, $P \neq N$.

Define the stereographic projection of $P(x,y,z)$ $\neq N$ as follows:

Join N to P. Extend it. It hits the (equitorial) plane $z=0$ at some point $(x,y,0)$.

$P \longmapsto x+iy$ is the map.

Stereographic projection

Analytically the line is :

Analytically, the line is :

$$t(x, y, z) + (1-t)(0, 0, 1).$$

We need $tz + 1-t = 0$ or $t = \frac{1}{1-z}$.

$\therefore \quad x = \frac{x}{1-z}$ and $y = \frac{y}{1-z}$. (Note : $z \neq 1$.)

Finally, $N \longmapsto \infty$.

(E.g.: Under the above map, $(0, 0, -1) \longmapsto (0, 0)$ or $0 + 0i$.)

To sum it up: Define $\theta : S^2 \longrightarrow \hat{\mathbb{C}}$ by

$$\theta(x, y, z) = \begin{cases} \frac{x}{1-z} + \frac{iy}{1-z} & ; \quad z \neq 1, \\ \infty & ; \quad \text{else}. \end{cases}$$

Check : θ is a bijection.

To see that it is onto, let $z = x + iy \in \mathbb{C}$ be arbit.

Check that

$$P(x, y, z) := \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right)$$

maps to z. (As usual, $|z| = x^2 + y^2$.)

Q. What happens to P above as $|z| \longrightarrow \infty$?

Evidently $P \longrightarrow N\,(0, 0, 1)$.

→ Using the above, we can define a topology on $\hat{\mathbb{C}}$.

In fact, we now define a metric on $\hat{\mathbb{C}}$ as follows:

For $w, z \in \hat{\mathbb{C}}$, define the distance between w and z to be the length of the straight line segment joining $\theta^{-1}(w)$ and $\theta^{-1}(z)$, i.e.,

$$d(w, z) := \|\theta^{-1}(w) - \theta^{-1}(z)\|_3 .$$

$$= \frac{\sqrt{2}\,|w-z|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$ after calculations (both $z, w \neq \mathbb{C}$)

If $w = \infty$ and $z \neq \infty$, we get $d(z, \infty) = \dfrac{\sqrt{2}}{\sqrt{1+|z|^2}}$.

Fix $z \in \hat{\mathbb{C}}$, $r > 0$.

$$B_d(z, r) := \{ w \in \hat{\mathbb{C}} : d(z, w) < r \}.$$

Describe the above set when $z = \infty$

Describe the open nbds in $\hat{\mathbb{C}}$.

Def^n. A domain in $\mathbb{C}$ is an open connected subset of $\mathbb{C}$.

(Nonempty!)

Domain

Remark. Automatically path-connected. Ω will usually denote a domain

Def^n. Let $\Omega \subseteq \mathbb{C}$ be a domain, $z_0 \in \Omega$, let $f : \Omega \to \mathbb{C}$.

We say that f is (complex) differentiable at z_0 if

Complex differentiable

$$\lim_{\mathbb{C} \ni h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists (and is finite).

The above quantity is then denoted $f'(z_0)$ and called the (complex) derivative of f at z_0.

Def^n. $\Omega \subseteq \mathbb{C}$ domain, $f : \Omega \to \mathbb{C}$ function.

f is said to be complex analytic/holomorphic on Ω if f is complex differentiable at each point of Ω.

Complex analytic, holomorphic

$$\mathcal{O}(\Omega) := \{ f : \Omega \to \mathbb{C} \text{ is holomorphic} \}.$$

script O

$\mathbb{R}$-valued

script $\mathcal{O}$

Obs. Let $f \in \mathcal{O}(\Omega)$. Write

$$f(z) = u(z) + i v(z)$$

$\mathbb{R}$-valued

$$= u(x+iy) + i v(x+iy)$$

$$= u(x,y) + i v(x,y)$$

treat $\Omega \subseteq \mathbb{R}^2$

Fix $z_0 = x_0 + i y_0 \in \Omega$.

$$f'(z_0) = \lim_{\mathbb{C} \ni h \to 0} \frac{f'(z_0+h) - f(z_0)}{h} \text{ exists.}$$

let $h \to 0$, $\mathbb{R}$

$$= \lim_{\mathbb{R} \ni h \to 0} \left\{ \left[\frac{u(x_0+h, y_0) - u(x_0,y_0)}{h} \right] + i \left[\frac{v(x_0, y_0+h) - v(x_0,y_0)}{h} \right] \right\}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0).$$

Similarly let $h \to 0$ along $i\mathbb{R}$.

Then, $$f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i \frac{\partial u}{\partial y}(x_0,y_0).$$

In particular, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{on } \Omega.$$

Cauchy-Riemann equations

Def^n. Define the operators

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right)$$

if $f \in \mathcal{O}(\Omega)$

$$\equiv 0$$

Check: If f is complex diff. at z_0, it is also real differentiable as a function $\Omega^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^2$.

Lecture 2 (06-01-2022)

06 January 2022 14:01

Integration:

Integration

Let Ω be a domain in $\mathbb{C}$, and $\gamma : [a,b] \longrightarrow \Omega$ is piecewise - $\mathcal{C}^1$. For any $f \in \mathcal{C}^0(\Omega)$, $(f : \Omega \to \mathbb{C})$

$$\int_\gamma f := \int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\, \gamma'(t)\, dt.$$

Index of a point wrt. a path:

Fix $\gamma : [a, b] \longrightarrow \mathbb{C}$ is piecewise $\mathcal{C}^1$. Assume γ is closed, i.e., $\gamma(a) = \gamma(b)$. Let $\Omega := \mathbb{C} \setminus \mathrm{im}(\gamma)$.

Then, Ω has possibly many connected components, out of which exactly one is unbounded.

Let $z_0 \in \Omega$. We define

$$\mathrm{Ind}_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z_0}\, d\xi$$

well-defined since $z \notin \mathrm{im}(\gamma)$.

$$= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt.$$

Properties:

(1) Ind_γ is an integer-valued function on Ω.

(2) Thus, Ind_γ is constant on the connected components of Ω.

(3) $\mathrm{Ind}_\gamma \equiv 0$ on the unbounded component.

Prop^n. (Cauchy's Theorem)   Cauchy's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $f : \Omega \to \mathbb{C}$ be continuous.

TFAE:

(i) $\int_\gamma f = 0$ for every closed γ in Ω.

(ii) $\exists F \in \mathcal{O}(\Omega)$ such that $F' \equiv f$ on Ω.

Consequently, $f \in \mathcal{O}(\Omega)$ (since once differentiable is always differentiable).

Example. Let γ be ... in $\mathbb{C}$.

If $a \notin \text{im}(\gamma)$, then evaluate

$$I_n := \int_\gamma (z-a)^n \, dz \quad \text{for} \quad n \in \mathbb{Z}.$$

If $n \neq -1$, we have an antiderivative for the integrand on $\mathbb{C} \setminus \{a\}$. $\therefore I_n = 0$.

If $n = -1$, then we simply have $I_n = 2\pi i \, \text{Ind}_\gamma(a)$.

Def^n. Path homotopy   Path homotopy

Given $\gamma_0, \gamma_1 : [0,1] \to \Omega$ two closed paths in Ω based at x_0.

A path homotopy between γ_0 and γ_1 is a function

$$H : [0,1] \times [0,1] \to \Omega$$

s.t. ① H is continuous,

② $H(s,0) = \gamma_0(s) \quad \forall s \in [0,1]$,

③ $H(s,1) = \gamma_1(s) \quad \forall s \in [0,1]$

④ $H(0,t) = x_0 = H(1,t) \quad \forall t \in [0,1]$.

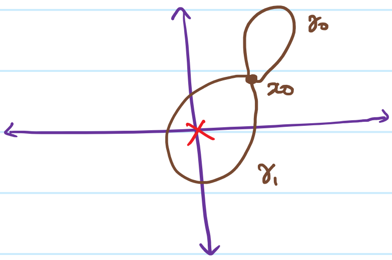
Recall: $\gamma_0 \sim \gamma_1$, (equiv. rel'n) path-homotopic, null-homotopic ($\gamma \sim 0$).

EXAMPLES. (1) $\Omega = \mathbb{C}$.

Any two loops are homotopic.

Indeed, $H(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$ does the job.

② $\Omega = \mathbb{C}\setminus\{0\}$.
$x_0 = 1+i$.
The drawn loops are not homotopic.



Theorem. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let γ_0, γ_1 be loops based at the same point with $\gamma_0 \sim \gamma_1$. Then,

$$\int_{\gamma_0} f = \int_{\gamma_1} f \qquad \text{for all } f \in \mathcal{O}(\Omega).$$

EXAMPLE. The paths $\gamma_1, \gamma_2 : [0,1] \to \Omega = \mathbb{C}\setminus\{0\}$ defined as

$$\gamma_1(t) := \frac{1}{\gamma_0(t)} := e^{-2\pi i t}$$

cannot be path homotopic (in Ω) since $f = (z \mapsto \frac{1}{z}) \in \mathcal{O}(\Omega)$ and

$$\int_{\gamma_0} f = 2\pi i \neq -2\pi i = \int_{\gamma_1} f.$$

Corollary. Let Ω be a domain and γ be a loop in Ω with $\gamma \sim 0$. Then,

$$\int_\gamma f = 0 \quad \forall f \in \mathcal{O}(\Omega).$$

Def^n. An open set $\Omega \subseteq \mathbb{C}$ is said to be simply-connected if Ω is connected and $\gamma \sim 0$ for every loop γ in Ω.

Simply-connected, simply connected

(NON-)EXAMPLES:

- $\mathbb{C}$, $D(0,1)$, convex sets, star-shaped domains, $\mathbb{C}\setminus[0,\infty)$
  ↳ s-c
- $\mathbb{C}\setminus\{0\}$, $D(0,1)\setminus\{0\}$, $D(0,a)\setminus D(0,b)$ for $a>b>0$

↳ not s-c.

Corollary. Let Ω be a s-c domain in $\mathbb{C}$ and let γ be a loop in Ω. Then,

$$\int_\gamma f = 0 \qquad \forall\, f \in \mathcal{O}(\Omega).$$

Cor. Let Ω be a s-c domain in $\mathbb{C}$. Let $f \in \mathcal{O}(\Omega)$. Then, $\exists F \in \mathcal{O}(\Omega)$ s.t. $F' = f$ on Ω.

Cor. Let Ω be a s-c domain in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$ be st. $f(z) \neq 0$ $\forall z \in \Omega$. Then, $\exists g \in \mathcal{O}(\Omega)$ s.t.

Analytic branch of logarithm

$$f = \exp \circ g.$$

(g is an analytic branch of logarithm of f.)

Proof. Since $f \neq 0$, $\frac{f'}{f} \in \mathcal{O}(\Omega)$.

$\because \Omega$ is s-c, $\exists h \in \mathcal{O}(\Omega)$ s.t. $h' = \frac{f'}{f}$.

Let $\tilde{g} := \exp \circ h$

Then, $\tilde{g} \neq 0$. $\therefore \frac{f}{\tilde{g}} \in \mathcal{O}(\Omega)$.

$$\tilde{g}^2 \left(\frac{f}{\tilde{g}}\right)' = f'\tilde{g} - f\tilde{g}'$$

$$= f'\tilde{g} - f\,(\exp \circ h) \cdot h'$$

$$= f'\tilde{g} - f\tilde{g}h' = \tilde{g} \cdot (f' - fh') = 0.$$

$$\therefore \frac{f}{\tilde{g}} \equiv c \neq 0.$$

$$\Rightarrow f = c \cdot \tilde{g} = c \cdot \exp \circ h = \exp \circ (h + c')$$

$=: g.$ $\square$

Lecture 3 (10-01-2022)

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Maximum Principle

① Let $\Omega \subseteq \mathbb{C}$ be a domain, and $f \in \mathcal{O}(\Omega)$.
Let $a \in \Omega$ such that $\exists r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$.
Then,

$$|f(a)| \leq \max_{0 \leq \theta \leq 2\pi} |f(a + re^{i\theta})|.$$

Moreover, equality holds iff f is constant.

② Let Ω be a bounded open set in $\mathbb{C}$. (Maximum Modulus Theorem)
Let $f \in C^0(\overline{\Omega}) \cap \mathcal{O}(\Omega)$. Then,

Maximum Modulus Theorem

$$|f(z)| \leq \max_{\partial\Omega} |f| \qquad \forall z \in \Omega.$$

In words, $|f|$ attains its maximum on the boundary.

Equivalently:

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

EXAMPLE: $\mathbb{H} := \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$.

Define $f(z) = \exp(-z^2)$ on $\overline{\mathbb{H}}$.

$f \in \mathcal{O}(\mathbb{H}) \cap C^0(\overline{\mathbb{H}})$.

Note that $|f(z)| \leq 1$ for $z \in \mathbb{R} = \partial\mathbb{H}$.

But

$$|f(iy)| = e^{y^2} \text{ grows rapidly on } i\mathbb{R}.$$

Thus, MMT need not hold if Ω is unbounded.

Now, we wish to formulate a similar theorem for unbounded.

- Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$.
For $a \in \overline{\Omega}$, define

$$\limsup_{\Omega \ni z \to a} f(z) := \lim_{r \to 0^+} \sup\{|f(z)| : z \in \Omega \cap D(a,r)\}.$$

this limit exists in $[0, \infty]$.

If a is the point at infinity, $D(a,r)$ is the neighbourhood of a in the metric d on the extended complex plane.

The extended boundary of Ω in $\mathbb{C} \cup \{\infty\}$ is denoted by $\partial_\infty \Omega$.
Note:

$$\partial_\infty \Omega = \begin{cases} \partial\Omega & ; \ \Omega \text{ is bounded,} \\ \partial\Omega \cup \{\infty\} & ; \text{ else.} \end{cases}$$

③ MMT: Let Ω be a domain in $\mathbb{C}$, $f \in \mathcal{O}(\Omega)$.
(Not necessarily bounded!)

Suppose that $\displaystyle\limsup_{z \to a} |f(z)| \leq M$ for all $a \in \partial_\infty \Omega$.

Then, $|f| \leq M$ on Ω.

Proof. Let $\delta > 0$, and define $S := \{z \in \Omega : |f(z)| > M + \delta\}$.
We will show that $S = \emptyset$ and be done.
Note that $|f|$ is continuous and thus, S is open.
We claim that S is bounded in $\mathbb{C}$.
If Ω is bounded, then done.
Suppose Ω is unbounded. Then, $\infty \in \partial_\infty \Omega$.
But $\displaystyle\limsup_{\Omega \ni z \to \infty} |f(z)| < M$.

$\therefore \exists R > 0$ s.t. $|f(z)| < M + \delta$ for all $|z| \geq R$, $z \in \Omega$.
(Check: $D(\infty, r)$ is the complement of a compact set)

$\therefore S \subseteq D(0, R)$.

Applying MMT (2) to $f|_S$, we see that

$$|f(z)| \leq \max_{\partial S} |f| \qquad \forall z \in S.$$

But by def^n of S it follows that $|f| = M+\delta$ on ∂S.

$$\therefore \quad |f(z)| \leq M+\delta \qquad \forall z \in S.$$

But by def^n of S, we have $|f(z)| > M+\delta$ for $z \in S$.

$\therefore S = \phi$. ∎

Remark. In our previous example, we have $\limsup_{\mathbb{H} \ni z \to \infty} f(z) = \infty$.

Thus, this MMT did not apply!

Generalisations of MMT to unbounded domains.

Phragmén - Lindelöf Theorems. Phragmén-Lindelöf, Phragmen-Lindelof

\# Liouville's Theorem: Bounded + entire $\Rightarrow$ constant

Also, recall the following exercise (using Cauchy's estimate, for example):

If $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)| \leq 1 + |z|^{1/3}$, then f is constant.

$\hookrightarrow$ "Generalisation" of Liouville.

Similarly, we generalise MMT.

(Phragmén - Lindelöf)

Theorem A. Let $\Omega \subseteq \mathbb{C}$ be simply-connected, and $f \in \mathcal{O}(\Omega)$. Fix $M > 0$.

Let $\partial_\infty \Omega = I \cup II$ be such that

(1) $\limsup_{\Omega \ni z \to a} |f(z)| \leq M$ for all $a \in I$, and

(2) $\exists\, \phi \in \mathcal{O}(\Omega)$, nonvanishing and bounded on Ω such that

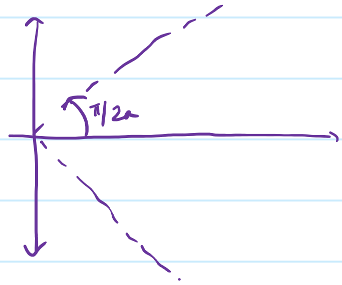
$$\limsup_{\Omega \ni z \to a} \left| f(z)\,(\phi(z))^{\eta} \right| \leq M$$

for all $a \in \mathbb{I}$ and for all $\eta > 0$.

Then, $|f| \leq M$ on Ω.

EXAMPLE. Fix $a \geq 1/2$. Let $\Omega = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{2a} \right\}$.

Let $f \in \mathcal{O}(\Omega)$ be s.t.



(a) $\overline{\lim}_{z \to \xi \in \partial\Omega} |f(z)| \leq M$, and

(b) $|f(z)| \leq A \exp(|z|^b)$ for $|z| \gg 1$,

where A and b are positive constants such that $b < a$.

using above theorem

Then, $|f(z)| \leq M \quad \forall\, z \in \Omega$.

Clearly Ω is s-c.

Now, we find $\phi \in \mathcal{O}(\Omega)$ as in P-L.

Consider $\phi(z) = \exp(-z^c)$, where $c > 0$ is chosen later.

Note that this is holo on Ω.

Also, $\phi(z) \neq 0 \quad \forall\, z \in \Omega$

$$|\phi(z)| = |\exp(-z^c)| \qquad z = re^{i\theta},\ |\theta| < \pi/2a.$$

$$= |\exp(-r^c e^{i\theta c})|$$

$$= \exp(-r^c \cos(c\theta)) \leq 1.$$

if $c < a$, then $\cos(c\theta) \geq 0$.

Thus, ϕ is bdd.

Take $I = \partial\Omega$ and $II = \{\infty\}$.

Now, fix $\eta > 0$ and for $z = re^{i\theta} \in \Omega$.

For large z, we have

$$|f(z)\,\phi(z)^\eta| \leq A\exp(|z|^b)\,|\exp(-z^c)|^\eta$$
$$= A\exp(r^b - \eta r^c \cos c\theta)$$
$$\leq A\exp(r^b - \eta r^c\,\delta).$$

$\delta := \inf\limits_{0 \leq \theta < \pi/2a} \cos(c\theta)$.

The above goes to 0 if $c > b$.

Thus, we can choose any $c \in (b, a)$.

We are now done. ∎

Proof of P-L. Since Ω is s-c and ϕ nonvanishing, ϕ has an analytic log, and ϕ^η makes sense as a holo. function.

$\phi^\eta = \exp(\eta \log\phi)$

Let $K > 0$ be s.t. $|\phi| \leq K$ on Ω.

Consider $g_\eta(z) := g(z) := \dfrac{f(z)\,\phi(z)^\eta}{K^\eta}$. $g \in \mathcal{O}(\Omega)$.

Note: $|g(z)| \leq |f(z)|$.

Thus,

$$\limsup_{z \to \xi \in I} |g(z)| \leq M.$$

On II:

$$\limsup_{z \to \xi \in II} \left|\frac{f(z)\,\phi(z)^\eta}{K^\eta}\right| \leq \frac{M}{K^\eta}.$$

Now, MMT (3) from earlier applied to g, we get that

$$|g_\eta| \leq \max\left(M, \frac{M}{K^\eta}\right) \quad \text{on } \Omega.$$

$$\Rightarrow \quad |f(z)| \leq |\phi(z)|^{-\eta} \max\left(MK^\eta, M\right)$$

$\forall z \in \Omega, \forall \eta > 0.$

Fix $z \in \Omega$ and let $\eta \to 0^+$ to conclude. $\blacksquare$

Lecture 4 (13-01-2022)

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(Phragmén - Lindëlof)

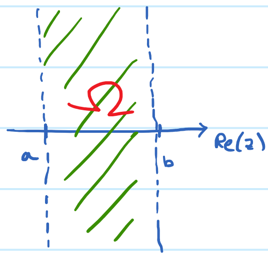
Theorem B. Fix reals $a < b$, and $B > 0$.

Let $\Omega = \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$, and $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

Assume that:

$$|f| \leq B \quad \text{on} \quad \Omega,$$

$$|f| \leq 1 \quad \text{on} \quad \partial\Omega.$$



Then,

$$|f| \leq 1 \quad \text{on} \quad \Omega.$$

Remark: Note that the above is a type of MMT.

Idea: Introduce a typical multiplicative factor g_ε with $\lim_{\varepsilon \to 0} g_\varepsilon = 1$, such that $|fg_\varepsilon| < M$ on the boundary of a BOUNDED subdomain Ω_ε of Ω. Then, apply usual MMT on Ω_ε.
Moreover, pick the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ nicely enough to cover all of Ω. Then take $\varepsilon \to 0$.

Proof. For each $\varepsilon > 0$, define $g_\varepsilon : \bar{\Omega} \to \mathbb{C}$ by

$$g_\varepsilon(z) := \frac{1}{1 + \varepsilon(z-a)}.$$

↳ denominator is 0 iff $z = a - \frac{1}{\varepsilon} \notin \bar{\Omega}$.

For $z \in \partial\Omega$, we have:

$$|f(z)\, g_\varepsilon(z)| \leq \frac{1}{|1 + \varepsilon(z-a)|} \leq \frac{1}{|\mathrm{Re}(1 + \varepsilon(z-a))|}$$

$$= \frac{1}{|1 + \varepsilon(\mathrm{Re}(z) - a)|}$$

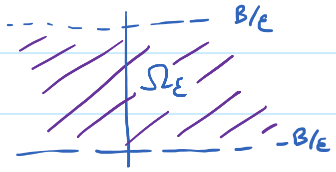
$$|1 + \varepsilon(\mathrm{Re}(z) - a)| \leq 1.$$

For $z = x + iy \in \overline{\Omega}$, we have:

(Note: $|f| \leq B$ on $\overline{\Omega}$ by continuity.)

$$|f(z)\, g_\varepsilon(z)| \leq \frac{B}{|1 + \varepsilon(z - a)|} \leq \frac{B}{|\mathrm{Im}(1 + \varepsilon(z - a))|}$$

$$= \frac{B}{|\varepsilon y|} = \frac{1}{|y|} \frac{B}{\varepsilon}. \qquad (*)$$



Now, define $\Omega_\varepsilon := \{ z \in \overline{\Omega} : -B/\varepsilon < y < B/\varepsilon \}$.

We have proven above that $|f g_\varepsilon| \leq 1$ on $\partial \Omega_\varepsilon$.
By the usual MMT, we have $|f g_\varepsilon| \leq 1$ on Ω_ε.

But also, by $(*)$, we see that $|f g_\varepsilon(z)| \leq 1$ if $|y| > \frac{B}{\varepsilon}$ as well!

Thus, for all z and for all $\varepsilon > 0$, we have the inequality. Fix z and let $\varepsilon \to 0^+$ to conclude. ∎

Theorem C. Fix reals $a < b$, and $B > 0$.
Let $\Omega = \{ z \in \mathbb{C} : a < \mathrm{Re}(z) < b \}$, and $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega}) \setminus \{0\}$.
Assume that $|f| < B$. Define $M : [a, b] \to [0, \infty)$ by

$$M(x) := \sup_{y \in \mathbb{R}} |f(x + iy)|.$$

Then $\log \circ M$ is a convex function on (a, b).

Remarks: (i) For $a \leq x < v < y \leq b$:

$$M(v)^{(y-x)} \leq M(x)^{(y-v)} \cdot M(y)^{(v-x)}.$$

$$M(v)^{(y-x)} \leq M(x)^{(y-v)} \cdot M(y)^{(v-x)}.$$

(ii) Since $\log \circ M$ is convex on (a, b), we get

$$M(x) \leq \max\{M(a), M(b)\} \quad \forall\, x \in [a, b].$$

In particular, if $M(a) = M(b)$, then we get Theorem B.

(iii) By continuity, we have $|f| \leq B$ on $\partial\Omega$. Thus, $\sup_{\partial\Omega} |f| < \infty$.

By the above, we get

$$|f| \leq \sup_{\partial\Omega} |f| \quad \text{on } \Omega.$$

<u>Proof</u> Suffices to show that

$$M(v)^{b-a} \leq M(a)^{b-v} \cdot M(b)^{v-a}$$

for $v \in (a, b)$.

What if $M(a)$ or $M(b) = 0$?
Take care of this separately.

Consider the entire function g defined as

$$g(z) := M(a)^{\frac{b-z}{b-a}} \cdot M(b)^{\frac{z-a}{b-a}}.$$

$(\lambda^z := \exp(z \log \lambda))$

Also note that g is nonvanishing.

$$\cdot \quad |g(z)| = M(a)^{\frac{b-x}{b-a}} \cdot M(b)^{\frac{x-a}{b-a}}.$$

The above is continuous as a function of x and is non-vanishing. Thus, $\exists\, C > 0$ s.t. $\frac{1}{\cdot} \leq C$ on $\bar{\Omega}$.

$|g(z)|$

Now, consider $\frac{f}{g} \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

$$\left|\frac{f(a+iy)}{g(a+iy)}\right| = \left|\frac{f(a+iy)}{M(a)}\right| \leq 1 \qquad \forall y \in \mathbb{R}.$$

$||^{ly}$
$$\left|\frac{f}{g}\right| \leq 1 \quad \text{on} \quad \partial\Omega.$$

Moreover, $\left|\frac{f}{g}\right| \leq CB$ on Ω

Thus, by Theorem B, we have $\left|\frac{f}{g}\right| \leq 1$ on Ω or

$|f| \leq |g|$ on Ω.

Expanding out, we get

$$|f(x+iy)|^{b-a} \leq M(a)^{b-x} \, M(b)^{x-a}$$

$\forall x \in (a,b)$
$\forall y \in \mathbb{R}$.

Take sup over $y \in \mathbb{R}$ and we are done. ∎

Consequences of MMT.

Schwarz Lemma: Let $f \in \mathcal{O}(D(0,1))$ such that $f(0) = 0$ and $|f| \leq 1$.

Then,

(a) $|f'(0)| \leq 1$ and

(b) $|f(z)| \leq |z| \quad \forall z \in D(0,1)$.

Moreover if equality holds either in (a) or for some $z \neq 0$ in (b), then $\exists \lambda \in S^1$ s.t. $f(z) = \lambda z$.

Sketch: Define $g : D(0,1) \to \mathbb{C}$ by $g(z) := \frac{f(z)}{z}$ holomorphically.
Fix $r \in (0,1)$. On $|z| = r$, we have

$$|g(z)| = \left|\frac{f(z)}{z}\right| \le \frac{1}{r}.$$

$\therefore$ $|g| \le \frac{1}{r}$ on $D(0,r)$ by MMT.

Let $r \to 1$ appropriately to get $|g| \le 1$ on $D(0,1)$.

This gives (a) and (b). Equality in MMT $\Rightarrow$ $\frac{f(z)}{z}$ is const. ∎

Lecture 5 (17-01-2022)

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- $\mathbb{D} := D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

- $\mathrm{Aut}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{D} \mid f \text{ is bijective, } f, f^{-1} \in \mathcal{O}(\mathbb{D})\}$.

  ↳ group under composition

Aut(D)

Automorphisms of $\mathbb{D}$ fixing the origin:

Automorphisms of the disc

Theorem. If $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = 0$, then f is a rotation, i.e., $\exists \lambda \in \partial\mathbb{D}$ s.t. $f(z) = \lambda z$ $\forall z \in \mathbb{D}$.

Proof. Let $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0) = 0$.
By Schwarz, $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ $\forall z$.
Moreover, $f^{-1} \in \mathrm{Aut}(\mathbb{D})$ also fixes the origin.
By Schwarz again, $|f^{-1\prime}(0)| \leq 1$ and $|f^{-1}(z)| \leq |z|$ $\forall z$.
Thus, it follows that $|f(z)| = |z|$ $\forall z$ and hence, $\exists \lambda \in S^1$ s.t. $f = (z \mapsto \lambda z)$. ∎

Möbius transforms:

Möbius, Mobius

Let $\alpha \in \mathbb{D}$, and consider $z \stackrel{\varphi_\alpha}{\longmapsto} \dfrac{z - \alpha}{1 - \bar{\alpha} z}$, $z \in \mathbb{D}$.

- Note that φ_α makes sense on $\mathbb{C} \setminus \{1/\bar{\alpha}\} \supseteq \mathbb{D}$.

Moreover, φ_α is holomorphic on $\mathbb{D}$, i.e., $\varphi_\alpha \in \mathcal{O}(\mathbb{D})$.

- $\varphi_\alpha(\alpha) = 0$.

- $\varphi_\alpha(\mathbb{D}) = ?$

  Check: $|\varphi_\alpha(e^{it})| \leq 1$ for $t \in \mathbb{R}$.

  Thus, by MMT $\varphi_\alpha(\mathbb{D}) \subseteq \mathbb{D}$.

  Also, $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ has inverse as $\varphi_{-\alpha}$.

  Thus, $\varphi_\alpha \in \mathrm{Aut}(\mathbb{D})$ $\forall \alpha \in \mathbb{D}$.

Theorem. $\mathrm{Aut}(\mathbb{D}) = \{\lambda \varphi_\alpha : \lambda \in \partial\mathbb{D},\ \alpha \in \mathbb{D}\}$.

Proof. (⊇) is clear.

(⊆) Let $f \in \mathrm{Aut}(\mathbb{D})$.

Put $\alpha := f^{-1}(0)$.

Then, $(f \circ \varphi_{-\alpha}) \in \mathrm{Aut}(\mathbb{D})$ and $(f \circ \varphi_{-\alpha})(0) = 0$.

Thus, $f \circ \varphi_{-\alpha}$ is r_λ (rotation by λ) for some $\lambda \in \partial \mathbb{D}$.

$\Rightarrow f = r_\lambda \circ \varphi_\alpha$. ∎

Let $\alpha, \beta \in \mathbb{D}$. Let $f : \mathbb{D} \longrightarrow \overline{\mathbb{D}}$ be holomorphic and $f(\alpha) = \beta$. Among all such f, what is the maximum possible value of $|f'(\alpha)|$?

$$\begin{tikzcd} \alpha \quad \mathbb{D} \arrow[r, "f"] & \overline{\mathbb{D}} \quad \beta \arrow[d, "\psi_\beta"] \\ 0 \quad \mathbb{D} \arrow[u, "\psi_{-\alpha}"] \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowright"] & \overline{\mathbb{D}} \quad 0 \end{tikzcd}$$

$g := \psi_\beta \circ f \circ \psi_{-\alpha}$.

$g(0) = 0$.

By Schwarz, we have $|g'(0)| \leq 1$.

Using chain rule, we have:

$$g'(0) = \psi_\beta'\big(f(\psi_{-\alpha}(0))\big) \cdot f'(\psi_{-\alpha}(0)) \cdot \psi_{-\alpha}'(0)$$

$$= \psi_\beta'(f(\alpha)) \cdot f'(\alpha) \cdot \psi_{-\alpha}'(0)$$

$$= \psi_\beta'(\beta) \cdot f'(\alpha) \cdot \psi_{-\alpha}'(0)$$

$$\psi_\beta(z) := \frac{z-\beta}{1-\bar\beta z}$$

$$\Rightarrow \psi_\beta'(z) = \frac{1-\bar\beta z - (z-\beta)(-\bar\beta)}{(1-\bar\beta z)^2}.$$

$$= \frac{1-\bar\beta\beta}{(1-\bar\beta\beta)^2} \cdot f'(\alpha) \cdot \frac{1-(\alpha\bar\alpha)}{1^2}$$

$$= \frac{1-|\alpha|^2}{1-|\beta|^2} \cdot f'(\alpha)$$

$$\therefore |f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}.$$

Note that equality is possible. For example, $f = \varphi_{-\beta} \circ \varphi_\alpha$.

In fact, it happens iff $\exists \lambda \in S^1$ s.t.

$$f = \varphi_{-\beta} \circ r_\lambda \circ \varphi_\alpha .$$

Ex. Calculate $\mathrm{Aut}(\mathbb{D}\setminus\{0\})$.

Towards the Riemann-Mapping Theorem

$$\mathcal{O}(\Omega) \subseteq \mathcal{C}^0(\Omega; \mathbb{C}).$$

want to make this a metric space.

let us consider $\Omega = \mathbb{D}$.

There is a sequence $\{K_n\}$ of compact sets in $\mathbb{C}$ s.t. :

(1) $\mathbb{D} = \bigcup_{n=1}^{\infty} K_n^\circ$,

(2) $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$,

(3) for each compact $K \subset \mathbb{D}$, $\exists n \in \mathbb{N}$ s.t. $K \subseteq K_n$.

One can take $K_n := \overline{D(0, 1-\frac{1}{n})}$, for example.

Claim. One can do the above for any open $\Omega \subseteq \mathbb{C}$.

Given any open $\Omega \subseteq \mathbb{C}$, $\exists$ a sequence $\{K_n\}_n$ of compact subset of $\mathbb{C}$ s.t.

(Compact exhaustion)

(1) $\Omega = \bigcup_{n=1}^{\infty} K_n^\circ$,

(2) $K_n \subseteq K_{n+1}^\circ$ $\forall n \in \mathbb{N}$,

(3) for any compact $K \subseteq \Omega$, $\exists n \in \mathbb{N}$ s.t. $K \subseteq K_n$.

Proof. For each $n \in \mathbb{N}$, let

$$K_n := \overline{D(0,n)} \cap \{ z \in \Omega : \operatorname{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/n \}.$$

Check that K_n satisfies (1) − (3).

Using the above, we define a metric on $\mathcal{C}^0(\Omega; \mathbb{C})$.

Fix some $\{K_n\}_n$ as given by compact exhaustion.
Let $f, g \in \mathcal{C}^0(\Omega; \mathbb{C})$.
Define
$$\rho_n(f, g) := \sup_{z \in K_n} |f(z) - g(z)|.$$

Finally, define
$$\rho(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f,g)}{1 + \rho_n(f,g)}.$$

Ex. ① $(\mathcal{C}^0(\Omega; \mathbb{C}), \rho)$ is a metric space.

② A sequence $\{f_k\}_{k \geq 1}$ converges to f in $(\mathcal{C}^0(\Omega; \mathbb{C}), \rho)$ iff $f_k \longrightarrow f$ uniformly on compact subsets of Ω.
What are open sets in $(\mathcal{C}^0(\Omega; \mathbb{C}), \rho)$?

This ex. shows that the topology does not depend on $\{K_n\}_{n \geq 1}$.

Lecture 6 (20-01-2022)

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$$\mathcal{O}(\Omega) \subseteq \mathcal{C}^0(\Omega; \underset{}{\mathbb{C}}).$$

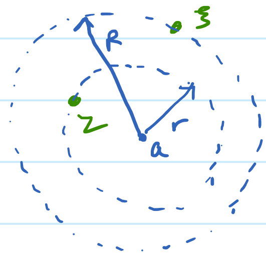
metric space

↳ subspace topology

Prop^n. $\mathcal{O}(\Omega)$ is closed in $\mathcal{C}^0(\Omega;\mathbb{C})$.
That is, if $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$ and $f_n \longrightarrow f$ in $\mathcal{C}^0(\Omega;\mathbb{C})$, then $f \in \mathcal{O}(\Omega)$.
Moreover, $f_n^{(k)} \longrightarrow f^{(k)}$ in $\mathcal{O}(\Omega)$ for all $k \geq 1$.

Proof. To show $f \in \mathcal{O}(\Omega)$, we may assume Ω is a disc and use Morera's theorem and that $\int_T f_n = 0$ for every triangle $T \subseteq \Omega$.

We now show that $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of Ω. Suffices to prove it for $k=1$ and use induction.



$$(f_n' - f')(z) = \frac{1}{2\pi i} \int_{|\xi - a| = R} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2}\, d\xi$$

for all $z \in \overline{D(a, r)}$.

$[\; D(a,r) \subsetneq D(a,R) \subseteq \Omega. \;]$

$$\Rightarrow \; |f_n'(z) - f'(z)| \leq \frac{1}{2\pi} \int_{|\xi - a| = R} \frac{|f_n(\xi) - f(\xi)|}{|\xi - z|^2}\, d\xi$$

$$\leq \frac{1}{(R-r)^2} \Big(\sup_{\partial D(a,R)} |f_n - f| \Big)$$

$\downarrow$ 0 as $n \to \infty$.

$$\Rightarrow \; |f_n'(z) - f'(z)| \longrightarrow 0 \text{ uniformly for } z \in \overline{D(a,r)}.$$

Thus, $f_n' \to f'$ uniformly on closed discs.

Now, given any arbitrary $K \subseteq \Omega$, we can cover it by finitely many closed discs contained in Ω. ∎

Normal Families

Normal family

Defn. Let $\Omega \subseteq \mathbb{C}$ be a domain, and $\mathcal{F} \subseteq \mathcal{O}(\Omega)$.
$\mathcal{F}$ is said to be normal if for every sequence $(f_n)_n \in \mathcal{F}^{\mathbb{N}}$, it is possible to extract a subsequence $(f_{n_k})_k$ such that either

(a) $(f_{n_k})_k$ converges uniformly on compact subsets of Ω, or

(b) given any pair of compact sets $K \subset \Omega$, $L \subset \mathbb{C}$, $\exists\, k_0 = k_0(K, L) \in \mathbb{N}$ s.t.

$$f_{n_k}(K) \cap L = \emptyset \qquad \forall\, k \geq k_0.$$

($f_{n_k} \to \infty$ uniformly on compact subset of Ω.)

EXAMPLES. (i) $\Omega_1 = D(0,1)$.

$$\mathcal{F}_1 = \{ \underset{z \in \Omega}{z \mapsto z^n} : n \in \mathbb{N} \}.$$

→ normal because of (a)

(ii) $\Omega_2 = \{ z \in \mathbb{C} : |z| > 1 \}$.

$$\mathcal{F}_2 = \{ z \mapsto z^n : n \in \mathbb{N} \}.$$

→ normal because of (b)

(iii) $\Omega_3 = \{ z \in \mathbb{C} : \frac{1}{2} < |z| < 2 \}$.

$$\mathcal{F}_3 = \{ z \mapsto z^n : n \in \mathbb{N} \}.$$

The above is not normal. Consider the sequence $f_n = (z \mapsto z^n) \in \mathcal{F}_3$.

Consider $K = \overline{D(1,\varepsilon)}$ for some small $\varepsilon > 0$ s.t. $K \subset \Omega$.

Consider $K \cap \Omega_1$ and $K \cap \Omega_2$ to see $\mathcal{F}_3$ is NOT NORMAL.

(iv) Let $\Omega \subseteq \mathbb{C}$ be a domain.
$\mathcal{F} = \{ z \mapsto z^n : n \in \mathbb{N} \}$ is NOT NORMAL
if $\partial D(0,1) \subseteq \Omega$.

REMARKS. (i) If (a) is true and $f_{n_k} \longrightarrow f$, then $f \in \mathcal{O}(\Omega)$.
(ii) However, f above need not be in $\mathcal{F}$.

Theorem (Montel's Theorem) Montel's theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ be locally uniformly bounded on Ω, i.e., for all compact $K \subseteq \Omega$, $\exists M = M(K) > 0$ such that

$$|f(z)| \leq M \qquad \forall f \in \mathcal{F}, \ \forall z \in K.$$

Then, $\mathcal{F}$ is a normal family.
In fact, $\mathcal{F}$ is normal and satisfying (a) of the def^n.

EXAMPLE. Let $\Omega \subseteq \mathbb{C}$ be a domain.
Then, given any subset $\mathcal{F} \subseteq \{ f \in \mathcal{O}(\Omega) : f(\Omega) \subseteq D(0,1) \}$, Montel's theorem asserts that $\mathcal{F}$ is normal!

Recall:
Theorem (Arzelà - Ascoli Theorem)

Let $\mathcal{F} \subset \mathcal{C}^0(\Omega; \mathbb{C})$.
Every sequence in $\mathcal{F}$ admits a convergent subsequence in $(\mathcal{C}(\Omega;\mathbb{C}), \rho)$ iff:
(i) $\mathcal{F}$ is pointwise bounded, i.e., $\exists M : \Omega \longrightarrow [0, \infty)$ s.t.

$$|f(z)| \leq M(z) \qquad \forall z \in \Omega, \quad \text{and}$$

(ii) $\mathcal{F}$ is equicontinuous at each point of Ω.

Proof of Montel's Theorem: Let $\mathcal{F}$ be as given.
It suffices to show that $\mathcal{F}$ is

Proof of Montel's theorem. Let Ω, $\mathcal{F}$ be as given.

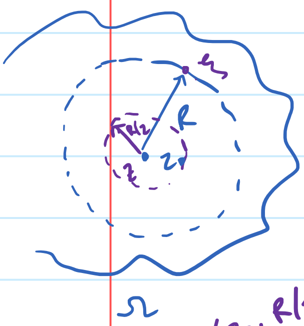
It suffices to show that $\mathcal{F}$ is equicontinuous at each $z \in \Omega$.

That is : $\forall z_0 \in \Omega \;\; \forall \varepsilon > 0 \;\; \exists \delta = \delta(z_0, \varepsilon) > 0$ s.t.

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$$

$$\forall z \in \Omega \;\; \forall f \in \mathcal{F}.$$

Let $\overline{D(z_0, R)} \subseteq \Omega$. Then, $\exists M > 0$ s.t.



$$|f(z)| \le M \quad \forall z \in \overline{D(z_0, R)} \quad \forall f \in \mathcal{F}.$$

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - z_0| = R} \frac{f(\xi)}{\xi - z} \, d\xi \qquad \forall z \in D(z_0, R).$$

For $z \in D(z_0, R/2)$:

$$|f(z_0) - f(z)| \le \frac{1}{2\pi} \left| \int_{|\xi - z_0| = R} f(\xi) \left(\frac{1}{\xi - z_0} - \frac{1}{\xi - z} \right) d\xi \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\xi - z_0| = R} \frac{f(\xi)(z_0 - z)}{(\xi - z_0)(\xi - z)} \, d\xi \right|$$

(we took $z \in D(z_0, R/2)$.)

$$\le \frac{1}{2\pi} \cdot (2\pi R) \cdot \frac{M \cdot |z_0 - z|}{R \cdot R/2}.$$

Thus, for all $f \in \mathcal{F}$ and for all $z \in D(z_0, R/2)$, we have

$$|f(z) - f(z_0)| \le \left(\frac{2M}{R} \right) \cdot |z - z_0|.$$

Equicontinuity follows. ∎

Lecture 7 (24-01-2022)

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EXAMPLE. Montel's Theorem fails on $\mathbb{R}$.

Indeed, consider the family $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$, where $f: \mathbb{R} \to \mathbb{R}$ is defined as $f_n(x) := \sin(nx)$.

Clearly, $\mathcal{F}$ is locally uniformly bounded as $|f_n(x)| \leq 1 \quad \forall x \in \mathbb{R} \ \forall n \in \mathbb{N}$.

However, given any $\delta > 0$, pick n s.t. $x = \frac{\pi}{2n} < \delta$.

Then, $|f_n(x) - f_n(0)| = \left|\sin\left(\frac{\pi}{2}\right)\right| = 1$.

Thus, no δ exists for $\varepsilon = 1$.

Thus, $\mathcal{F}$ is not equicontinuous.

Hurwitz's theorem

Theorem (Hurwitz's Theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that $\exists\, a \in \Omega$, $r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$ such that f has no zeroes on $\partial D(a,r)$.
Then, $\exists\, N \in \mathbb{N}$ such that f and f_n have the same number of zeroes (counting multiplicities) in $D(a,r)$ for all $n \geq N$.

Remark. Note that if f is not identically zero, one can find $a \in \Omega$, $r > 0$ as stated. In fact, for any $a \in \Omega$, we can find an $r > 0$ since zeroes are isolated!

Proof. Since $f \neq 0$ on $\partial D(a,r)$, $\min\limits_{\partial D(a,r)} |f| =: \delta > 0$.

Since $f_n \to f$ uniformly on compact subsets of Ω, it follows that $\exists\, N \in \mathbb{N}$ s.t.

$$|f_n(z) - f(z)| < \delta/2 \qquad \forall z \in \partial D(a,r) \quad \forall n \geq N.$$

$2 \quad \forall n \geq N.$

Thus, $|f_n(z) - f(z)| < |f(z)| \quad \forall z \in \partial D(a,r)$ and $n \gg 0$.

Now, by Rouché's theorem, we are done. ∎

Corollary 1. Let Ω be a domain in $\mathbb{C}$, $f_n \in \mathcal{O}(\Omega)$ $\forall n$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that each f_n is non-vanishing on Ω.
Then, either $f \equiv 0$ or f is also nonvanishing.

Corollary 2. Let $\Omega \subseteq \mathbb{C}$ be a domain, $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$, $f_n \to f$ in $\mathcal{O}(\Omega)$.
Suppose that each f_n is injective on Ω, then f is injective on Ω.

Riemann Mapping Theorem

Theorem (RMT). Let $\Omega \subsetneq \mathbb{C}$ be simply-connected.
Then, Ω is biholomorphic to $D(0,1)$.

Remark ① $\mathbb{C}$ cannot be biholo. to $D(0,1)$, by Liouville.
② If Ω is biholo. to $D(0,1)$, then Ω is homeomorphic to $D(0,1)$ and thus, simply-connected.

QUESTION Is this Riemann map unique? $f: \Omega \to D(0,1)$

No. These will precisely "differ" by $\mathrm{Aut}(\mathbb{D})$.

Proof of RMT Let $\Omega \subsetneq \mathbb{C}$ be as specified.
Fix $p \in \Omega$.
Let

$$\mathcal{F} = \{ f \in \mathcal{O}(\Omega) : f(p) = 0,\ f \text{ is injective},\ f(\Omega) \subseteq D(0,1) \}.$$

If we can find $f_0 \in \mathcal{F}$ such that $f(\Omega) = D(0,1)$, then we are done since f_0^{-1} is also holomorphic.

Steps: (I) $\mathcal{F} \neq \emptyset$.

(II) $\sup_{f \in \mathcal{F}} |f'(p)| = |f_0'(p)|$ for some $f_0 \in \mathcal{F}$.

(III) f_0 (as above) is onto.

Motivation: Suppose we a compact exhaustion $(K_n)_{n \in \mathbb{N}}$ of Ω with $p \in K_n$ $\forall n$.

By choosing f as in (II), we get a function which "starts out fastest" at p. Then, $\bigcup_{n=1}^{\infty} f_0(K_n) = D(0,1)$ is likely.

(I) To show: $\mathcal{F} \neq \emptyset$.

(a) If Ω is bounded, then $z \longmapsto \frac{z-p}{M}$ works for an appropriate $M > 0$.

(b) As $\Omega \subsetneq \mathbb{C}$, pick $Q \in \mathbb{C} \setminus \Omega$.

Let $\phi(z) := z - Q$ is nonvanishing on Ω.

As Ω is simply-connected, $\exists$ a holomorphic square root of ϕ.

$\exists h \in \mathcal{O}(\Omega)$ s.t. $(h(z))^2 = \phi(z)$ $\forall z \in \Omega$.

Note that since ϕ is injective, we get
$h(z_1) \neq h(z_2)$ and $h(z_1) \neq -h(z_2)$
for $z_1 \neq z_2 \in \Omega$.

In particular, h is nonconstant on Ω.
Thus, h is an open map.

Let $b \in h(\Omega)$. Then, $D(b, r) \subseteq h(\Omega)$ for some $r > 0$.
Then, $D(-b, r) \cap h(\Omega) = \emptyset$ by earlier observation.

For $z \in \Omega$, define $f(z) := \frac{r}{\quad}$.

$2(b+h(z))$

Then, $f \in \mathcal{O}(\Omega)$ and $|f(z)| \leq 1/2$.

Clearly, f is injective.

$f(p) = 0$ not guaranteed but just compose with appropriate Möbius transform.

Thus, $\mathcal{F} \neq \emptyset$.

Lecture 8 (27-01-2022)

27 January 2022 14:00

(II) To show: $\exists\, g \in \mathcal{F}$ s.t. $\sup\limits_{f \in \mathcal{F}} |f'(p)| = |g'(p)|$.

Since $\mathcal{F} \neq \emptyset$, $\lambda := \sup\limits_{f \in \mathcal{F}} |f'(p)| > 0$.

(Injective $\Rightarrow$ f' never vanishing in $\mathbb{C}$ Analysis!)

Thus, $\exists\, (f_n)_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ s.t. $|f_n'(p)| \to \lambda$ as $n \to \infty$.

($\lambda = \infty$ is not ruled out yet.)

Note that Montel's theorem ensures that $\mathcal{F}$ is a normal family. Thus we may assume $(f_n)_n$ itself converges to g in $\mathcal{O}(\Omega)$. Then, $f_n' \longrightarrow g'$ in $\mathcal{O}(\Omega)$.

In particular,

$$|g'(p)| = \lambda.$$

(Also shows that $\lambda < \infty$ (!))

Now, we show that $g \in \mathcal{F}$ to conclude!

- As $g := \lim\limits_n f_n$, it follows that $g(p) = 0$.

- As $f_n(\Omega) \subseteq D(0,1)$, we have $g(\Omega) \subseteq \overline{D(0,1)}$.

  (WTS: $g(\Omega) \subseteq D(0,1)$.)

  By MMT, if $g(\Omega) \cap \partial D(0,1) \neq \emptyset$, then g is constant. But $g(p) = 0$, thus it can't happen.

  Thus, $g(\Omega) \subseteq D(0,1)$.

- It follows from corollary 2 of Hurwitz's theorem that g is injective on Ω (g is not constant since $\underbrace{|g'(p)|}_{= \lambda \neq 0}$).

(III) We show that $g(\Omega) = D(0,1)$.

Suppose not. Then, $g(\Omega) \subsetneq D(0,1)$. Pick $a \in D(0,1) \setminus g(\Omega)$.

We construct $s \in \mathcal{F}$ s.t. $|s'(p)| > |g'(p)|$, giving us the desired contradiction.

Define $\rho = \psi_a \circ g$.

$$\rho(z) = \frac{g(z) - a}{1 - \bar{a}\, g(z)} \quad ; \quad z \in \Omega.$$

$\rho(p) = -a$.

- $\rho \in \mathcal{O}(\Omega)$, $\rho(\Omega) \subseteq D(0,1)$.
- ρ never vanishes on Ω.

As Ω is simply-connected, it follows that $\exists h \in \mathcal{O}(\Omega)$ s.t.
$\rho(z) = (h(z))^2 \quad \forall z \in \Omega$.

Then, $h(\Omega) \subseteq D(0,1)$. $(h(p))^2 = -a$.

Let $s := \psi_{h(p)} \circ h : \Omega \longrightarrow D(0,1)$.

g injective $\Rightarrow$ ρ injective $\Rightarrow$ h injective $\Rightarrow$ s injective.

- $s \in \mathcal{O}(\Omega)$,
- $s(p) = 0$,
- $s(\Omega) \subseteq D(0,1)$,
- s injective.

Thus, $s \in \mathcal{F}$.

$$s(z) = (\psi_{h(p)} \circ h)(z) = \frac{h(z) - h(p)}{1 - \overline{h(p)}\, h(z)}.$$

$$s'(z) = \frac{h'(z)\left(1-\overline{h(p)}\,h(z)\right) - \left(h(z)-h(p)\right)\left(-\overline{h(p)}\,h'(z)\right)}{\left(1-\overline{h(p)}\,h(z)\right)^2}.$$

$$\therefore\ s'(p) = \frac{h'(p)}{1-|h(p)|^2}.$$

$$(h(z))^2 = \rho(z) = (\varphi_a \circ g)(z)$$
$$= \frac{g(z)-a}{1-\bar{a}g(z)}.$$

$$\Rightarrow\ 2h(z)h'(z) = \frac{1}{(1-\bar{a}g(z))^2}\left(g'(z)(1-\bar{a}g(z)) - (g(z)-a)(-\bar{a}g'(z))\right)$$

$(g(p) = 0.)$

$$\Rightarrow\ 2h(p)h'(p) = g'(p)(1-|a|^2).$$

$$\therefore\ s'(p) = \frac{(1-|a|^2)\,g'(p)}{2h(p)\left(1-|h(p)|^2\right)}$$

$\left((h(p))^2 = -a.\right)$

$$= \frac{(1-|a|^2)\,g'(p)}{2h(p)\,(1-|a|)}$$

$$= \frac{1+|a|}{2h(p)}\,g'(p).$$

$$\Rightarrow\ \frac{|s'(p)|}{|g'(p)|} = \frac{1+|a|}{2\sqrt{|a|}} \overset{\text{AM GM}}{>} 1.$$

$\left(|a| \neq 1 \text{ since } a \in D(0,1).\right)$

This is the desired contradiction! ☒

Remark. The only property of simple-connectedness that we used was that every nonvanishing function has a holomorphic square root.

Example. $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Then, $\mathbb{H} \subsetneq \mathbb{C}$ is a simply-connected domain.
We have an explicit biholomorphism $f: \mathbb{H} \to D(0,1)$
given by $z \mapsto \dfrac{z-i}{z+i}$.

Next up: **Weierstrass Factorisation Theorem**

Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{O}$. Then, f is either identically zero on Ω or $Z(f) := \{z \in \Omega : f(z) = 0\}$ is discrete in Ω.

Q. Let $A \subseteq \mathbb{C}$ be discrete. Can we find $f \in \mathcal{O}(\Omega)$ such that $Z(f) = A$?

Note: A must be countable. If finite, consider polynomials.
Now, assume $(a_n)_{n\in\mathbb{N}}$ is an enumeration of A.

Naive guess: $f(z) = (z-a_1)(z-a_2)\cdots(z-a_n)\cdots$.
How to make sense of f?

Another attempt: Construct $f_1, f_2, \ldots \in \mathcal{O}(\mathbb{C})$ s.t. $Z(f_n) = \{a_n\}$
and put $f = \prod_n f_n$.
↳ need to make sense of infinite products.

Infinite Products.

Infinite products

Def^n. Suppose that $(u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Define the sequence $(p_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ by

$$p_n := (1+u_1)\cdots(1+u_n).$$

If $\lim_{n\to\infty} p_n =: p$ exists (in $\mathbb{C}$), then we write

∞

$$p = \prod_{n=1}^{\infty} (1+u_n).$$

p_n are called the partial products of the infinite product $\prod_{n=1}^{\infty}(1+u_n)$.

In this case, we say that $\prod_{n=1}^{\infty}(1+u_n)$ converges (to p).

- Suppose that $z_n \neq 0$ $\forall n$. Assume $z := \prod_{n=1}^{\infty} z_n$ exists and $z \neq 0$.

Let $p_n := z_1 \cdots z_n$. Then, $\lim_n (z_n) = \lim_n \left(\frac{p_{n+1}}{p_n}\right) = \frac{\lim_n p_{n+1}}{\lim_n p_n} = \frac{z}{z} = 1.$

(Each p_n is nonzero and $p_n \to z \neq 0$.)

Lemma Let $u_1, \ldots, u_N \in \mathbb{C}$. Define

$$p_N := \prod_{n=1}^{N}(1+u_n), \qquad p_N^* := \prod_{n=1}^{N}(1+|u_n|).$$

Then,

(i) $p_N^* \leq \exp(|u_1| + \cdots + |u_N|)$,

(ii) $|p_N - 1| \leq p_N^* - 1$.

Lecture 9 (31-01-2022)

31 January 2022 14:03

Theorem Let X be a metric space. Let $u_n : X \to \mathbb{C}$ be a sequence of functions such that $\sum_{n=1}^{\infty} |u_n|$ converges uniformly to a bounded function.
(Say, bounded by $M > 0$.)

Then, (1) $\prod_{n=1}^{\infty} (1+u_n)$ converges uniformly on X.

Define $f(x) := \prod_{n=1}^{\infty} (1+u_n(x))$ for $x \in X$.

(2) For $x_0 \in X$: $f(x_0) = 0 \iff u_M(x_0) = -1$ for some $M \in \mathbb{N}$.

(3) For every permutation $\sigma \in S_{\mathbb{N}}$, the infinite product

(Rearrangement)

$$\prod_{k=1}^{\infty} \left(1 + u_{\sigma(k)}(x)\right)$$ converges to $f(x)$, for all $x \in X$.

Proof. (1) Let $p_N(x) := \prod_{n=1}^{N} (1+u_n(x))$, $x \in X$.

We will show that $(p_N)_{N=1}^{\infty}$ is uniformly Cauchy on X.

For $M > N$, note

$$|p_M(x) - p_N(x)| = \left| p_N(x) \cdot \prod_{n=N+1}^{M} (1+u_n(x)) - p_N(x) \right|$$

$$= |p_N(x)| \cdot \left| \prod_{n=N+1}^{M} (1+u_n(x)) - 1 \right|$$

last lec's last lemma

$$\leq |p_N(x)| \left[\prod_{n=N+1}^{M} (1+|u_n(x)|) - 1 \right]$$

—"—

$$\leq |p_N(x)| \left[\exp\left(\sum_{n=N+1}^{M} |u_n(x)| \right) - 1 \right]$$

this term is uniformly Cauchy since $\sum |u_n|$ converges uniformly

Cauchy uniformly since $\sum |u_n|$ converges

$$\leq \exp(M) \cdot \left(\text{small}\right). \quad ✓$$

(2) Let f denote the limit. Let $x \in K$ be s.t. $p_N(x) \neq 0 \ \forall x$.

From the above, given $\varepsilon = 1/4$, we can get N_0 s.t.
$$|p_M(x) - p_{N_0}(x)| < 2|p_{N_0}(x)|\varepsilon \qquad \forall M > N_0.$$

Then, $$|f(x)| \geq (1 - 2\varepsilon)\, p_{N_0}(x).$$
In particular, $f(x) \neq 0$.

Thus, $f(x) = 0 \Rightarrow p_N(x) = 0$ for some N ← finite product
$\Rightarrow 1 + u_n(x) = 0$ for some n
$\Rightarrow u_n(x) = -1$ for some n. ✓

(3) Exercise.

Theorem. Let Ω be a domain in $\mathbb{C}$. Let $(f_n)_n \in \mathcal{O}(\Omega)^{\mathbb{N}}$ be such that no f_n is identically zero.
Suppose that $\sum_{n=1}^{\infty} |1 - f_n|$ converges uniformly on compact subsets of Ω.

(1) Then, $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of Ω.

Consequently $f := \prod_{n=1}^{\infty} f_n$ is holomorphic.

(2) Let $a \in \Omega$. If $f(a) = 0$, then $f_n(a) = 0$ for some n.
Moreover, this is true for only finitely many n.
Lastly,
$$\operatorname{ord}_f(a) = \sum_{n=1}^{\infty} \operatorname{ord}_{f_n}(a).$$

multiplicity

this is only nonzero for finitely many.

<u>Proof</u>. (1) follows from earlier by taking $u_n := f_n - 1$.

(2) Each f_n has countably many zeroes. By (2) of earlier thm,

$$Z(f) \subseteq \bigcup_{n=1}^{\infty} Z(f_n).$$

$\therefore$ $Z(f)$ is countable. $\therefore$ $f \not\equiv 0$ on Ω

$\therefore$ $Z(f)$ is discrete in Ω. Let $a \in \Omega$ be s.t. $f(a) = 0$.

Pick $r > 0$ s.t. $f(z) \neq 0$ for $z \in D(a,r) \setminus \{a\}$.

Consequently: • each f_n is nonzero on $D(a,r) \setminus \{a\}$,

• $f_{n_0}(a) = 0$ for some $n_0 \in \mathbb{N}$.

As $\sum |1 - f_n|$ converges uniformly on $\{a\}$, we have $f_n(a) \to 1$.

$\therefore$ $f_n(a) = 0$ only for finitely many n.

To conclude: $A := \{n \in \mathbb{N} : f_n(a) = 0\}$ is a finite nonempty subset of $\mathbb{N}$.

Write $f(z) := \prod_{n \in \mathbb{N}} f_n(z)$ ↓ rearrangement

$$= \prod_{n \in A} f_n(z) \underbrace{\prod_{n \notin A} f_n(z)}_{\text{holomorphic, nonvanishing on } D(a,r)}$$

$$\therefore \operatorname{ord}_f(a) = \sum_{n \in A} \operatorname{ord}_{f_n}(a).$$ ☒

Lecture 10 (03-02-2022)

03 February 2022 14:00

If we can find $g_k \in \mathcal{O}(\Omega)$ for $k \in \mathbb{N}$ s.t.

(i) g_k has no zeroes on Ω, and

(ii) $\sum_{k=1}^{\infty} |1-(z-z_k)g_k(z)|$ converges uniformly on compact ...,

$$\text{then } z \mapsto \prod_{k=1}^{\infty} (z-z_k)g_k(z) \quad \in \mathcal{O}(\Omega)$$

and the zeroes are precisely $\{z_k\}_{k=1}^{\infty}$.

Guess: $g_k = \exp(h_k)$ for some $h_k \in \mathcal{O}(\Omega)$.
($\because$ we want $g_k \neq 0$.)

Elementary Factors :

Weierstrass elementary factors

Def^n. $E_0(z) = 1 - z$ for $z \in \mathbb{C}$.
For $p \in \mathbb{N}$, define

$$E_p(z) := (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

These functions are called (Weierstrass) Elementary factors.
Below, we have $p \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$.

- Each E_p vanishes precisely at 1.
  1 is a simple zero (order $= 1$) for each E_p.
- $E_p(0) = 1$

- For $|z|<1$,

$$E_p(z) = (1-z)\exp\left(\sum_{k=1}^{p} \frac{z^k}{k}\right)$$

Heuristic!

$$= (1-z)\exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k}\right)\exp\left(-\sum_{k=p+1}^{\infty} \frac{z^k}{k}\right)$$

$$\left(\sum_{k=1} \frac{\;}{k} \right) \quad \left(\sum_{k=p+1} \frac{\;}{k} \right)$$

$\hookrightarrow$ branch of log of $z \mapsto \frac{1}{1-z}$ on $D(0,1)$

$$= (1-z) \cdot \frac{1}{1-z} \cdot \exp\left(- \sum_{k=p+1}^{\infty} \frac{z^k}{k} \right)$$

$$= \exp\left(- \sum_{k=p+1}^{\infty} \frac{z^k}{k} \right)$$

$$= 1 - \frac{z^{p+1}}{p+1} + \text{higher order}$$

HAND-WAVY: Thus, if p is large, we expect $E_p \sim 1$.
More precisely:

Lemma For every $p \geq 0$,

$$|1 - E_p(z)| \leq |z|^{p+1} \qquad \text{if} \quad |z| \leq 1.$$

Proof. Fix $p \geq 1$. ($p = 0$ is clear.)

Write $E_p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$.

$$\Rightarrow E_p'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

(This expansion is valid on $\mathbb{C}$ since E_p is entire.
$0 = E_p(1) = 1 + \sum_{n=1}^{\infty} a_n$.)

OTOH, $E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$

$$\Rightarrow E_p'(z) = -\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$$

$$+ (1-z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)\left(1 + z + \cdots + z^{p-1}\right)$$

$$= \exp\left(z + \cdots + \frac{z^p}{p} \right) \left[(-1) + (1-z)\left(\frac{1 - z^p}{1-z} \right) \right]$$

$$= \exp\left(\qquad \right) \left[(-1) + (1 - z^p) \right]$$

$$= -z^p \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right).$$

$\therefore E_p'$ has a zero of order p at the origin.
Thus, $a_1 = \cdots = a_p = 0$.

Thus, $E_p(z) = 1 + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots$.

Also, equating
$$-z^p \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) = \sum_{n=p+1}^{\infty} n a_n z^{n-1}$$

shows us that $a_n \in \mathbb{R}\ \forall n$ and $a_n \leq 0\ \forall n \geq p+1$.

Coefficients here are +ve

For $|z| \leq 1$:
$$|E_p(z) - 1| = \left|\sum_{n=p+1}^{\infty} a_n z^n\right|$$
$$= |z|^{p+1} \left|\sum_{n=p+1}^{\infty} a_n z^{n-p-1}\right|$$
$$\leq |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n|$$
$\because a_n \leq 0\ \forall n \geq p+1$
$$= -|z|^{p+1} \sum_{n=p+1}^{\infty} a_n$$
$$= -|z|^{p+1} \left(\cancel{E_p(1)}^{0} - 1\right)$$
$$= |z|^{p+1}.$$
☒

Remark. The function $z \mapsto E_p\left(\frac{z}{a}\right)$ has a simple zero at $z = a$ (and no other zeroes).

(Weierstrass Product Theorem)

Theorem Let $(a_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$ be such that $a_n \neq 0\ \forall n \geq 1$ and
$$|a_n| \to \infty \text{ as } n \to \infty.$$

(Note: the sequence need not consist of distinct points.
However, $|a_n| \to \infty$ forces that no point appears inf. often.)

IF $(p_n)_n \in \mathbb{N}_0^{\mathbb{N}}$ is such that
$$\sum^{\infty} \left(\frac{r}{\cdot}\right)^{p_n+1} < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n + 1} < \infty$$

for every $r > 0$, THEN:

(i) $\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ converges in $\mathcal{O}(\mathbb{C})$.

Write f for the above function.

(ii) $f \in \mathcal{O}(\mathbb{C})$ and $Z(f) = \{a_n : n \in \mathbb{N}\}$.

(iii) The multiplicity of any zero is precisely the number of times that it appears in the sequence.

Remarks: (1) Since $|a_n| \to \infty$ as $n \to \infty$, for every $r > 0$, $\exists N_0 = N_0(r) \in \mathbb{N}$ s.t. $|a_n| > 2r$ for all $n \geq N_0$.

Thus,

$$\left(\frac{r}{|a_n|}\right) < \frac{1}{2} \qquad \forall\, n \geq N_0.$$

In turn,

$$\left(\frac{r}{|a_n|}\right)^{p_n+1} < \left(\frac{1}{2}\right)^{p_n+1} \qquad \forall\, n \geq N_0.$$

Thus, $p_n = n-1$ ALWAYS works for any $(a_n)_n$ with $|a_n| \to \infty$.

(2) Suppose that $\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$.

Then, $p_n \equiv 0$ works!

In this case, $f(z) = E_0\left(\frac{z}{a_n}\right)$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \text{ works.}$$

(3) If $\sum \frac{1}{|a_n|} = \infty$ but $\sum \frac{1}{|a_n|^2} < \infty$, then $p_n \equiv 1$ works.

$$\therefore f(z) = \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right)$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n}\right).$$

(4) To create a zero of order k at the origin, simply multiply with z^k.

Thus, given the theorem and the remarks, we have completely answered the desired question on $\mathbb{C}$.

Proof of the theorem. Let $(p_n)_n$ be as given.

We wish to use the theorem from last lecture. Will show

$$\sum_{n=1}^{\infty} \left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \text{ converges uniformly on compact } \subseteq \mathbb{C}.$$

Suffices this to prove for the compact sets $\overline{D(0,r)}$ for all $r > 0$.

By the earlier lemma,

$$\left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \leq \left|\frac{z}{a_n}\right|^{p_n+1} \quad \text{for } |z| \leq |a_n|.$$

Fix $r > 0$. Then, for $n \gg 0$, $r < |a_n|$.

Thus, for $z \in \overline{D(0, r)}$ and $M > N \gg 0$, we have:

$$\sum_{n=N}^{M} \left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \overset{|z/a_n| \leq 1}{\leq} \sum_{n=N}^{M} \left|\frac{z}{a_n}\right|^{p_n+1}$$

$$\overset{|z| \leq r}{\leq} \sum_{n=N}^{M} \left|\frac{r}{a_n}\right|^{p_n+1} \longrightarrow 0.$$

Thus, we are done. ∎

Lecture 11 (07-02-2022)

07 February 2022 14:03

EXAMPLE : Construct $f \in \mathcal{O}(\mathbb{C})$ with

(i) simple zeroes at $\mathbb{Z}$,

(ii) zeroes of order 2 at $\pm i\sqrt{n}$ for $n \in \mathbb{N}$,

no other zeroes.

Let us first construct one with (i).

Note : $\sum \frac{1}{n^2} < \infty$. Can take $p_n \equiv 1$.

Can take

$$f_1(z) = z \cdot \prod_{n=1}^{\infty} E_1\left(\frac{z}{n}\right) \cdot \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right).$$

For (ii) : Note $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^3 < \infty$. Can take $p_n \equiv 2$

Thus, can take $$f_2(z) = \prod_{n=1}^{\infty} E_2\left(\frac{z}{i\sqrt{n}}\right) \cdot \prod_{n=1}^{\infty} E_2\left(-\frac{z}{i\sqrt{n}}\right).$$

f_2^2 satisfies (ii).

The final desired function is $f = f_1 f_2^2$.

Weierstrass Factorisation Theorem

Theorem. Let $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ and let $(a_n)_{n \geq 1}$ be the nonzero zeroes of f, listed with multiplicity. Suppose f has a zero at the origin of order $m \geq 0$.

Then, $\exists g \in \mathcal{O}(\mathbb{C})$ and $(p_n)_n \in \mathbb{N}_0^{\mathbb{N}}$ such that

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof. Since zeroes are isolated, $|a_n| \to \infty$.

Proof. Since zeroes are isolated, $|a_n| \to \infty$.

As discussed last line, $\exists (p_n)_n$ s.t. $\sum \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \quad \forall r > 0$.

(e.g.: $p_n = n-1$.)

Thus, $h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ is holo on $\mathbb{C}$ and has SAME zeroes as f (with mult).

Thus, f/h is entire and nonvanishing. $\therefore \exists g \in \mathcal{O}(\mathbb{C})$ s.t.

$$f/h = \exp \circ g.$$

$\boxtimes$

Theorem. Let $\Omega \subsetneq \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$ be an open set.

Suppose $A \subset \Omega$ has no limit points in Ω.

Let $m : A \to \mathbb{N}$ be any function.

Then, $\exists f \in \mathcal{O}(\Omega)$ such that $\mathcal{Z}(f) = A$, and f has a zero of multiplicity $m(\alpha)$ for every $\alpha \in A$.

Proof. It suffices to prove the theorem in the special case where:

Ω is a deleted neighbourhood of ∞ and $\infty \notin \bar{A}$.

$\Omega = \mathbb{C} \setminus K$ for some compact $K \subseteq \mathbb{C}$.

Justification.

Let Ω_1 and A_1 be as in the hypothesis of the theorem.

Fix $\infty \neq a \in \Omega_1 \setminus A_1$. Define

$$T(z) = \frac{1}{z-a}.$$

T is a linear fractional transformation from $\hat{\mathbb{C}}$ onto itself.

T is a homeomorphism of Ω_1 onto $T(\Omega_1) =: \Omega$.

Define $A := T(A_1)$. Then, A is BOUNDED as well has no limit points in Ω.

Now, Ω and A satisfy the requirements of the special case.

Now, if theorem holds for special case, we can translate it back.

Now, we prove the theorem for the special case.

If $A = \{a_1, \dots, a_n\}$, take

$$f(z) := (z-a_1)^{m_1} \cdots (z-a_{n_0})^{m_n}$$

If $A = \{a_1, \dots, a_n\}$, take
$$f(z) := \frac{(z-a_1)^{m_1} \cdots (z-a_{n_0})^{m_n}}{(z-b)^{m_1+\cdots+m_n}}$$
for some $b \in \mathbb{C} \setminus \Omega$.

Suppose $|A| = \infty$. Let $(z_n)_n$ be an enumeration with the multiplicities taken care of.

For each n, $\exists\, w_n \in \mathbb{C} \setminus \Omega$ such that
$$|w_n - z_n| = \text{dist}(z_n, \mathbb{C} \setminus \Omega).$$
this is a nonempty compact set.

Lecture 12 (10-02-2022)

10 February 2022 13:52

Recall: Had reduced theorem to special case.

We now prove it for the special case:

$\Omega = \mathbb{C} \setminus K'$ for $K' \neq \emptyset$ compact, (if $\Omega = \mathbb{C}$, we already know.)

$\infty \notin \bar{A}$.

Had done it for finite A.

$(z_n)_{n \geq 1}$: enumeration of A, with multiplicities.

$(w_n)_{n \geq 1}$: satisfy $\operatorname{dist}(z_n, \mathbb{C} \setminus \Omega) = |z_n - w_n|$.

↳ lie in $\mathbb{C} \setminus \Omega$

If $|z_n - w_n| \not\to 0$, then $\exists$ subsequence s.t. $|z_{n_k} - w_{n_k}| \geq \delta > 0$.

But A is bounded. $\exists (z_{n_{k_m}})$ s.t. $z_{n_{k_m}} \to z_0 \in \mathbb{C} \setminus \Omega$.

But then $|z_{n_{k_m}} - w_{n_{k_m}}| \to 0$. $\to\leftarrow$

Thus, $|z_n - w_n| \longrightarrow 0$.

Note that if $a \in \Omega$, $b \notin \Omega$, then $z \mapsto E_p\left(\frac{a-b}{z-b}\right)$ is holo on Ω and has a simple zero at a.

Claim: $z \mapsto \prod_{n=1}^{\infty} E_n\left(\frac{z_n - w_n}{z - w_n}\right)$ converges in $\mathcal{O}(\Omega)$.

From the claim, everything follows.

Proof. Suffices to show that

$$z \mapsto \sum_{n=1}^{\infty} \left| 1 - E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| \quad \text{converges in } \mathcal{O}(\Omega).$$

Fix $K \subseteq \Omega$. Then, $\operatorname{dist}(K, \mathbb{C} \setminus \Omega) =: \delta > 0$.

For $z \in K$:

$$\left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{|z_n - w_n|}{\delta} \longrightarrow 0.$$

$$\therefore \left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{1}{2} \quad \forall\ n \gg 0.$$

$$\therefore \left| 1 - E_n\left(\frac{z_n - w_n}{z - w_n}\right) \right| \leq \left(\frac{1}{2}\right)^{n+1} \quad \forall n >> 0.$$ ∎

EXAMPLE. Consider $f(z) = \sin(\pi z)$, $z \in \mathbb{C}$.

$f \in \mathcal{O}(\Omega)$ and $\mathcal{Z}(f) = \mathbb{Z}$.

$$\mathbb{Z} = \{a_n\}_{n \geq 1} \cup \{0\}$$

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

By Weierstrass factorisation, $\exists\, h \in \mathcal{O}(\mathbb{C})$ with $\mathcal{Z}(h) = \mathbb{Z}$.

Then, $\frac{f}{h} = \exp \circ g$ for some $g \in \mathcal{O}(\mathbb{C})$.

One explicit construction of h is:

$$h(z) = z \cdot \prod_{n=1}^{\infty} E_1\left(\frac{z}{n}\right) \cdot \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right).$$

$$= z \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)\exp\left(\frac{z}{n}\right) \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)\exp\left(-\frac{z}{n}\right).$$

abs. conv.

$$= z \prod_{n=1}^{\infty}\left[\left(1 - \frac{z}{n}\right)\exp\left(\frac{z}{n}\right)\left(1 + \frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right]$$

$$= z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

$$\therefore \sin(\pi z) = z \exp(g(z)) \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right), \quad z \in \mathbb{C}.$$

Remark. Using some more analysis, one can determine g somewhat. (log derivative?) We don't do it here.

Harmonic Functions:

Defn. Let $\Omega \subseteq \mathbb{C}$ be open.

Let $u : \Omega \to \mathbb{R}$ be C^2.

u is said to be harmonic on Ω if

$$\underbrace{\Delta u}_{} := \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{} u = 0.$$

Laplacian operator

Harmonic function, Laplacian operator

We define two more operators:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{i\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right).$$

CALCULATION:

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial}{\partial x} - \frac{i\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y} - i\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2}\right).$$

If $u \in \mathcal{C}^2(\Omega)$, then

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(u) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = \frac{1}{4}\Delta u.$$

mixed partials cancel!

For harmonic u, $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} u = 0$.

EXAMPLES. Harmonic functions:

(1) $u(x,y) = ax + by + c$,

(2) $u(x,y) = 2xy$,

(3) $u(x,y) = x^3 - 3xy^2$,

(4) if $f \in \mathcal{O}(\Omega)$, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic, by the Cauchy-Riemann equations.

NON harmonic:

(5) $u(x,y) = x^2 + y^2$.

LAPLACIAN IN POLAR: Define $u(z) = \log(|z|)$.

Write $z = re^{i\theta}$, $|z| = r$.

$x = r\cos\theta$, $y = r\sin\theta$.

(Exercise) $$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Then, $$\Delta u = \frac{\partial^2}{\partial r^2}\log(r) + \frac{1}{r}\frac{\partial}{\partial r}\log(r)$$
$$= -\frac{1}{r^2} + \frac{1}{r}\cdot\frac{1}{r} = 0.$$

Question: Let $\Omega \subseteq \mathbb{C}$ be a domain.
Let $u : \Omega \to \mathbb{R}$ be harmonic on Ω.
Does there exist another harmonic $v : \Omega \to \mathbb{R}$ s.t.
$$f := u + iv \in \mathcal{O}(\Omega).$$

In such a case, v is said to be a harmonic conjugate of u.

Observation: Let $\Omega \subseteq \mathbb{C}$ be a domain.
Suppose u is harmonic on Ω.
Define $g : \Omega \to \mathbb{C}$ by
$$g(z) := \frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial x}(z).$$

Note that $\operatorname{Re}(g) = u_x$, $\operatorname{Im}(g) = -u_y$.
Since $u \in C^2(\Omega)$, the above two have continuous (first) partials on Ω.
Moreover, $\Delta u \equiv 0 \implies g$ satisfies the CR equations on Ω.

CONCLUSION. g is holomorphic. (For any domain Ω, and any harmonic u.)

Now, if v is a harmonic conjugate of u, then CR equations tell us:

$$\nabla v = (-u_y, \; u_x).$$

CANNOT ALWAYS SOLVE THIS!

EXAMPLE. Let $\Omega = \mathbb{C} \setminus \{0\}$.

Define $u : \Omega \to \mathbb{R}$ by $u(z) = \log|z|$ or

$$u(x,y) = \frac{1}{2}\log(x^2+y^2).$$

$\Delta u \equiv 0$. Suppose $\exists v : \Omega \to \mathbb{R}$ harmonic s.t.

$$\nabla v = (-u_y, \; u_x).$$

Then,
$$(\nabla v)(x,y) = \left(-\frac{y}{x^2+y^2}, \; \frac{x}{x^2+y^2}\right).$$

$\rightarrow\leftarrow$ (Integrate along unit circle!)

Alter: Then, $f = \operatorname{Log}|z| + iv$ is holomorphic.

Why is this a contradiction?

Lecture 13 (14-02-2022)

14 February 2022 14:04

- Let $u : \Omega \to \mathbb{R}$ be harmonic with Ω a domain.
  If $v_1, v_2 : \Omega \to \mathbb{R}$ are harmonic conjugates of Ω, then

$$i(v_1 - v_2) = (u + iv_1) - (u + iv_2) \in \mathcal{O}(\Omega).$$

But $i(v_1 - v_2)$ is purely imaginary valued. Thus, $v_1 \equiv v_2 + c$ for some constant $c \in \mathbb{R}$.

- Last time, we saw that not every harmonic function has a harmonic conjugate.
  Let $\Omega \subseteq \mathbb{C}$ be a domain, $u : \Omega \to \mathbb{R}$ be harmonic. Define $g : \Omega \to \mathbb{C}$ by

$$g := u_x - iu_y.$$

Then, g is holomorphic on Ω.
SUPPOSE f is an antiderivative of g.
Let $f = \tilde{u} + i\tilde{v}$.
Then, $\underset{\underset{g}{\shortparallel}}{f'} = \tilde{u}_x + i\tilde{v}_x = \tilde{u}_x - i\tilde{u}_y.$

$\therefore \tilde{u} \equiv u + c.$

Thus, $\tilde{v}$ is a harmonic conjugate of u!

Thus, u has a harmonic conjugate whenever g has an antiderivative. As g is holomorphic, this does happen whenever Ω is simply-connected.

Consequences:

- Let Ω be an open set in $\mathbb{C}$, $u : \Omega \to \mathbb{R}$ be harmonic.

Let $a \in \Omega$. Then, $\exists r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$.

As the disc is simply-connected, $\exists f \in \mathcal{O}(D(a,r))$ s.t. $\mathrm{Re}(f) = u$. Thus, u is C^∞-smooth on $D(a,r)$. In turn, u is C^∞-smooth on Ω. (In fact, real analytic!)

$$f(a) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(\xi)}{\xi - a}\, d\xi$$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{\cancel{re^{i\theta}}} \cancel{r}\cancel{i}\cancel{e^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

Taking the real part gives:

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

Thus, u satisfies the mean value property.

Prop^n. Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be domains and $f : \Omega_1 \to \Omega_2$ be holomorphic. Let $u : \Omega_2 \to \mathbb{R}$ be harmonic.
Then, $u \circ f$ is also harmonic.

$$\begin{tikzcd} \Omega_1 \arrow[r, "f"] \arrow[dr, dashed] & \Omega_2 \arrow[d, "u"] \\ & \mathbb{R} \end{tikzcd}$$

Proof. For any $a \in \Omega_2$, we can find a disc $D(a, r_a)$ compactly contained in Ω_2 and $g_a \in \mathcal{O}(D(a, r_a))$ s.t. $\mathrm{Re}(g_a) = u|_{D(a,r_a)}$.
Now, given $b \in \Omega_1$, set $a := f(b)$.
By continuity, $\exists \delta > 0$ s.t. $f(D(b,\delta)) \subseteq D(a, r_a)$.
Thus, $\mathrm{Re}(g_a \circ f) = u \circ f$ is harmonic on $D(b, \delta)$. ∎

Def^n. Let $\Omega \subseteq \mathbb{C}$ be open, and $u : \Omega \to \mathbb{R}$.
u has the mean value property if whenever $D(a, \delta) \subset\subset \Omega$, then

u has the mean value property if whenever $D(a,\delta) \subset\subset \Omega$, then
compactly contained

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+\delta e^{i\theta})\,d\theta.$$

- We showed: harmonic $\Rightarrow$ MVP.

We will show: MVP $\Rightarrow$ harmonic.

Maximum Principle

Thm. Let $\Omega \subseteq \mathbb{C}$ be a domain, $u: \Omega \to \mathbb{R}$ have the MVP on Ω.
If $\exists\, p_0 \in \Omega$ s.t.

$$u(p_0) = \sup_{z\in\Omega} u(z),$$

then u is constant.

Proof. Let $E = \{\xi \in \Omega : u(\xi) = \sup_{z\in\Omega} u(z)\}$.

$E \neq \emptyset$ as $p_0 \in E$. E is clearly closed (in Ω).
We show E is open and thus, $E = \Omega$ as Ω is connected.

Let $p \in E$. Pick $R > 0$ s.t. $D(p, R) \subset\subset \Omega$.
Fix $r \in (0, R)$. By MVP, we have

$$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p+re^{i\theta})\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(p)\,d\theta = u(p).$$

Thus, $u(p+re^{i\theta}) = u(p) \quad \forall\, \theta \in [0, 2\pi]$.
Thus, u is constant on $D(p, R)$. $\therefore D(p,R) \subseteq E$. ∎

Similarly, we have the minimum principle

Thm. (Global version)
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, and $u : \Omega \to \mathbb{R}$ have MVP.
Suppose $u \in C^0(\overline{\Omega})$. Then,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \quad \text{and}$$

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Proof. $\max_{\overline{\Omega}} u$ attained somewhere. If its interior, then constant ... $\square$

Corollary. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain in $\mathbb{C}$.
Suppose $u_1, u_2 \in C^0(\overline{\Omega})$ are s.t. u_1, u_2 have the MVP on Ω.
If $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then $u_1 \equiv u_2$.

Proof. $u_1 - u_2$ has the MVP and is 0 on $\partial\Omega$... $\square$

Lecture 15 (28-02-2022)

28 February 2022 13:49

Recall: Dirichlet problem on $D(0,1) = \mathbb{D}$.

Given $f : \partial D(0,1) \longrightarrow \mathbb{R}$ continuous, we wish to find

$$u : \overline{\mathbb{D}} \to \mathbb{R}$$

such that (i) $u \in C^0(\overline{\mathbb{D}})$,

(ii) u is harmonic on $\mathbb{D}$,

(iii) $u|_{\partial \mathbb{D}} = f$.

We defined u as follows:

$$u(z) := \begin{cases} f(z) & ; z \in \partial \mathbb{D}, \\ \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) f(e^{it})\, dt & ; z \in \mathbb{D}. \end{cases}$$

u was seen to be harmonic as it was the real part of the holomorphic $F : \mathbb{D} \to \mathbb{C}$ defined by

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(e^{it})\, dt.$$

Q. Suppose we are given a harmonic $u : \mathbb{D} \longrightarrow \mathbb{R}$.

We wish to explicitly find $F \in \mathcal{O}(\mathbb{D})$ s.t. $\operatorname{Re}(F) = u$.

(Such an F exists since u is harmonic and $\mathbb{D}$ simply connected.)

Prop^n. Let $u : \overline{\mathbb{D}} \longrightarrow \mathbb{R}$, $u \in \mathcal{C}^0(\overline{\mathbb{D}})$, u is harmonic on $\mathbb{D}$.

Then, $u|_{\mathbb{D}}$ is the real part of $F : \mathbb{D} \to \mathbb{C}$ is defined as

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it})\, dt.$$

Proof. Follows from the Dirichlet problem with $f = u|_{\partial\mathbb{D}}$.
(We know that the solution is unique) ◻

Poisson kernel on $\mathbb{D} = D(0,1)$.

$$P : D(0,1) \times \partial D(0,1) \longrightarrow \mathbb{R}$$
$$(z, s) \longmapsto \frac{1-|z|^2}{|z-s|^2}.$$

Poisson kernel on $D(a,R)$.

s again!

$$\tilde{P} : D(a,R) \times \partial\mathbb{D} \longrightarrow \mathbb{R}$$
$$(z, \zeta) \longmapsto P\left(\frac{z-a}{R}, s\right).$$

Generalised Poisson Integral Formula:

Prop^n. Let u be harmonic on $D(a,R)$ and continuous on $\overline{D(a,R)}$.
Then, for any $z \in D(a,r)$, we have

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{P}(z, e^{it})\, u(a+Re^{it})\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - |z-a|^2}{|z-a-Re^{it}|^2}\, u(a+Re^{it})\, dt.$$

Obs1. Suppose further that $u \geq 0$ (cont. on $\overline{D}$, harmonic on D).

$$\frac{R^2 - |z-a|^2}{|z-a-Re^{it}|^2} \leq \frac{R^2 - |z-a|^2}{(R-|z-a|)^2} = \frac{R+|z-a|}{R-|z-a|}$$

$$\frac{R^2 - |z-a|^2}{|z-a-Re^{it}|^2} \geq \frac{R^2-|z-a|^2}{(|z-a|+R)^2} = \frac{R-|z-a|}{R+|z-a|}$$

That is: $\dfrac{R-|z-a|}{R+|z-a|} \leq \dfrac{R^2-|z-a|^2}{|z-a-Re^{it}|^2} \leq \dfrac{R+|z-a|}{R-|z-a|}$.

Can multiply with $u(e^{it}) \geq 0$ to integrate and get:

$$u(a)\left(\frac{R-|z-a|}{R+|z-a|}\right) \leq u(z) \leq u(a)\left(\frac{R+|z-a|}{R-|z-a|}\right).$$

Harnack's Inequality

(We can relax u to not extend continuously on ∂D)

Obs 2. Let $(u_n)_n$ be a seq. of nonnegative harmonic functions on $D(a,R)$.

- Assume that $u_n(a) \to 0$.

  Then, Harnack's inequality tells us that $u_n(z) \to 0$ for all $z \in D(a,R)$. Moreover, this is uniform on every CC subdisk.
- OTOH, if $(u_n(a))_n$ is bounded, then $(u_n)_n$ is locally uniformly bounded.

Theorem. Let $\Omega \subseteq \mathbb{C}$ be a domain.

Let $u_n : \Omega \to \mathbb{C}$ be a sequence of nonnegative harmonic functions.

- If $\exists z_0 \in \Omega$ s.t. $u_n(z_0) \to \infty$, then $u_n \to \infty$ uniformly on compact subset.
- If $\exists z_0 \in \Omega$ s.t. $(u_n(z_0))_n$ is bdd, then $(u_n)_n$ is bdd uniformly on compact subsets.

Proof. Let $B = \{z \in \Omega : (u_n(z))_n \text{ is bdd}\}$.

By Obs 2, both B and $\Omega \setminus B$ are open.

Thus, either $B = \Omega$ or $B = \emptyset$.

Thus, if $(u_n(z))_n$ is bdd for some z_0, it is bdd for all. Uniformity part follows from Obs. as well.

Now, let $A := \{z \in \Omega : u_n(z) \xrightarrow{n} \infty\}$.

A is open by Obs 2. Suppose $A \neq \phi$.
For A^c: suppose $z_0 \in A^c$. Then, $(u_{n_k}(z_0))_k$ is bdd for some subseq.
Then $(u_{n_k}(z))_k$ is bdd for all z.
Thus, $A^c = \phi$. ☒

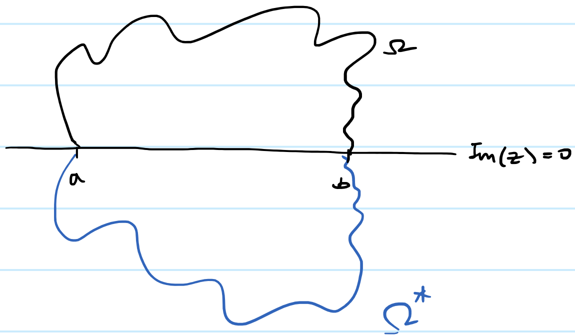
Prop^n. Let $\Omega \subseteq \mathbb{C}$ be a domain.
Let $u \in \mathcal{C}^0(\Omega)$ and suppose that u has the mean value property.
Then, u is harmonic.
(Only assumed continuity and got real analyticity!)

Proof. Fix $a \in \Omega$, $r > 0$ s.t. $\overline{D(a,r)} \subseteq \Omega$. Let $D := D(a,r)$.
Define $f = u|_{\partial D}$.
Then, solve the Dirichlet problem on D with boundary data f.
We get $\tilde{u}$.
$u - \tilde{u}$ both have MVP and agree on ∂D.
$\therefore u \equiv \tilde{u}$ on D. ☒

Schwarz Reflection Principle for Harmonic Functions

Schwarz Reflection Principle for Harmonic Functions

Prop^n.



Let u be harmonic on Ω (where Ω is as shown). (open)
Define
$$\Omega^* = \{ z \in \mathbb{C} : \bar{z} \in \Omega \}.$$

Assume that for all $x \in (a,b)$, we have $\lim_{\Omega \ni z \to x} u(z) = 0$.

Then, we can extend u to $u^* : \Omega \cup \Omega^* \cup (a,b) \to \mathbb{R}$ as

$$u^*(z) = \begin{cases} u(z) & ; \quad z \in \Omega \end{cases}$$

$$u^*(z) = \begin{cases} u(z) & ; \quad z \in \Omega \\ 0 & ; \quad z \in (a,b) \\ -u(\bar{z}) & ; \quad z \in \Omega^* \end{cases}$$

Then, u^* is harmonic on $\underbrace{\Omega \cup (a,b) \cup \Omega^*}_{=\Omega'}$.

Proof. (0) u^* is continuous on Ω'. (Check.)

(1) u^* is harmonic on Ω^*. Use MVP.

(2) Suppose $x \in (a,b)$. If $r>0$ is s.t. $\overline{D(x,r)} \subseteq \Omega'$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x+re^{it})\,dt = \frac{1}{2\pi}\int_{0}^{\pi} u(x+re^{it})\,dt - \frac{1}{2\pi}\int_{-\pi}^{0} u(x+re^{-it})\,dt$$

$$= 0.$$

Thus, we are done. $\square$

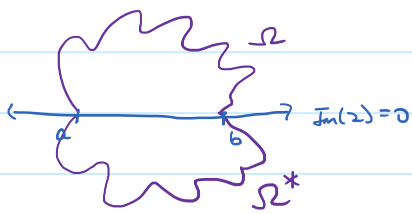
Lecture 16 (03-03-2022)

03 March 2022 13:54

Schwarz Reflection Principle For Holomorphic Functions

Theorem Let $G \subseteq \mathbb{C}$ be a domain in $\mathbb{C}$ such that $G \cap \mathbb{R} = (a, b)$.
Let $\Omega = \{z \in G : \operatorname{Im}(z) > 0\}$.
Suppose $F \in \mathcal{O}(\Omega)$ and

$$\lim_{\Omega \ni z \to x} \operatorname{Im}(F(z)) = 0$$



for all $x \in (a, b)$.
Then, $\exists F^* \in \mathcal{O}(\Omega \sqcup (a, b) \sqcup \Omega^*)$ s.t. $F^*|_{\Omega} = F$.

Note: Did not assume that $\operatorname{Re} \circ F$ has a limit on (a, b).
But it follows as a consequence.

Furthermore, F^* is given as

$$F^*(z) := \begin{cases} F(z) & ; \ z \in \Omega, \\ \lim\limits_{\Omega \ni \xi \to z} F(z) & ; \ z \in (a, b), \\ \overline{F(\bar{z})} & ; \ z \in \Omega^*. \end{cases}$$

(Part of the theorem is that the limit on the RHS does exist.

Proof. Observe: If $H \in \mathcal{O}(\Omega \cup (a, b) \cup \Omega^*)$ restricts to F on Ω, then it must the case that $H(z) = \overline{F(\bar{z})}$.

Proof. Indeed, it must be the case that $H(x + 0i) \in \mathbb{R}$ $\forall x \in [a, b]$.
That is, $H(z) = \overline{H(z)}$ for all $z \in [a, b]$.
$\overline{H(z)} = \overline{H(\bar{z})}$

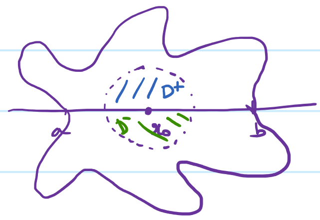
But $z \mapsto \overline{H(\bar{z})}$ is also a hol function on $\Omega \cup (a, b) \cup \Omega^*$.
Thus, $H(z) = \overline{H(\bar{z})}$ for ALL $z \in \Omega \cup (a, b) \cup \Omega^*$.
In particular, $H(z) = \overline{F(\bar{z})}$ for all $z \in \Omega^*$. ∎

Now, let $v = \operatorname{Im} \circ F : \Omega \to \mathbb{R}$. By the reflection principle for harmonic functions, we see that v extends to a harmonic function v^* on $\Omega \cup (a,b) \cup \Omega^*$.

$\left(\lim_{\Omega \ni z \to x} v(x) = 0 \ \forall x \in [a,b] \right.$ is true by hypothesis.)

Fix $x_0 \in (a,b)$ and $r > 0$, let D^+ and D^- be as shown:

$D := D(x_0, r) = D^+ \cup D^- \cup (x_0 - r, x_0 + r) \subseteq \Omega$.



$v^*|_D$ has a harmonic conjugate $u^* : D \to \mathbb{R}$.

Actually, just assume that $\operatorname{domain}(v^*) = D$.

Clearly,

$$F|_D - (u^* + iv^*) \in \mathcal{O}(D^+) \quad \text{and}$$

real-valued on D^+. By adjusting u^* by adding a const,

$$F \equiv u^* + iv^* \text{ on } D^+.$$

Extend F to $F_0 : D \to \mathbb{C}$ by

$$F_0 := u^* + iv^*.$$

Then, $F_0 \in \mathcal{O}(D)$ and $F_0|_{D^+} = F$.

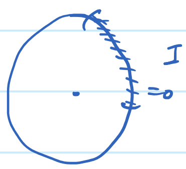
By the observation, $F_0(z) = \overline{F(\bar{z})}$ on D^-.

Conclude that this is enough. ∎

EXAMPLE Let $F \in \mathcal{O}(\mathbb{D}) \cap C^0(\bar{\mathbb{D}})$. $(\mathbb{D} := D(0,1))$

Suppose that there is an arc $I \subseteq \partial\mathbb{D}$ such that

$$F|_I \equiv 0.$$



Then, $F \equiv 0$.

If $I = \partial\mathbb{D}$, then MMT would give us that.

Assume $I \subsetneq \partial\mathbb{D}$. Wlog, $1 \notin I$.

Let $\mathbb{H} = \{x + iy : y > 0\}$. Map $\psi : \mathbb{D} \to \mathbb{H}$ by

$$\psi(z) = -i\left(\frac{z+1}{z-1}\right).$$

Check: ψ maps $\partial\mathbb{D} \setminus \{1\}$ bijectively onto $\mathbb{R}$.

Use reflection formula by appropriately composing with ψ. ∎

biholo

Towards the Runge's Theorem

Let $f \in \mathcal{O}(\mathbb{D})$. Then, f can be written as a limit of polynomials. (limit in $\mathcal{O}(\mathbb{D})$.) Simply truncate the power series centered at 0.
Indeed, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, take $f_N(z) := \sum_{n=0}^{N} a_n z^n$.

Then, $f_N \to f$ uniformly on COMPACT SUBSETS. Need not be uniform on $\mathbb{D}$, such as $f(z) := \dfrac{1}{1-z}$.

Then, $f_N(z) = \dfrac{1-z^{N+1}}{1-z}$ and $\sup\limits_{z \in \mathbb{D}} |f_N - f(z)| = \sup\limits_{z \in \mathbb{D}} \left| \dfrac{z^{n+1}}{1-z} \right|$
$= \infty$.

Q. Now, let Ω be any domain in $\mathbb{C}$. Suppose $f \in \mathcal{O}(\Omega)$.
Is f a limit (in $\mathcal{O}(\Omega)$) of polynomials?

Ans. No. Take $\Omega = \mathbb{D} \setminus \{0\}$ and $f = (z \mapsto 1/z)$.
If $p_N \to f$ in $\mathcal{O}(\Omega)$, then $0 = \lim\limits_N \int_{|z|=1/2} p_N = \int_{|z|=1/2} f = 2\pi i$. $\to\leftarrow$

Or, look at $\sup\limits_{0<|z|<1/2} |f(z) - p_N(z)| = \infty$.

Theorem. (Runge's Theorem)
Let $K \subseteq \mathbb{C}$ be compact.
Let f be holomorphic on a neighbourhood Ω of K.
Suppose $E \subseteq \hat{\mathbb{C}} \setminus K$ containing (at least) one point from each connected component of $\hat{\mathbb{C}} \setminus K$.
Then, for any $\varepsilon > 0$, there is a rational function R such that

$$\sup_{z \in K} |f(z) - R(z)| < \varepsilon$$

and $\text{Poles}(f) \subseteq E$.

Note: $K \to$ compact. K^c open. Connected components: open and disjoint.
Thus, only countably many components

Corollary: Let $K \subseteq \mathbb{C}$ be compact such that $\hat{\mathbb{C}} \setminus K$ is connected. Let $\epsilon > 0$.
Then, taking $E = \{\infty\}$ ($\infty \notin K$) shows that we can find a polynomial P s.t. $\|P - f\|_K < \epsilon$.

Exercise Let $K \subseteq \mathbb{C}$ be compact.
Show that $\hat{\mathbb{C}} \setminus K$ is connected iff $\mathbb{C} \setminus K$ has no bounded components.
(Maybe compactness not needed?)

If $G = \{z : 0 \leq \operatorname{Im} z \leq 1\}$, then $\mathbb{C} \setminus G$ is not connected
but $\hat{\mathbb{C}} \setminus G$ is.

Towards the proof of Runge's Theorem:

Lemma. Every nonempty open set $\Omega \subseteq \mathbb{C}$ is the union of a sequence $(K_n)_{n \geq 1}$ of compact sets such that:

(i) $\Omega = \bigcup_{n=1}^{\infty} K_n$,

(ii) $K_n \subseteq K_{n+1}^{o}$ for all $n \in \mathbb{N}$,

$\Rightarrow$ every compact $K \subseteq \Omega$ is contained in some K_n.

(iii) every connected component of $\hat{\mathbb{C}} \setminus K_n$ contains a component of $\hat{\mathbb{C}} \setminus \Omega$.

"K_n has no other holes than those forced upon it by Ω"

Proof. As before, we define

$$K_n := \{z \in \Omega : \operatorname{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/n\} \cap \overline{D(0, n)}.$$

Only need to check (iv).
Suffices to show that every component of $\hat{\mathbb{C}} \setminus K$ intersects $\hat{\mathbb{C}} \setminus \Omega$.

Lecture 17 (07-03-2022)

07 March 2022 14:00

Let V be a component of $\hat{\mathbb{C}} \setminus K_n$.

If V is unbounded, then $\infty \in V \cap (\hat{\mathbb{C}} \setminus \Omega)$.

Suppose now that V is bounded.

By definition of K_n, $\exists z \in V$ s.t. $\operatorname{dist}(z, \mathbb{C} \setminus \Omega) < \frac{1}{n}$.

(Think about it. Note that V is different from the unique unbounded component that contains $\mathbb{C} \setminus \overline{D(0,n)}$)

By def^n, $\exists w \in \mathbb{C} \setminus \Omega$ s.t. $|z - w| < 1/n$.

Since discs are connected, we see that $w \in D(z, \frac{1}{n}) \subseteq V$. ☒

Theorem. (Runge's Theorem ver. 2)

Let $\Omega \subseteq \mathbb{C}$ be an open set. Let A be a set intersecting each component of $\hat{\mathbb{C}} \setminus \Omega$. Let $f \in \mathcal{O}(\Omega)$. Then, there is a sequence of rational functions $(R_n)_{n \geq 1}$ with poles in A s.t.

$$R_n \longrightarrow f$$

uniformly on compact subsets of Ω.

Corollary. If $\hat{\mathbb{C}} \setminus \Omega$ is connected, then R_n can be chosen to be polynomials.

(Take $A = \{\infty\}$.)

Proof of Runge Ver 2 using original Runge: Let $\Omega \subseteq \mathbb{C}$ be open and take a compact exhaustion $(K_n)_{n \geq 1}$ as provided by the above lemma. Note that A contains one point of every component of each $\hat{\mathbb{C}} \setminus K_n$ as well.

(By property (iv) of exhaustion)

By Runge (original), we can get rational R_n with $\operatorname{Poles}(R_n) \subseteq A$ and $\|f - R_n\| < 1/n$.

Now conclude ... ☒

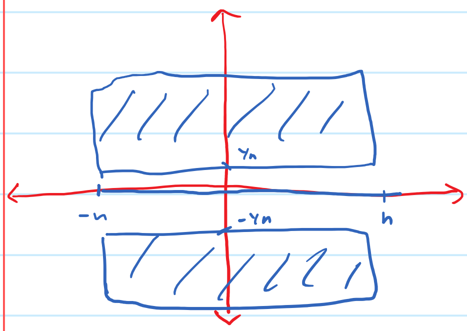
EXAMPLES ① Is there a sequence $(P_n)_{n \geq 1}$ of polynomials such that

$$\lim_{n\to\infty} p_n(z) = \begin{cases} -1 & ; \ \operatorname{Im} z > 0 \\ 0 & ; \ \operatorname{Im} z = 0 \\ 1 & ; \ \operatorname{Im} z < 0 \end{cases}$$

Call this $f(z)$.

Let $\Omega = \mathbb{C}$. $K_n = \{ z \in \mathbb{C} : \frac{1}{n} \le |\operatorname{Im} z| \le n, \ |\operatorname{Re} z| \le n \}$

$\cup \{ x \in \mathbb{R} : |x| \le n \}$.



Define

$$f_n(z) := \begin{cases} -1 & \operatorname{Im}(z) > 1/2n \\ 0 & |\operatorname{Im}(z)| < 1/4n \\ 1 & \operatorname{Im}(z) < -1/2n \end{cases}$$

Note: f_n is defined on an open nbd Ω_n of K_n and is holomorphic on it.

Also, $\hat{\mathbb{C}} \setminus K_n$ is connected. Also, $\mathbb{C} = \bigcup_{n\ge 1} K_n$.

Thus, by Runge, $\exists$ polynomial p_n s.t.

$\| p_n - f_n \|_{K_n} < 1/n$.

Now, given $z \in \mathbb{C}$, $\exists N \in \mathbb{N}$ s.t. $z \in K_n$ $\forall n \ge N$.

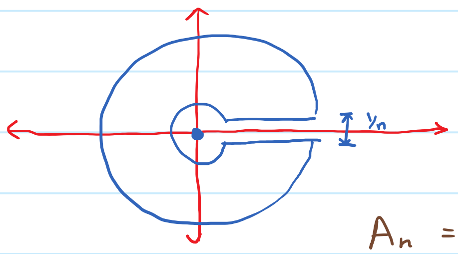
$\therefore$ $|p_n(z) - f_n(z)| < 1/n$ for all $n \ge N$.

Letting $n \to \infty$ gives the desired result as

$f_n(z) = f(z)$ for all $n \ge N$. $\boxtimes$

② Is there a sequence of polynomials $(p_n)_{n\ge 1}$ s.t. $p_n(0) \to 1$ as $n \to \infty$ and $p_n \to 0$ in $\mathcal{O}(\mathbb{C} \setminus [0,\infty))$, i.e., on compact subsets of $\mathbb{C} \setminus [0, \infty)$

Consider K_n as in the diagram:



$K_n = \{0\} \cup A_n$, where

$A_n = \{ z \in \mathbb{C} : \frac{1}{n} \le |z| \le n \} \setminus \{ z : \operatorname{Re} z > 0, \ |\operatorname{Im} z| < \frac{1}{2n} \}$.

Note: $\hat{\mathbb{C}} \setminus K_n$ is connected $\forall n$.

Define

$$f_n(z) := \begin{cases} 1 & ; \quad z \in D(0, 1/4n), \\ 0 & ; \quad z \in \mathbb{C} \setminus D(0, \frac{1}{3n}). \end{cases}$$

As before f_n are defined and holo on an open nbd of K_n $\forall n \in \mathbb{N}$.

Can construct a polynomial p_n s.t.

$$\| f_n - p_n \|_{K_n} < 1/n. \quad \text{Done as before.} \quad \boxtimes$$

Now, we give a proof of original Runge's theorem.

Proof OUTLINE:

Step I: Find a "cycle" in $\Omega \setminus K$ for which the Cauchy integral formula holds:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \qquad \text{for all } z \in K.$$

Step II: Break γ into finitely many pieces $\{\gamma_k\}_k$ so that the integral on the RHS can be uniformly approximated on K by a Riemann sum

$$f(z) \approx \frac{1}{2\pi i} \sum_k \frac{f(q_k)}{q_k - z} \cdot (\gamma_k(1) - \gamma_k(0)),$$

where q_k are the sample points on trace γ_k.
The poles of RHS are within $\{p_k\} \subseteq \Omega \setminus K$.

Step III. Pole pushing: Approximate these rational functions with rational functions having poles in E.

Step I. Set $\delta := dist(K, \mathbb{C} \setminus \Omega) > 0$.
Choose N s.t. $2^N < \delta/2$.

Step I. Set $\delta := \operatorname{dist}(K, \mathbb{C} \setminus \Omega) > 0$.

Choose N s.t. $2^N < \delta/2$.

Consider a grid in $\mathbb{C}$ consisting of closed rectangles with vertices at $\frac{1}{2^N}\mathbb{Z} \times \frac{1}{2^N}\mathbb{Z}$.

Let $\mathcal{G}$ be the set of all such rectangles intersecting K.

Note that $\mathcal{G}$ is finite. For each $Q \in \mathcal{G}$ and $z \notin \partial Q$

(Note: $Q \subseteq \Omega$)

$$\frac{1}{2\pi i} \int_{\partial Q} \frac{f(\xi)}{\xi - z} \, d\xi = \begin{cases} f(z) & \text{if } z \in Q, \\ 0 & \text{else.} \end{cases}$$

Each ∂Q is oriented positively.

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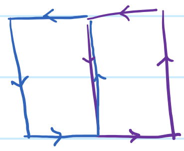
If $z \in K$ is fixed and $z \notin \partial Q$ for any Q, then z is in precisely one such rectangle Q.
For such a z, we have

$$\frac{1}{2\pi i} \sum_{Q \in \mathcal{G}} \int_{\partial Q} \frac{f(\xi)}{\xi - z} \, d\xi = f(z).$$



The integration over any edge shared by two rectangles in $\mathcal{G}$ will cancel out.
Thus, instead of integrating over individual ∂Q, simply integrate over the oriented boundary of the rectangles without repeating. This gives us the desired γ and we have

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} \, dz = f(z) \quad \text{for ALL } z \in K.$$

STEP II. γ is the sum n horizontal or vertical oriented segments in $\Omega \setminus K$, each having length $l < \frac{\delta}{2}$.
Let $d := \mathrm{dist}(K, \mathrm{trace}(\gamma)) > 0$.
Subdivide each of these segments into n_0 (to be chosen later) subsegments with consistent orientation, each having length l/n_0.
We now have segments

$$\gamma_1, \ldots, \gamma_{nn_0}.$$

As f is uni. cont. on $\mathrm{trace}(\gamma)$, we choose $n_0 \gg 0$ s.t.
$|f(p) - f(q)| < \varepsilon$ (as in the theorem statement) whenever $p, q \in \mathrm{trace}(\gamma_k)$ for all $k \in \{1, \ldots, nn_0\}$.

$\forall k$: Fix a sample point q_k on $\mathrm{trace}(\gamma_k)$. For $z \in K$, $\xi \in \mathrm{trace}(\gamma_k)$:

$$\left|\frac{f(\xi)}{\xi - z} - \frac{f(q_k)}{q_k - z}\right| \le \left|\frac{f(\xi)}{\xi - z} - \frac{f(\xi)}{q_k - z}\right| + \left|\frac{f(\xi)}{q_k - z} - \frac{f(q_k)}{q_k - z}\right|$$

$$\le \frac{|f(\xi)|\,|q_k - \xi|}{|\xi - z|\,|q_k - z|} + \frac{|f(\xi) - f(q_k)|}{|q_k - z|}$$

$M := \sup_\gamma |f|$

$$\le \frac{M \cdot (\ell/n_0)}{d \cdot d} + \frac{\varepsilon}{d}$$

$$\le \frac{M\delta}{2d^2} \cdot \frac{1}{n_0} + \frac{\varepsilon}{d}$$

can assume $n_0 \gg 0$.

$$\le \frac{2\varepsilon}{d}.$$

$$f(z) = \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{k=1}^{nn_0} \int_{\gamma_k} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Let $$\tilde{R}(z) = \sum_{n=1}^{nn_0} \frac{f(q_k)}{q_k - z}\left(\gamma_k(1) - \gamma_k(0)\right)$$

$$= \sum_{k=1}^{nn_0} \frac{A_k}{q_k - z}$$ → rational function with poles outside K

We have shown that $\exists$ rational $\tilde{R}$ s.t.

$$\|f - \tilde{R}\|_K < M' \cdot \varepsilon.$$

M' did not depend on ε.

Step II

Lemma (Pole pushing lemma)

Let $K \subseteq \mathbb{C}$ be compact. Let $P \in \hat{\mathbb{C}} \setminus K$ and U be the connected component of $\hat{\mathbb{C}} \setminus K$ containing P. If $\varepsilon > 0$ and $Q \in U \setminus \{\infty\}$, then there is a rational function R with pole only at P s.t.

only at P s.t.

$$\sup_{z \in K} \left| \frac{1}{z-Q} - R(z) \right| < \varepsilon.$$

<u>Proof of Lemma</u>. Assume $P \neq \infty$. Note that $\mathcal{U}\setminus\{\infty\}$ is connected.
(If $\mathcal{U}$ is bounded, $\mathcal{U}\setminus\{\infty\} = \mathcal{U}$.)

Let $S \subseteq \mathcal{U}\setminus\{\infty\}$ be the set of points in $\mathcal{U}\setminus\{\infty\}$ which satisfy the above conclusion. $P \in S$.

<u>Claim</u>: S is open in $\mathcal{U}\setminus\{\infty\}$.

<u>Proof</u>. Pick $Q \in S$. ($\therefore Q \neq \infty$.)

$\operatorname{dist}(Q, K) = d > 0$. Let $r < d$.

For $Q' \in D(Q,r) \cap \mathcal{U}$ and $z \in K$, we have

$$\frac{1}{z-Q'} = \frac{1}{z-Q+Q-Q'}$$

$$= \frac{1}{(z-Q)\left(1 - \frac{Q'-Q}{z-Q}\right)}$$

$|Q'-Q| < r$

$|z-Q| > r$

$$= \left(\frac{1}{z-Q}\right)\left(1 + \frac{Q'-Q}{z-Q} + \left(\frac{Q'-Q}{z-Q}\right)^2 + \cdots\right)$$

Note: $\displaystyle\sup_{z \in K} \left|\frac{Q'-Q}{z-Q}\right| < 1.$

Thus, the above convergence is uniform on K.

$$\Rightarrow \frac{1}{z-Q'} = \sum_{n=0}^{\infty} \frac{(Q'-Q)^n}{(z-Q)^{n+1}}, \quad z \in K.$$

The partial sums of the series are polynomials in $\frac{1}{z-Q}$ which uniformly approximate $\frac{1}{z-Q'}$ on K.

By assumption $\frac{1}{z-Q}$ can be uniformly approximated on K by rat'l f^ns with poles only at P. Conclude. ⊠

Claim. S is closed in $U\setminus\{\infty\}$.

Proof. Let $(Q_j)_{j\geq 1} \in S^{\mathbb{N}}$ such that $Q_j \to Q_0 \in U\setminus\{\infty\}$.

NTS: $Q_0 \in S$.

Note that $\frac{1}{z-Q_j} \to \frac{1}{z-Q_0}$ uni. on K. Conclude. $\boxtimes$

Thus, $S = U\setminus\{\infty\}$, as desired.

Now, suppose $P = \infty$.

Choose $r>0$ s.t. $\{z \in \mathbb{C} : |z| \geq r\} \subseteq U$.

Then,

$$\frac{1}{z-r} = -\sum_{n=0}^{\infty} \frac{z^n}{r^{n+1}} \quad \text{uniformly on } K.$$

The partial sums of RHS are polynomials which uniformly approximate $\frac{1}{z-r}$ on K.

By first part, we are done by taking $P' = r$. $\boxtimes$

(Back to the proof of Runge.)

Let $E \subseteq \hat{\mathbb{C}}\setminus K$ be as in theorem statement.

For each $k \in \{1,\dots,n_0\}$, pick $p_k \in E$ s.t. p_k and q_k are in the same connected component.

Now, use pole pushing on each... $\boxtimes$

Lecture 19 (14-03-2022)

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Mittag-Leffler Theorem

Recall: • Let $\Omega \subseteq \mathbb{C}$ be open. A function f is meromorphic on Ω if for every $a \in \Omega$, there exists a disc $D(a, \delta) \subseteq \Omega$ s.t. either (i) f is holomorphic on $D(a, \delta)$ or
(ii) f is holomorphic on $D(a, \delta) \setminus \{a\}$ and a is a pole of f.

• Meromorphicity at ∞ is translated to meromorphicity at 0 by the usual $z \mapsto f\left(\frac{1}{z}\right)$ business.

• A meromorphic function may have infinitely many poles. For example, $z \overset{f}{\longmapsto} \frac{1}{\sin \pi z}$ has poles at $\mathbb{Z}$.

However, the set of poles is a closed and discrete subset of Ω. Note that the above function f is meromorphic on $\mathbb{C}$ but not on $\hat{\mathbb{C}}$. The poles have a limit point, namely ∞.
(∞ is NOT an isolated singularity of f.)

• Let $\Omega \subseteq \mathbb{C}$ be open. If f is meromorphic on Ω, then (using the Weierstrass factorisation theorem) $f = \frac{g}{h}$ for some $g, h \in \mathcal{O}(\Omega)$.

Exercise: Describe meromorphic functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

If f is a meromorphic function with a pole at z_0, then the Laurent series expansion of f around z_0 is of the form:

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k, \qquad m \geq 1, \quad a_{-m} \neq 0.$$

The principal (singular) part of f at z_0 is given by

$$P(f, z_0; z) = \sum_{k=-m}^{-1} a_k (z - z_0)^k.$$

Note that $f - P(f, z_0; -)$ is holomorphic at z_0.

Consider the following problem:

Let $\Omega \subseteq \mathbb{C}$ be open and we are given a subset $A \subseteq \Omega$ s.t. A has no limit point in Ω. Write $A = \{a_k\}_k$.

Suppose that for each k, we are given a polynomial in $\frac{1}{z - a_k}$, say $S_k(z) = \sum_{j=1}^{m_k} \frac{A_{j,k}}{(z - a_k)^j}$.

Is there a meromorphic function f defined on Ω with $\text{Poles}(f) = A$ and $P(f, a_k; z) = S_k(z)$ $\forall k$.

Ans. Yes. (This is the Mittag-Leffler Theorem.)

Note: If A is finite, we can simply add the S_k and be done.

Theorem (Mittag-Leffler)

Let $\Omega \subseteq \mathbb{C}$ be open, and $A \subseteq \Omega$ be s.t. A has no limit point in Ω. Suppose that for each $\alpha \in A$, we are given:

- $m(\alpha) \in \mathbb{Z}^+$, and
- $P_\alpha(z) = \sum_{j=1}^{m(\alpha)} \frac{A_{j,\alpha}}{(z-\alpha)^j}$ for $A_{j,\alpha} \in \mathbb{C}$.

Then, $\exists f$ meromorphic on Ω s.t. $\text{Poles}(f) = A$ and the principal part of f at α is P_α ($\forall \alpha \in A$).

Proof. Let $(K_n)_{n=1}$ be a compact exhaustion of Ω satisfying the conditions as in the end of Lec 16.

For $n \geq 1$, define

$$A_n := A \cap (K_n \setminus K_{n-1}). \qquad (K_0 := \emptyset.)$$

Note $A_n \subseteq K_n$ has no limit point in K_n. Thus, each A_n is finite. Thus, we may define

$$Q_n(z) = \sum_{\alpha \in A_n} P_\alpha(z) \quad ; \quad n = 1, 2, 3 \ldots$$

Each Q_n is a rational function having poles precisely at A_n.
In particular, Q_n has no poles in K_{n-1} $(\forall n \geq 2)$.
$\therefore$ Q_n is holomorphic on a nbd of K_{n-1}.
Choose $E \subseteq \hat{\mathbb{C}} \setminus \Omega$ containing a point of each conn. comp. of $\hat{\mathbb{C}} \setminus \Omega$. Then, it also contains a point of ... of $\hat{\mathbb{C}} \setminus K_n$ $\forall n$.
By Runge's theorem (first ver), there exist rational functions $(R_n)_{n \geq 1}$ with poles in E s.t.

$$\sup_{z \in K_{n-1}} |(Q_n - R_n)(z)| < \frac{1}{2^n} \qquad \text{for all } n \geq 2.$$

<u>Claim</u>: $f(z) := Q_1(z) + \sum_{n=2}^{\infty} (Q_n - R_n)(z)$ has the desired properties.

<u>Proof</u>. (I) Convergence. Let $K \subseteq \Omega$ be compact.
Then, $\exists N$ s.t. $K \subseteq K_n^\circ$ for all $n \geq N$.
Then,

$$f(z) = \overbrace{Q_1(z) + \cdots + Q_N(z)}^{\text{poles at } A_1 \cup \cdots \cup A_N} - (\overbrace{R_2(z) + \cdots + R_N(z)}^{\text{poles outside } \Omega})$$
$$+ \underbrace{\sum_{n=N+1}^{\infty} (Q_n - R_n)(z)}_{\text{this converges uniformly by Weierstrass - M test.}}$$

$(Q_n - R_n)$: holo on $K_N^\circ \supseteq K$

This solves convergence issue and shows that

f is holomorphic on $\Omega \setminus A$.

(2) Behaviour on A.

Let K be as above. Assume $\alpha \in K \cap A$.

$$f(z) - [Q_1(z) + \cdots + Q_N(z)] = \sum_{n \geq N+1} (Q_n - R_n)(z) - (R_2 + \cdots + R_N)(z).$$

The RHS is holo on $K_N^\circ \supseteq K$.

The statement about principal part also follows. $\boxtimes$

EXAMPLE ① $\Omega = \mathbb{C}$. Let A and $\{P_\alpha\}_{\alpha \in A}$ be as earlier.

Can choose K_n to be $\overline{D}(0, n)$. $(K_0 := \emptyset)$ A_n as before.

$$Q_n(z) := \sum_{\alpha \in A_n} P_\alpha(z)$$

$$= \sum_{\substack{\alpha \in A \\ n-1 < |\alpha| \leq n}} P_\alpha(z).$$

Note that each Q_n is holomorphic on a nbd of K_{n-1}.

Thus, the truncations of power series give us polynomial approximations. These act as R_n.

Then, $f(z) = Q_1(z) + \sum_{n=2}^{\infty} (Q_n - R_n)(z)$, $z \in \mathbb{C}$ does the job.

② Find an f when $\Omega = \mathbb{C}$, $A = \mathbb{Z}^+$, $P_n(z) = \dfrac{1}{z-n}$ $\forall n \in \mathbb{Z}^+$.

Around 0, we have the power series expansion:

$$\frac{1}{z-n} = -\frac{1}{n}\left(\frac{1}{1-\frac{z}{n}}\right) = -\frac{1}{n}\left(1 + \frac{z}{n} + \frac{z^2}{n^2} + \cdots\right).$$

GUESS: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \sum_{n=1}^{\infty}\left(\frac{z}{n(z-n)}\right).$

Fix $R > 0$ and consider $\overline{D}(0, R)$. Let $N \in \mathbb{N}$ be s.t. $N > 2R$.

Then, for $z \in \bar{D}(0, R)$, and $n > N$,

$$\left| \frac{z}{(z-n)(n)} \right| \leq \frac{R}{|n-z|\,|n|} \leq \frac{R}{(n-|z|)\,n}$$

$$\leq \frac{2R}{n^2}.$$

As $\sum_{n \geq N} \frac{1}{n^2}$ converges, we are done. ☐

Lecture 20 (17-03-2022)

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Introduction To Several Complex Variables

Notations:

- $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

$\alpha \cdot = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ or $\mathbb{Z}^n$.

$|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$\alpha! = \alpha_1! \cdots \alpha_n!$,

$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathbb{C}$.

- $[n] = \{1, \ldots, n\}$
- $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, $a \in \mathbb{C}$, $r > 0$.
- Ball $B^n(\vec{a}, r) = \{z \in \mathbb{C}^n : |z - \vec{a}| < r\}$, $\vec{a} \in \mathbb{C}^n$, $r > 0$.
- Polydisc $D^n(\vec{a}, \vec{r}) = D(a_1, r_1) \times \cdots \times D(a_n, r_n)$,

for $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$,

$\vec{r} = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$.

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Let $\Omega \subseteq \mathbb{C}^n$ be open. Let $f : \Omega \to \mathbb{C}$.

Some possible definitions:

(A) f is holomorphic on Ω iff $f \in \mathcal{C}^1(\Omega)$ and $\dfrac{\partial f(z)}{\partial \bar{z}_j} = 0$

for all $z \in \Omega$ and $j \in [n]$.

(B) f is holomorphic on Ω iff for each $a \in \Omega$ and any polydisc $D^n(a, \vec{r}) \subset\subset \Omega$, we have

↳ compactly contained

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D(a_n, r_n)} \cdots \int_{\partial D(a_2, r_2)} \int_{\partial D(a_1, r_1)} \frac{f(\omega)}{\prod_{j=1}^{n} (\omega_j - a_j)} \, d\omega_1 \, d\omega_2 \cdots d\omega_n.$$

(Cauchy Integral Formula.)

(C) f is holomorphic on Ω iff for each $a \in \Omega$ and any polydisk $D^n(a, \vec{r}) \subset\subset \Omega$, f admits a power series expansion:

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (z - a)^\alpha \qquad \forall z \in D^n(a, \vec{r}),$$

where the RHS converges absolutely and uniformly on each compact subset of $D^n(a, \vec{r})$.

As in one-variable, we shall see (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C).

Let $f(z) = u(z) + iv(z)$ be $\mathcal{C}^1$-smooth.

(A) $\Rightarrow$ (B): Let $D^n(a, \vec{r}) \subset\subset \Omega$.

Consider the function

$$\xi \longmapsto f(a_1, \ldots, a_{n-1}, \xi)$$

defined on $D(a_n, r_n)$. This is holomorphic (in the usual sense) on a nbd of $\overline{D(a_n, r_n)} \subseteq \mathbb{C}$.

Hence,

$$f(a) = \frac{1}{2\pi i} \int_{\partial D(a_n, r_n)} \frac{f(a_1, \ldots, a_{n-1}, w_n)}{w_n - a_n} \, dw_n.$$

For each $w_n \in \partial D(a_n, r_n)$, consider the function

$$\xi \longmapsto f(a_1, \ldots, a_{n-2}, \xi, w_n).$$

The above is holo. on a nbd of $\overline{D(a_{n-1}, w_{n-1})}$.

Thus, we can now proceed inductively and get the result. $\boxtimes$

(B) $\Rightarrow$ (A): Suppose that f satisfies (B).

$$(\xi_1, \ldots, \xi_n) \longmapsto \frac{1}{\prod_{j=1}^{n} (\xi_j - a_j)}$$

is $\mathcal{C}^1$ on a nbd of $\partial D(a_1, r_1) \times \cdots \times \partial D(a_n, r_n)$.

To evaluate $\frac{\partial f}{\partial \bar{z}_j}$, we may differentiate under the integral sign to check that (A) holds. $\boxtimes$

(B) $\Rightarrow$ (C): Fix $a \in \Omega$ s.t. $D(a, \vec{r}) \subset\subset \Omega$.

Let $w = (w_1, \ldots, w_n)$ be s.t. $|w_j - a_j| = r_j$ $\forall j \in [n]$.

$z_j \in D(a_j, r_j)$.

$$\frac{1}{w_j - z_j} = \frac{1}{w_j - a_j + a_j - z_j} = \frac{1}{(w - a_j)\left(1 - \frac{z_j - a_j}{w_j - a_j}\right)}$$

$$= \frac{1}{(w_j - a_j)}\left(1 + \left(\frac{z_j - a_j}{w_j - a_j}\right) + \left(\frac{z_j - a_j}{w_j - a_j}\right)^2 + \cdots\right)$$

$$= \frac{1}{w_j - a_j} \sum_{m=0}^{\infty} \left(\frac{z_j - a_j}{w_j - a_j}\right)^m.$$

By (B), we know

$$f(z) = \frac{1}{\;} \int \cdots \int \frac{f(w)}{\;} \, dw_1 \cdots dw_n$$

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D(a_n, r_n)} \cdots \int_{\partial D(a_1, r_1)} \frac{f(w)}{\prod (w_j - z_j)} \, dw_1 \cdots dw_n$$

for $z \in D^n(a, \vec{r})$.

Now, plugging in the values of $\dfrac{1}{w_j - z_j}$ and switching Σ and $\int$ gives the result. $\square$

(C) $\Rightarrow$ (A): Differentiate term by term. $\square$

(D) f is holomorphic on Ω iff f is holomorphic in each variable separately. (Not even assuming f continuous a priori.)

That is: for each $a \in \mathbb{C}$ and $j \in [n]$, consider the subset

$$\{\xi \in \mathbb{C} : (z_1, \ldots, z_{j-1}, \xi, z_{j+1}, \ldots, z_n) \in \Omega\}$$

and demand that

$$\xi \longmapsto f(z_1, \ldots, z_{j-1}, \xi, z_{j+1}, \ldots, z_n) \text{ is}$$

hol. on the above subset.

Theorem (HARTOG'S LEMMA) (B) $\Leftrightarrow$ (D).

Theorem Fix $n \geq 2$.

Suppose $\Omega \subseteq \mathbb{C}^n$ open. Let $K \subseteq \Omega$ be compact, $\Omega \setminus K$ connected. Then, for $f \in \mathcal{O}(\Omega \setminus K)$, there exists $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \setminus K} = f$.

Corollary. Let $\Omega \subseteq \mathbb{C}^n$ be open, $n \geq 2$.
Then, there does not exist $f \in \mathcal{O}(\Omega)$ having a compact zero set.

The above do not hold for $n = 1$.

Proof of Corollary: Assume $f \not\equiv 0$. Suppose $K = Z(f)$ is compact.
Note that $\Omega \setminus K$ is nonempty connected set (will see this later). Then, $\exists\, F \in \mathcal{O}(\Omega)$ s.t. $F = 1/f$ on $\Omega \setminus K$.
But this means that F blows up as we approach K. $\rightarrow\leftarrow$

Exercises: ① Suppose $\Omega \subseteq \mathbb{C}^n$ and $f \in \mathcal{O}(\Omega)$.
Let $D(z_0, \vec{r}) \subset\subset \Omega$.
Then, f admits a power series expansion around z:

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (z - z_0)^\alpha.$$

Find an expression for c_α in terms of the derivatives of f at the point z_0.

$$c_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial z^\alpha} f(z_0).$$

② Find an integral representation for

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} f.$$

③ Cauchy's estimate's :

Let $z \in \Omega$ and $D(z, \vec{r}) \subset\subset \Omega$. Given $f \in \mathcal{O}(\Omega)$, show that

$$\left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} f(z) \right| \leq \frac{\overset{\text{indep of } z,\, \vec{r},\, f,\, \Omega.}{M_\alpha} \sup_{w \in D^n(z, \vec{r})} |f(w)|}{r^{|\alpha|}}.$$

Recommended Texts:

- Steven Krantz — Function Theory of SCV
- Hörmander
- Grauert and Fritzche
- Raghavan Narasimhan - Chicago lecture series

Lecture 21 (21-03-2022)

21 March 2022 14:05

Convergence domains of one-variable power series are always discs (or one point or $\mathbb{C}$).
But convergence domains of multivariable power series can be much more convoluted.

EXAMPLES. (i) $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z_1^n z_2^m$ converges absolutely on

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}.$$

(ii) $\sum_{n=0}^{\infty} z_1^n z_2^n$ converges in $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2| < 1\}$.

(Looking at the largest open sets.)

$n = 1$: (non degenerate) power series < entire functions; holomorphic functions on $D(a, r) \cong D(0,1)$.

For $n \geq 2$, the study of power series leads to function theory on different types of domains.

- For $n \geq 2$ variables, it is difficult to construct holomorphic functions with specified properties.

No analogues of Weierstrass, Mittag-Leffler, Riemann Mapping, Rouché's theorems.

Theorem 1. Let $\Omega \subseteq \mathbb{C}$ be an open set bounded by a simple closed curve. Then, $\exists f \in \mathcal{O}(\Omega)$ with the following property:
if $\hat{\Omega}$ is any open subset of $\mathbb{C}$ s.t. $\Omega \subseteq \hat{\Omega}$ and $\hat{\Omega} \cap \partial\Omega \neq \phi$, then there is no $F \in \mathcal{O}(\hat{\Omega})$ extending f.

Proof. Let $A \subseteq \Omega$ be a countable set s.t.

(i) A has no limit point in Ω,
(ii) A accumulates at every boundary point of Ω.
(Why can such an A be constructed? Exercise.)

Now, use Weierstrass' Theorem to get $f \in \mathcal{O}(\Omega)$ s.t. $Z(f) = A$.
Clearly, f cannot be extend to any open set intersecting $\partial\Omega$. ∎

Q. Does a similar result hold for holomorphic functions of SCV?
Ans: "Yes" for some domains and "No" for others.

Let $\phi \in \mathcal{O}(\mathbb{D})$ be nonextendable, as given by Theorem 1.
Let $\Omega := D(0,1) \times D(0,1)$
$= D^2((0,0), (1,1)) \subseteq \mathbb{C}^2$.

Define $f : \Omega \longrightarrow \mathbb{C}$
$(z_1, z_2) \longmapsto \phi(z_1)\,\phi(z_2)$.

There is no $\hat{\Omega}$ to which f can be continued analytically.

Similarly, take any $G \subseteq \mathbb{C}$ satisfying Thm 1. Then, $G \times \cdots \times G$ has above property.

Defn. A connected open set in $\mathbb{C}^n$ is called a domain of holomorphy if either of the following two (equivalent) properties hold:
(i) $\exists f \in \mathcal{O}(\Omega)$ that does not extend holomorphically across any part of the boundary.
(ii) for every $p \in \Omega$, $\exists f_p \in \mathcal{O}(\Omega)$ that does not extend holomorphically across the boundary at p.

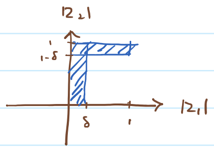
Clearly, (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) later.
Our earlier discussion gives us examples of domains of holomorphy. In fact, for $n=1$, Theorem 1 can be extended to ANY open connected subset of $\mathbb{C}$.

Precise meaning of "f extends holomorphically across the boundary at p": $\exists$ connected open nbd $U \ni p$, $\exists$ nonempty open $V \subseteq U \cap \Omega$, $\exists F \in \mathcal{O}(U)$ s.t. $F|_V = f|_V$.

Now, we give a non-example of a domain of holomorphy.

Theorem 2 (Hartog's): Fix $0<\delta<1$.

Let
$$\Omega = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < \delta, \ |z_2|<1\} \cup \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|<1, \ 1-\delta < |z_2| < 1\}.$$



Let $f \in \mathcal{O}(\Omega)$. Then, $\exists F \in \mathcal{O}(D^2((0,0),(1,1)))$ s.t. $F|_\Omega = f$.

Thus, Ω is NOT a domain of holomorphy.

Proof For each fixed $z_1 \in \mathbb{D}$,
$z_2 \longmapsto f(z_1,z_2)$ is holomorphic on the annulus $\{z_2 \in \mathbb{C} : 1-\delta < |z_2| < 1\}$.

Write a Laurent series expansion:
$$f(z_1,z_2) = \sum_{k=-\infty}^{\infty} a_k(z_1)\, z_2^k. \qquad (*)$$
$$a_k(z) = \frac{1}{2\pi i}\int_{|z_2|=r} \frac{f(z_1,z_2)}{z_2^{k+1}}\, dz_2$$
(for any $1-\delta<r<1$).

When $z_1 \in D(0,1)$ and $|z_2|=r$, $f(z_1,z_2)$ is jointly continuous on both variables and holo in the z_1 variable.

Hence each a_k is a holomorphic function on $\mathbb{D}$.
(Apply Morera's Theorem.)

Note that $|z_1|<\delta$, then $f(z_1,-)$ is holo on $\mathbb{D}$. Then, $(*)$ tells us that
$$a_k(z_1) = 0 \quad \forall\, k<0 \ \ \forall\, |z_1|<\delta.$$

But identity theorem (single variable) gives $a_k \equiv 0$
$\forall k < 0$.

Thus, the expansion

$$f(z_1, z_2) = \sum_{k=0}^{\infty} a_k(z_1) z_2^k$$

tells us that f can be extended to $D^2((0,0),(1,1))$,
as desired.

Only need to check that

$$F(z_1, z_2) = \sum_{k=0}^{\infty} a_k(z_1) z_2^k$$

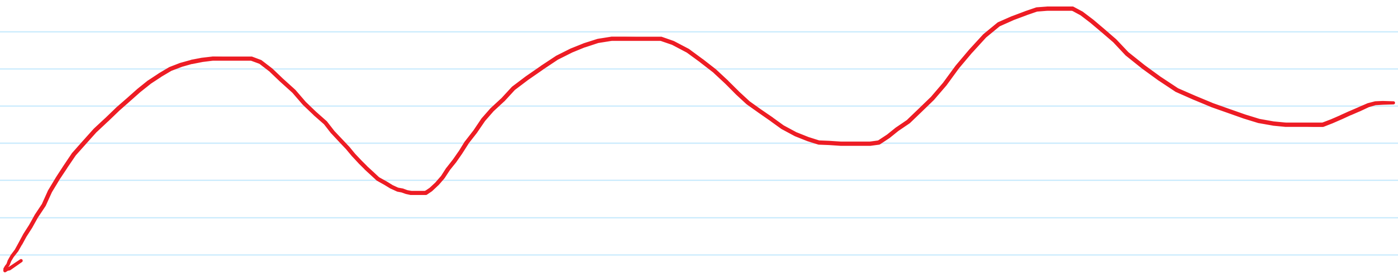
converges uni. on compact subsets of $D^2(\vec{0}, \vec{1})$.

Fix $0 < s < 1$. Then, $|f(z_1,z_2)| \le M$ for $|z_1| \le s$, $|z_2| = r$.

Then,

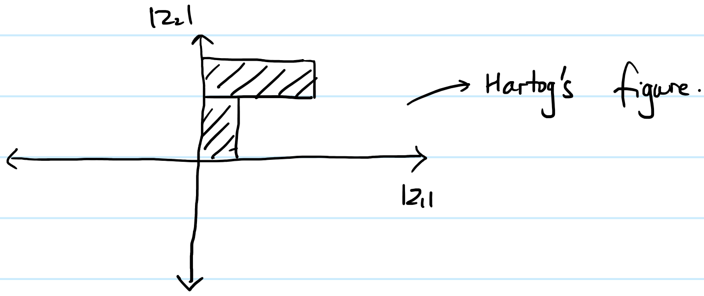
$$a_k(z_1) = \frac{1}{2\pi i} \int_{|z_2|=r} \frac{f(z_1,z_2)}{z_2^{k+1}} \, dz_2 .$$

$$\therefore |a_k(z_1)| \le \frac{M}{r^k} .$$ Conclude. ∎



Lecture 22 (24-03-2022)

24 March 2022 14:00



EXAMPLE : ① $\mathbb{B}^n = \{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}$.

We shall show that $\mathbb{B}^n$ is a domain of holomorphy using (ii) of def^n.

Fix $p \in \partial \mathbb{B}^n$. By applying a rotation, we may assume $p = (1, 0, \dots, 0)$.

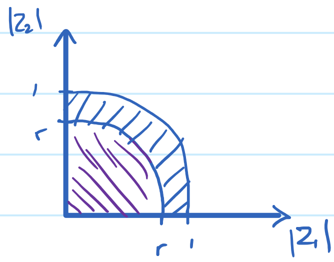
Then, $f(z_1, \dots, z_n) = \dfrac{1}{z_1 - 1}$ does the job. ☒

② (Not a domain of holomorphy.)

For $0 \le r < 1$, consider

$$\Omega = \{ z = (z_1, z_2) \in \mathbb{C}^2 : r^2 < |z_1|^2 + |z_2|^2 < 1\}.$$

Let $f \in \mathcal{O}(\Omega)$. Then, $\exists F \in \mathcal{O}(\mathbb{B}^2)$ s.t. $F|_{\Omega} = f$.



Proof. Let $\Omega_1 \subseteq \mathbb{C}$ be the projection of Ω onto first variable. (Ω_1 is open.)

For each fixed $z_1 \in \Omega_1$, as before we write

$$f(z_1, z_2) = \sum_{k=-\infty}^{\infty} a_k(z_1) z_2^k.$$

For each fixed $z_1 \in D(0,1)$, there is a nbd U of z_1 and a corresponding radius s s.t.

$$U \times \{ z_2 \in \mathbb{C} : |z_2| = s\}$$

is contained in a compact subset of Ω.

Thus, as last time, each $a_k(\cdot)$ admits a local integral representation

$$a_k(z_1) = \frac{1}{\quad} \int \frac{f(z_1, z_2)}{\quad} dz_2 \dots$$

representation

$$a_k(z_1) = \frac{1}{2\pi i} \int_{|z_2|=s} \frac{f(z_1, z_2)}{z_2^{k+1}} \, dz_2.$$

When $|z_1|$ is close 1, we have

$$a_k(z_1) = 0 \quad \forall k < 0$$

on an open subset of $D(0,1)$.

As before, this finishes the proof. ☒

Prop^n. Let $\Omega \subseteq \mathbb{C}^n$ be open.
If $(f_j)_j \in \mathcal{O}(\Omega)^{\mathbb{N}}$ converges uniformly on compact subsets to $f \in \mathbb{C}^\Omega$, then $f \in \mathcal{O}(\Omega)$.
Moreover,

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} f_j \text{ converges uniformly on compact subsets of } \Omega, \text{ to } \frac{\partial^{|\alpha|}}{\partial z^\alpha} f.$$

Proof. Use Cauchy Integral Formula. ☒

Theorem: Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $n \geq 2$.
Let $f \in \mathcal{O}(\Omega)$. Then f has <u>NO</u> isolated zeroes.

Proof. Suppose $p \in \Omega$ is an isolated zero of f.
Thus, $\exists\, r > 0$ s.t. $\mathbb{B}^n(p, r) \subseteq \Omega$ and

$$Z(f) \cap \mathbb{B}^n(p, r) = \{p\}.$$

Then, $g := 1/f$ is well defined and holomorphic on $\mathbb{B}^n(p,r) \setminus \{p\}$.

From our earlier example, $\exists\, G \in \mathcal{O}(\mathbb{B}^n(p,r))$ s.t.

$$G(z) = g(z) \quad \forall z \in \mathbb{B}^n(p,r) \setminus \{p\}.$$

Taking limit $z \to p$ gives a contradiction. ☒

(Contd.) Similarly, f cannot have isolated singularity. (Since holo. f^n on a a n punctured ball leads extension to full ball.)

Aside: $\Omega \subseteq \mathbb{C}^2$. Let $a \in \Omega$. Suppose $D^2(\vec{a}, \vec{r}) \subset\subset \Omega$.
Then, for all $z \in D^2(\vec{a}, \vec{r})$, we have

$$f(z) = \frac{1}{(2\pi i)^2} \int_{\partial D(a_1, r_1)} \int_{\partial D(a_2, r_2)} \frac{f(w_1, w_2)}{(w_1 - z_1)(w_2 - z_2)} \, dw_2 \, dw_1.$$

Thus, it is determined entirely by values on
$\partial D(a_1, r_1) \times \partial D(a_2, r_2)$.
Note that this is much smaller than the boundary of the polydisk. Indeed,

$$\partial D^2(\vec{a}, \vec{r}) = \partial D(a_1, r_1) \times \overline{D(a_2, r_2)} \cup \overline{D(a_1, r_1)} \times \partial D(a_2, r_2).$$

Theorem (Identity theorem)
Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Let $f, g \in \mathcal{O}(\Omega)$ be s.t.
$f \equiv g$ on a nonempty open subset of Ω.
Then, $f \equiv g$ on Ω.

Proof. WLOG, $g \equiv 0$. Let $U \overset{\neq \emptyset}{\subseteq} \Omega$ be s.t. $f|_U \equiv 0$.
Let

$$E = \left\{ z \in \Omega : \frac{\partial^{|\alpha|}}{\partial z^\alpha} f(z) = 0 \text{ for all } \alpha \in \mathbb{N}_0^n \right\}.$$

Clearly, $\emptyset \neq U \subseteq E$. Moreover, E is closed.
E is open since f is representable by power series. ☒

Theorem. (Open Mapping Theorem)

Any nonconstant holomorphic function $f : \Omega \to \mathbb{C}$ is open. ($\Omega \subseteq \mathbb{C}^n$)

Proof. Exercise. ☒

Theorem (Maximum Principle)

Theorem. (Maximum Principle)

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $f \in \mathcal{O}(\Omega)$.
Suppose that $|f|$ attains a local maximum at some $a \in \Omega$.
Then, f is constant.

Lecture 23 (28-03-2022)

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Proof. Suppose $|f|$ attains a local max at $a \in \Omega$. Let $\bar{D}^n(a, \vec{r}) \subset\subset \Omega$. Then,

$$z_1 \longmapsto f(z_1, a_2, \ldots, a_n)$$

is holo on $D(a_1, r_1)$ and attains a local max. Thus, this function is constant (max principle for one complex variable).

$$\therefore f(z_1, a_2, \ldots, a_n) = f(a_1, \ldots, a_n) \quad \forall z_1 \in D(a_1, r_1).$$

Now, fix z_1 and look at z_2, etc. to see that f is constant on $D^n(a, \vec{r})$. Then use identity theorem. ∎

Power Series.

For one variable:

Thm. If the power series $\sum_{n=0}^{\infty} c_n z^n$ converges for some $a \in \mathbb{C}^\times$, then the series converges absolutely on $D(0, |a|)$. Moreover, the convergence is uniform on compact subsets of $D(0, |a|)$.

As a consequence, if $D \subseteq \mathbb{C}$ is the region of convergence, then interior(D) is a union of open discs (centered at 0).

(Thus, is either an open disc centered at 0 or $\mathbb{C}$. (Assuming $D \neq \emptyset$.))

We also have

$$\text{radius of convergence} = \frac{1}{\limsup_n \sqrt[n]{|c_n|}}.$$

We wish to develop analogous results for SCV.

EXAMPLE: $\sum_{n=0}^{\infty} z_1^n z_2^{n!}$ converges absolutely on the following subsets of $\mathbb{C}^2$:

- $\{0\} \times \mathbb{C} \cup \mathbb{C} \times \{0\}$,

- $D(0,1) \times \bar{D}(0,1)$,
- $\bar{D}(0,1) \times D(0,1)$,
- $\left\{ \left(z_1, \frac{1}{z_1}\right) \in \mathbb{C}^2 : \ |z_1| > 1 \right\}$.

Thm. If the power series $\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha$ converges absolutely at $a \in \mathbb{C}^n$, then the series converges absolutely on the polydisc $D(0, |a_1|) \times \cdots \times D(0, |a_n|)$ with convergence uniform on compact subsets. (Assume that $a_1, \ldots, a_n$ are nonzero.)

Proof. By hypothesis, $|c_\alpha a^\alpha| \leq M$ for some $M > 0$ and all $\alpha \in \mathbb{N}_0^n$. Fix $0 < \lambda < 1$. For $z \in \bar{D}(0, \lambda|a_1|) \times \cdots \times \bar{D}(0, \lambda|a_n|)$, we have

$$|c_\alpha z^\alpha| \leq |c_\alpha \lambda^{|\alpha|} a^\alpha| \leq M \lambda^{|\alpha|}.$$

By comparison test, we need to look at

$$\sum_{\alpha \in \mathbb{N}_0^n} \lambda^{|\alpha|} = \sum_{\alpha_1 = 0}^{\infty} \cdots \sum_{\alpha_n = 0}^{\infty} \lambda^{\alpha_1 + \cdots + \alpha_n} = \frac{1}{(1-\lambda)^n}.$$

Thus, we are done. ∎

Corollary (1) The largest open set on which $\sum c_\alpha z^\alpha$ converges absolutely is a union of open polydiscs centered at origin.

(2) The above proof also shows that the convergence domain (defined below) is the interior of set $\mathcal{B}$ of points z for which the set $\{|c_\alpha z^\alpha|\}_\alpha$ is bounded.

$$\mathcal{B} = \left\{ z \in \mathbb{C}^n : \sup_\alpha |c_\alpha z^\alpha| < \infty \right\}.$$

$\mathcal{C} = \mathcal{B}^\circ$.

(or domain of convergence)

Defn. The convergence domain $\mathcal{C}$ of a multivariable power series $\sum c_\alpha z^\alpha$ is the largest open set on which the series converges absolutely.

Note that the convergence is uniform on compact subsets of the convergence domain.

$$\mathcal{C} = \bigcup_{r>0} \Big\{ z \in \mathbb{C}^n : \sum_\alpha |c_\alpha w^\alpha| < \infty \text{ for all } w \in D(z_1, r) \times \cdots \times D(z_n, r) \Big\}.$$

Properties of Domain of Convergence

① The domain of convergence is multicircular:
$(z_1, \dots, z_n) \in \mathcal{C} \Rightarrow (\lambda_1 z_1, \dots, \lambda_n z_n) \in \mathcal{C}$ whenever $|\lambda_j| = 1$ $\forall j$.

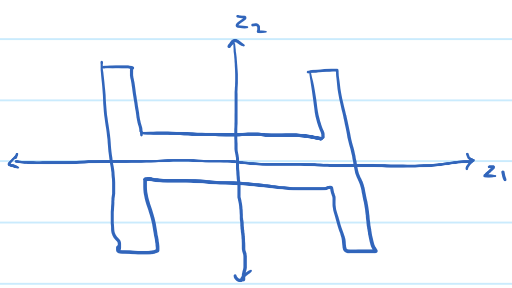
Defn. $\Omega \subseteq \mathbb{C}^n$ (open!) is said to be a Reinhardt domain / multicircular if
(i) $(z_1, \dots, z_n) \in \Omega \Rightarrow (\lambda_1 z_1, \dots, \lambda_n z_n) \in \Omega$ whenever $|\lambda_j| = 1$ $\forall j$,
(ii) $0 \in \Omega$.

EXAMPLES: (i) Union of polydiscs centered at 0 is a Reinhardt domain.

(ii) $\{ z \in \mathbb{C}^2 : |z_1| < 1+\varepsilon,\ |z_2| < \delta \} \cup \{ z \in \mathbb{C}^2 : 1-\varepsilon < |z_1| < 1+\varepsilon,\ |z_2| < 1+\delta \}$.

This is a Reinhardt domain.

Projection on $\mathbb{R}^2$:



② $(z_1, \dots, z_n) \in \mathcal{C} \Rightarrow (\lambda_1 z_1, \dots, \lambda_n z_n) \in \mathcal{C}$ even if $|\lambda_j| \le 1$ for all j.
The above H does not have this property!

Such a domain is called a complete Reinhardt domain.

③ The domain of convergence is logarithmically convex:

$\{(x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in \mathcal{C}^n\}$ is convex (in $\mathbb{R}^n$).

In fact, ① – ③ is sufficient for a domain $\Omega \subseteq \mathbb{C}$ to be a convergence domain of some power series

ASIDE: If $\sum |c_\alpha z^\alpha|$ and $\sum |c_\alpha w^\alpha|$ converges, then

$$\sum |c_\alpha| \, |z^\alpha|^t \, |w^\alpha|^{1-t} \quad \text{converges}$$

for $0 \le t \le 1$. (Hölder's inequality.)

Thus, if z, w belong to $\mathcal{C}$, then so does the point obtained by forming, in each coordinate, the geometric average of moduli with weights t and $(1-t)$, i.e.,

$$\left(|z_1|^t \, |w_1|^{1-t}, \dots, |z_n|^t \, |w_n|^{1-t}\right) \in \mathcal{C}.$$

This property of a Reinhart domain is called logarithmic convexity.

This proves ③.

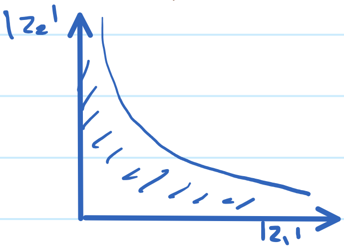
Lecture 24 (04-04-2022)

04 April 2022 14:02

\# Given $\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha$, its domain of convergence $\mathcal{C}$ is the largest open set in $\mathbb{C}^n$ where the series converges (absolutely).
Moreover, $\mathcal{C} = \mathring{\mathcal{B}}$ where
$$\mathcal{B} = \{ z \in \mathbb{C}^n : \sup_\alpha |c_\alpha z^\alpha| < \infty \}.$$

Abel's lemma: If $(z_1, \ldots, z_n) \in \mathcal{B}$, then the power series $\sum c_\alpha z^\alpha$ converges absolutely on $D^n(\vec{0}; |z_1|, \ldots, |z_n|)$ and uniformly on compact subsets. $\rightsquigarrow$ Consequently, $\mathcal{C}$ is a union of polydiscs.

EXAMPLE: (i) $\sum_{k=0}^{\infty} z_1^k z_2^k$ converges absolutely on $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2| < 1\}$.



Note that this is not a polydisk itself.
The domain of convergence is precisely the above.
It is the following union:
$$\bigcup_{r>0} D(z_1, r) \times D(z_2, 1/r).$$

(ii) $\sum_{k=0}^{\infty} z_1 z_2^k$ converges absolutely PRECISELY on
$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 1\}$$

However, $\mathcal{C} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < 1\}$
$= \mathbb{C} \times D(0,1)$.

(iii) $\sum_{\alpha \in \mathbb{N}_0^2} z_1^{\alpha_1 + 1} z_2^{\alpha_2}$.

This converges absolutely precisely! on $(\mathbb{C} \times \{0\}) \cup (D(0,1) \times D(0,1))$.
$\mathcal{C} = D^2(\vec{0}; 1, 1)$.

(iv) Find a power series whose domain of convergence is

Exercise:

(iv) Find a power series whose domain of convergence is

$$\mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C} : |z_1|^2 + |z_2|^2 < 1\}.$$

(v) Consider

$$\sum_{k=0}^{\infty} c_k z_1^k + \sum_{k=0}^{\infty} d_k z_2^k$$

Show that the domain of convergence of the above power series is a bidisc.

Recall: Logarithmic convexity: Consider the map

$$\mathbb{C}^n \ni z \xrightarrow{\lambda} (\log|z_1|, \ldots, \log|z_n|).$$

This is a mapping of the set $(\mathbb{C} \setminus \{0\})^n$ into $\mathbb{R}^n$.

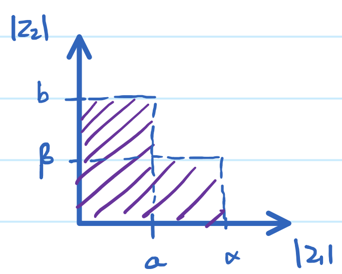
Defn. • The logarithmic image of a set $M \subseteq \mathbb{C}^n$ is $\lambda(M_0)$, where $M_0 := \{z \in M : z_1 \cdots z_n \neq 0\} = M \cap (\mathbb{C} \setminus \{0\})^n$.

By abuse of notation, the set is also denoted $\lambda(M)$.

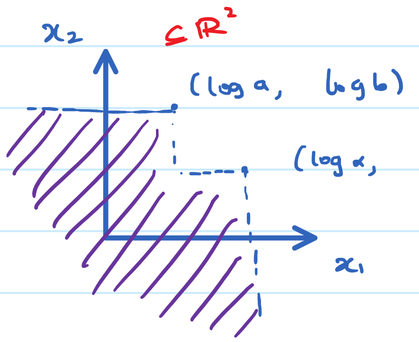
• M is said to logarithmically convex if $\lambda(M) \subseteq \mathbb{R}^n$ is convex.

EXAMPLE: $M = D^2(\vec{0}; a, b) \cup D^2(\vec{0}; \alpha, \beta)$

with $0 < a < \alpha$ and $0 < \beta < b$.



$\rightsquigarrow$



Evidently, M is NOT logarithmically convex.

Theorem A. Let $\Omega \subseteq \mathbb{C}^n$ be a complete Reinhardt domain (containing $\vec{0}$). Let $f \in \mathcal{O}(\Omega)$. Then, f admits a power series expansion on

Theorem A. Let $\Omega \subseteq \mathbb{C}^n$ be a complete Reinhardt domain (containing 0). Let $f \in \mathcal{O}(\Omega)$. Then, f admit a power series expansion on Ω.

(That is, $\exists (c_\alpha)_\alpha$ s.t. $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha \quad \forall z \in \Omega$.)

Thus, complete Reinhardt domains play the role of discs from single variable.

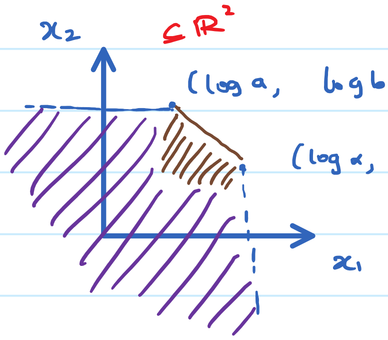
- $M = D^2(\vec{0}; a, b) \cup D^2(\vec{0}; \alpha, \beta)$ from earlier is a complete Reinhardt domain which is not logarithmically convex.

\# Let $\Omega \subseteq \mathbb{C}^n$ be a complete Reinhardt domain which is not logarithmically convex.
Let $f \in \mathcal{O}(\Omega)$. Then, by Theorem A, f can be represented in Ω by a power series (centered at $\vec{0}$).
Let $\mathcal{C}$ be the associated domain of convergence of the power series. Then, $\Omega \subseteq \mathcal{C}$. But $\mathcal{C}$ must be logarithmically convex. Thus, $\Omega \subsetneq \mathcal{C}$. Thus, Ω is not a domain of holomorphy. Moreover, $\mathcal{C}$ must contain the logarithmic convex hull of Ω, i.e., the smallest log. convex set containing Ω, i.e., the intersection of all log. convex sets containing Ω.

$$\hat{\Omega} := \{ z \in \mathbb{C}^n : |z_j| \le e^{x_j} \text{ for } (x_1, \dots, x_n) \in \widehat{\lambda(\Omega)} \}.$$

usual convex hull in $\mathbb{R}^n$

$\hat{\Omega}$ is the log. convex hull of Ω.

- 

Equation of the line:

$$\frac{y - \log b}{x - \log a} = \frac{\log \beta - \log b}{\log \alpha - \log a} =: c_0$$

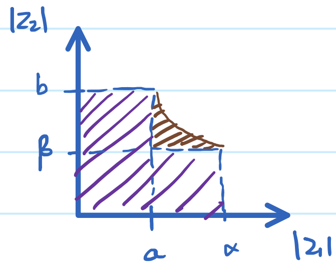
$$y - \log b = c_0 (x - \log a)$$

$$\Rightarrow \exp(y) = \exp(c_0 x - c_0 \log a + \log b)$$

$$\Rightarrow e^y = \frac{b}{a^{c_0}} e^{c_0 x}.$$

$$\Rightarrow \quad e^y = \frac{b}{a^6} e^{6x}.$$

Thus, $\hat{M}$ looks something like:



Prop^n. Every $f \in \mathcal{O}(M)$ can be extended holomorphically to

$$\hat{M} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \begin{array}{ll} |z_1| < \alpha, & \text{and} \\ |z_2| < b & \text{if } |z_1| < a, \\ |z_2| < \frac{b}{a^6} |z_1|^6 & \text{if } a \le |z_1| < \alpha \end{array} \right\}.$$

Theorem. Given a complete Reinhardt domain $\Omega \subseteq \mathbb{C}^n$, and $f \in \mathcal{O}(\Omega)$, f extends holomorphically to $\hat{\Omega}$.

FACT: Given a log. convex complete Rein. domain Ω, $\exists$ a power series having Ω as its domain of convergence.

NEXT CLASSES:

1. In one variable, we have that $\Omega \subsetneq \mathbb{C}$ simply-connected is biholo. to $D(0,1)$.
   However, $D(0,1) \times D(0,1)$ is not biholomorphic to $\mathbb{B}^2$.

2. Solutions of the $\bar{\partial}$-bar problem.
   - In one variable: $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow f$ holom.
   - Given f, find u s.t. $\frac{\partial u}{\partial \bar{z}} = f$. Can we find u?

Lecture 25 (07-04-2022)

07 April 2022 13:55

Recall Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{C}$ is simply connected, then Ω is biholomorphic to $D(0,1)$.

In $\mathbb{C}^2$, we have $D(0,1) \times D(0,1)$ and $\mathbb{B}^2$ are proper simply-connected

Theorem. $D(0,1) \times D(0,1)$ and $\mathbb{B}^2$ are not biholomorphic.

(Note that they are homeomorphic and in fact, diffeomorphic)

The above is called Poincaré's theorem. His original proof had a flaw since he assumed that a biholo. would extend continuously to the boundary. It was first correctly proved by H. Cartan (1936).

Proof. Let $f : D(0,1) \times D(0,1) \to \mathbb{B}^2$ be a biholomorphism.

($f : \Omega \to \mathbb{C}^n$ is holo. if each component is holo.)

Fix $e^{i\theta} \in \partial D(0,1)$.

Let $(a_j)_j \in D(0,1)^{\mathbb{N}}$ be s.t

$$a_j \to e^{i\theta} \text{ as } j \to \infty.$$

Consider the map $g_j : \mathbb{D} \to \mathbb{B}^2$ by

$$\zeta \mapsto f(a_j, \zeta).$$

(Let $\mathbb{D} := D(0,1)$.)

Note $(g_j)_j$ is uniformly bdd as $\mathbb{B}^2$ is bounded.

By Montel's theorem, some subsequence of $(g_j)_j$ converges uniformly on compact subsets of $\mathbb{D}$; let $g : \mathbb{D} \to \overline{\mathbb{B}^2}$ be the limit mapping.

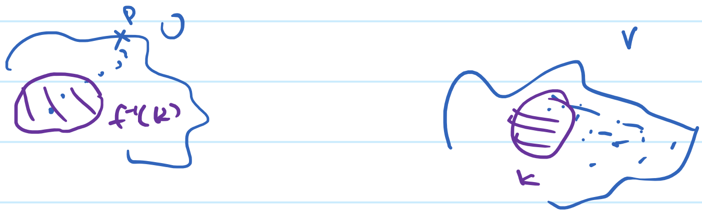
Claim 1: $g(\mathbb{D}) \subseteq \partial \mathbb{B}^2$.

FACT: Let U and V be bounded domains in $\mathbb{C}^n$ and $F : U \to V$ be a biholomorphism.

Then, for every compact $K \subseteq V$, $F^{-1}(K) \subseteq U$ is compact.

Then, if $(p_j)_j \in U^{\mathbb{N}}$ converges to $p \in \partial U$, then the set of limit points of $\{F(p_j) : j \in \mathbb{N}\}$ must lie in ∂V.

the set of limit points of $\{F(p_j) : j \in \mathbb{N}\}$ must lie in ∂V.



Proof of Claim 1: Fix $\zeta \in \mathbb{D}$.

$$(a_j, \zeta) \longrightarrow (e^{i\theta}, \zeta) \quad \text{as } j \to \infty.$$
$$\in \partial(\mathbb{D} \times \mathbb{D})$$

Thus, using the fact, we see that the set of limit points of $\{g_j(\zeta) : j \geq 1\}$ must lie in $\partial \mathbb{B}^2$.

As $g(\zeta) = \lim_k g_{j_k}(\zeta)$, we are done. ∎

Claim 2: g is a constant map.

Proof. For each $z \in \mathbb{D}$, we have
$$|g_1(z)|^2 + |g_2(z)|^2 = 1$$
$(g =: (g_1, g_2))$.

After composing with a unitary transformation, assume $g(0) = (1, 0)$.
But $|g_1(z)| \leq 1 \quad \forall z \in \mathbb{D}$.
By MMT, $g_1 \equiv 1$.
Consequently $|g_2| \equiv 0$ and hence, $g_2 \equiv 0$. ∎

Hence, $g' \equiv 0$. Hence,
$$\frac{\partial f_1}{\partial z_2}(a_j, \zeta) \longrightarrow 0$$
and
$$\frac{\partial f_2}{\partial z_2}(a_j, \zeta) \longrightarrow 0$$
as $j \to \infty$ along some subsequence.

Hence $w \longmapsto \dfrac{\partial f_1}{\partial z_2}(\zeta, w)$ and $w \longmapsto \dfrac{\partial f_1}{\partial z_2}(\zeta, w)$

extend continuously to $\overline{\mathbb{D}}$ and are 0 on $\partial\mathbb{D}$.

Thus, by MMT, they are 0 identically on $\mathbb{D}\times\mathbb{D}$.

But then, f_1 is constant. $\rightarrow\leftarrow$ ☒

Defn. A nonconstant holomorphic mapping $\varphi : \mathbb{D} \longrightarrow \mathbb{C}^n$ (with $\mathbb{D} = D(0,1)$) is called an *analytic disc*. Often, one refers to $\varphi(\mathbb{D})$ as an analytic disc.

These somewhat play the role of line segments.

Fix $p_0 \in \partial\mathbb{D}$. Define $\varphi_1 : \mathbb{D} \longrightarrow \mathbb{C}^2$ by
$$z \longmapsto (z, p_0).$$
$$\varphi_1(\mathbb{D}) = \mathbb{D}\times\{p_0\} \subseteq \partial(\mathbb{D}\times\mathbb{D}).$$
Similarly, can define φ_2 by $z \mapsto (p_0, z)$.
Then, $\varphi_2(\mathbb{D}) \subseteq \partial(\mathbb{D}\times\mathbb{D})$.

Through every boundary point of $\mathbb{D}\times\mathbb{D}$ (except those in $\partial\mathbb{D}\times\partial\mathbb{D}$), there is an analytic disc lying inside the boundary.

OTOH, $\partial\mathbb{B}^2$ does not contain any analytic disc (the argument from the proof earlier will give this).

This was essentially the heart of the above proof.

Solutions of the $\bar\partial$-problem on the plane.

$$\bar\partial := \frac{\partial}{\partial\bar z}.$$ (one variable.)

Q. If $\bar\partial g = 0$, then does there exist f s.t. $\bar\partial f = g$?

Lecture 26 (11-04-2022)

11 April 2022 13:59

Generalised Cauchy's Integral formulae

Let $\Omega \subseteq \mathbb{C}$ be a bounded domain.
Assume that $\partial\Omega$ is a simple closed curve which is piecewise smooth. Let $f \in \mathcal{C}^1(\Omega)$ be complex-valued.
Then, for any $z \in \Omega$,

This shows (taking $f(z)=\bar{z}$) that $\iint_\Omega \frac{1}{w-z}\, dA(w) < \infty$

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)}{w-z}\,dw - \frac{1}{\pi}\iint_\Omega \frac{1}{w-z}\frac{\partial f}{\partial \bar{w}}\, dA(w).$$

Proof Fix $\varepsilon > 0$ s.t. $\bar{D}(z,\varepsilon) \subseteq \Omega$.

Then,

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\left[\int_{\partial\Omega} - \int_{\partial D(z,\varepsilon)}\right]\frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{\partial D(z,\varepsilon)}\frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\left[\int_{\partial\Omega} - \int_{\partial D(z,\varepsilon)}\right]\frac{f(w)}{w-z}\,dw + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$\Omega_\varepsilon := \Omega \setminus D(z,\varepsilon)$

$F(w) := \frac{f(w)}{w-z} = U(w) + iV(w)$

$$= \frac{1}{2\pi i}\int_{\partial\Omega_\varepsilon} F(w)\,dw + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi i}\int_{\partial\Omega_\varepsilon}\big(U(w)+iV(w)\big)(ds + i\,dt) + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi i}\int [U(w) + iV(w)]\,ds \quad \ldots \quad + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi i} \int_{\partial \Omega_\varepsilon} [U(\omega) + iV(\omega)]\,ds + [iU(\omega) - V(\omega)]\,dt \quad + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

Green's theorem to Ω_ε

$$= \frac{1}{2\pi i} \iint_{\Omega_\varepsilon} \left(i\frac{\partial U}{\partial s} - \frac{\partial V}{\partial s} \right) - \Big(\qquad + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$$\left[\frac{\partial F}{\partial \bar\omega} = \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{i\partial}{\partial t}\right)(U+iV) \right.$$

$$= \frac{1}{2}\left[\left(\frac{\partial U}{\partial s} - \frac{\partial V}{\partial t}\right) + i\left(\frac{\partial U}{\partial t} + \frac{\partial V}{\partial s}\right)\right]\Bigg]$$

$$= \frac{1}{\pi} \iint_{\Omega_\varepsilon} \frac{\partial F}{\partial \bar\omega}\, dA(\omega) + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$$= \frac{1}{\pi} \iint_{\Omega_\varepsilon} \frac{1}{\omega - z} \frac{\partial f}{\partial \bar\omega}\, dA(\omega) + \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta .$$

As $\varepsilon \to 0$, the right integral goes to $f(z)$ and first converges to the (improper) integral

$$\frac{1}{\pi} \iint_{\Omega} \frac{1}{\omega - z} \frac{\partial f}{\partial \bar\omega}\, dA(\omega). \qquad \blacksquare$$

<u>Remark</u>. ① If f is holo, then $\frac{\partial f}{\partial \bar\omega} = 0$ and we get the usual CIF.

② An analogue of this exists for higher variables as well.

<u>$\bar\partial$ - problem in $\mathbb{C}$.</u>

compactly supported

compactly supported

If $\phi \in \mathcal{C}^1_c(\mathbb{C})$, then a solution of $\boxed{\dfrac{\partial u}{\partial \bar{z}} = \phi}$ is given by

$$u(z) := -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\phi(\omega)}{\omega - z} \, dA(\omega).$$

inhomogeneous $\bar{\partial}$-problem / CR equation

Remark. $u + f$ is also a solution for any $f \in \mathcal{O}(\mathbb{C})$.

Proof. Define $u(z) := -\dfrac{1}{\pi} \displaystyle\iint_{\mathbb{C}} \frac{\phi(\xi + z)}{\xi} \, dA(\xi)$.

Note that "the domain of integration is actually compact" since ϕ is compactly supported. Let $R > 0$ be s.t. $\operatorname{supp} \phi \subseteq D(0, R)$.

Check: u is $\mathcal{C}^1$-smooth.

Applying Cauchy integral formula to ϕ gives

$$\phi(z) = \cancel{\frac{1}{2\pi i} \int_{\partial D(0,R)} \frac{\phi(\omega)}{\omega - z} \, d\omega}^{\,0} - \frac{1}{\pi} \iint_{D(0,R)} \frac{1}{\omega - z} \cdot \frac{\partial \phi}{\partial \bar{\omega}}(\omega) \, dA(\omega)$$

$$= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{1}{\omega - z} \frac{\partial \phi}{\partial \bar{\omega}}(\omega) \, dA(\omega)$$

$$= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{1}{\xi} \frac{\partial \phi}{\partial \bar{\xi}}(\xi + z) \, dA(\xi)$$

Consider $f(z, \xi) = \phi(z + \xi)$

$$= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{1}{\xi} \frac{\partial \phi}{\partial \bar{z}}(\xi + z) \, dA(\xi)$$

$\mathbb{C}$

$$= \frac{\partial}{\partial \bar{z}} u(z).$$

$\bar{\partial}$-problem in $\mathbb{C}^n$:

Given $\phi_1, \dots, \phi_n : \mathbb{C}^n \longrightarrow \mathbb{C}$.
Solve

$$\frac{\partial u}{\partial \bar{z}_j} = \phi_j \quad \text{for} \quad j = 1, \dots, n.$$

Suppose $\phi_j \in \mathcal{C}^1(\mathbb{C}^n)$ for all j and we have a sufficiently nice solution u. Then,

$$\frac{\partial}{\partial \bar{z}_k}\left(\frac{\partial}{\partial \bar{z}_j} u\right) = \frac{\partial}{\partial \bar{z}_k} \phi_j$$

$$\|$$

$$\frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial}{\partial \bar{z}_k} u\right) = \frac{\partial}{\partial \bar{z}_j} \phi_k.$$

Thus, if we want a sufficiently nice solution, we will need

$$\boxed{\frac{\partial}{\partial \bar{z}_j} \phi_k = \frac{\partial}{\partial \bar{z}_k} \phi_j \quad \forall j, k.}$$ — (CC) compatibility condition

$\bar{\partial}$-problem. Fix n and k.
Given $\phi_1, \dots, \phi_n \in \mathcal{C}^k_c(\mathbb{C}^n)$ such that

$$\frac{\partial}{\partial \bar{z}_\ell} \phi_j = \frac{\partial}{\partial \bar{z}_j} \phi_\ell \quad \text{for all } j, \ell.$$

Then, $\dfrac{\partial u}{\partial \bar{z}_j} = \phi_j \quad \forall j$ admits a solution of class $\mathcal{C}^k$.

Proof. Consider

$$u(z_1, \dots, z_n) := -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\phi_1(\omega, z_2, \dots, z_n)}{\omega - z_1} \, dA(\omega).$$

Then, $\dfrac{\partial u}{\partial \bar{z}_1} = \phi_1$. Check: $u \in \mathcal{C}^k$.

For $j > 1$, note

$$\frac{\partial u}{\partial \bar{z}_j} = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial}{\partial \bar{z}_j} \frac{\phi_1(\omega, z_2, \dots, z_n)}{\omega - z_1} \, dA(\omega)$$

(cc)

$$= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{1}{\omega - z_1} \frac{\partial}{\partial \bar{z}_j} \phi_1(\omega, z_2, \dots, z_n) \, dA(\omega)$$

Compact support

$$= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{1}{\omega - z_1} \frac{\partial}{\partial \bar{z}_1} \phi_j(\omega, z_2, \dots, z_n) \, dA(\omega)$$

$$= -\frac{1}{\pi} \iint_{D(0,R)} \frac{1}{\omega - z_1} \frac{\partial \phi_j}{\partial \bar{z}_1}(\omega, z_2, \dots, z_n) \, dA(\omega)$$

Cauchy on $f(\cdot) = \phi_j(\cdot, z_2, \dots, z_n)$

$$= \frac{\partial \phi_j}{\partial z_1}(z_1, \dots, z_n) + \cancel{\int_{\partial D(0,R)} (\ )} \to 0$$

$$= \frac{\partial \phi_j}{\partial z_1}. \qquad \blacksquare$$

Theorem. (Hartog's Phenomenon) $n \geq 2$.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, and $K \subseteq \Omega$ be compact such that $\Omega \setminus K$ is connected. Then, any $f \in \mathcal{O}(\Omega \setminus K)$ extends to $F \in \mathcal{O}(\Omega)$.

Proof. Fix open V s.t. $K \subset V \subset \bar{V} \subset \Omega$.

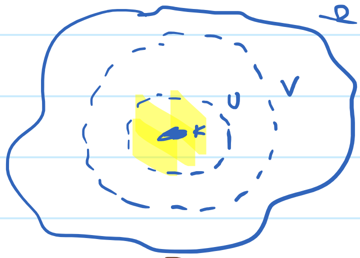
Let U be an open subset s.t. $K \subset U \subset \bar{U} \subset V$.

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ be s.t.

Let $\chi \in C_c^\infty(\mathbb{C}^n)$ be s.t.

$\chi(z) = 1$ for $z \in U$ and

$\operatorname{supp}(\chi) \subseteq V$.



Let $\tilde{f}(z) := (1-\chi(z)) f(z)$.

Then, $\tilde{f} \equiv 0$ on U and $\tilde{f}$ is holo on $\Omega \setminus \bar{V}$ (agrees with f).

Also, $\tilde{f} \in C^\infty(\Omega)$.

Thus, we have extended f smoothly.

Define $\phi_j \in C_c^\infty(\mathbb{C}^n)$ by

$$\phi_j(z) := \begin{cases} \dfrac{\partial \tilde{f}}{\partial \bar{z}_j}(z) & \text{for } z \in \Omega, \\ 0 & \text{else.} \end{cases}$$

($\tilde{f}$ is holo on $\bar{V}^c$ and thus $\dfrac{\partial \tilde{f}}{\partial \bar{z}_j} = 0$ on $\Omega \setminus \bar{V}$.

This gives smoothness on $\partial\Omega$)

Note ϕ_j vanishes on $\Omega^c \cup (\Omega \setminus \bar{V}) \cup U$

$\therefore \operatorname{supp} \phi_j \subseteq V \setminus \bar{U}$.

can choose U and V so that $V \setminus \bar{U}$ is bounded

Also, $\dfrac{\partial}{\partial \bar{z}_k} \phi_j = \dfrac{\partial}{\partial \bar{z}_j} \phi_k$ since $\tilde{f}$ is C^∞.

Thus, we get a solution $u \in C^\infty$ to the $\bar{\partial}$-bar problem with $\dfrac{\partial u}{\partial \bar{z}_j} = \phi_j$.

Define $F := \tilde{f} - u$.

Then, $\dfrac{\partial F}{\partial \bar{z}_j} = 0 \quad \forall j$.

Thus, F is holomorphic on Ω.

Now, need to check

$$F\big|_{\Omega \setminus K} = f.$$

Since $\Omega \setminus K$ is connected, it suffices to show that $\exists$ nonempty open set $A \subseteq \Omega \setminus K$ s.t.

$$F|_A = f.$$

Taking $A := \Omega \setminus \overline{V}$ does the job since $\tilde{f}|_A = f|_A$ & $u|_A \equiv 0$. ∎