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MA 5106

Introduction to Fourier Analysis

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Lecture 1 (06-01-2021)

Preliminaries

- Rectangle in $\mathbb{R}^d$: $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$. } closed
- Cube in $\mathbb{R}^d$: $Q = [a_1, b_1] \times \cdots \times [a_d, b_d]$ where $b_1 - a_1 = \cdots = b_d - a_d$.
- Volume of R: $|R| = \prod_{i=1}^{r} (b_i - a_i)$
- Exterior measure of $E \subseteq \mathbb{R}^d$: (Exterior measure)

$$m_*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j, \ Q_j \text{ are cubes} \right\}$$

Observations:

(1) Any singleton has exterior measure 0.

(2) Exterior measure of any (closed/open) rectangles is equal to its volume.

(3) $m_*(\mathbb{R}^d) = \infty$.

(4) $m_*(\text{Cantor set}) = 0$.

Properties:

(1) $E \subseteq F \Rightarrow m_*(E) \leq m_*(F)$

(2) $m_*\left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} m_*(E_j)$ (equality needn't hold even if disjoint)

Measurable set

(Measurable set, Lebesgue measurable set)

Def^n A set $E \subseteq \mathbb{R}^d$ is called (Lebesgue) measurable if for every $\epsilon > 0$, $\exists$ an open set O with $O \supseteq E$ s.t. $m_*(O \setminus E) = 0$.

If $E \subseteq \mathbb{R}^d$ is measurable, then (Lebesgue) measure of E is denoted by $m(E)$ and defined as

$$m(E) = m_*(E).$$

(Lebesgue measure)

Examples of measurable sets

(1) Any open set is measurable.

(2) E s.t. $m_*(E) = 0 \Rightarrow E$ is measurable

(3) Countable union of measurable sets are measurable.

(4) Complement of a meas. set is meas.

(5) Any closed set. Any countable intersection of meas. sets.

Thm. (1) Let $E_1, E_2, \dots$ be disjoint measurable sets. Then,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

(2) $m(E+h) = m(E)$ $\forall$ measurable $E \subseteq \mathbb{R}^d$, $\forall h \in \mathbb{R}^d$

$$\left(E+h := \{ y+h \mid y \in E \}\right)$$

" $E+h$ is also measurable" is implicit. Similar for next ones.

(3) $m(cE) = c^d\, m(E)$, $c > 0$

$(cE := \{cy \mid y \in E\})$

(4) $m(-E) = m(E)$. $(-E := \{-y \mid y \in E\})$

Def^n. $f : \mathbb{R}^d \to [-\infty, \infty]$ is said to be measurable if for any $a > 0$,

$$f^{-1}([-\infty, a)) \subseteq \mathbb{R}^d$$

is measurable. (Measurable function)

Examples

(1) Any continuous function is measurable.

(2) If f is measurable and $p : \mathbb{R} \to \mathbb{R}$ is measurable, then $p \circ f$ is measurable.

(3) If $\{f_n\}_n$ is a sequence of measurable functions, then the functions

$$\sup f_n, \quad \inf f_n, \quad \limsup f_n, \quad \liminf f_n.$$

are all measurable.

(4) Limit (Pointwise) of a sequence of measurable functions is measurable.

(5) If f, g are measurable, then so are $f \pm g$, $f \cdot g$.

Ex. Characteristic function. Let $E \subseteq \mathbb{R}^d$.

Ex. Characteristic function. Let $E \subseteq \mathbb{R}^d$.

Define

$$\chi_E(x) := \begin{cases} 1 & ; \text{ if } x \in E \\ 0 & ; \text{ if } x \notin E \end{cases}$$

Then, χ_E is a measurable $f^n \iff E$ is measurable.

Note $f^{-1}([-\infty, a)) = \begin{cases} E^c & ; \quad 0 < a \leq 1 \\ \mathbb{R}^d & : \quad 1 < a \end{cases}$

Thus, χ_E is a meas. $f^n \iff E^c$ is meas $\iff E$ is.

Def^n. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be simple if (Simple function)

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}.$$ ($a_k \in \mathbb{R}$ constants) ($\exists\, m(E_k) < \infty$)

Thm. Let f be a non-negative measurable function on $\mathbb{R}^d$. Then, $\exists$ an increasing seq. of non-neg simple functions $\{\varphi_k\}_k$ s.t.

$$\lim_{k \to \infty} \varphi_k = f \quad \text{pointwise.}$$

$$\left(\phi_k(x) \leq \phi_{k+1}(x) \quad \forall x \right)$$

Integration

(Integration)

(1) Let f be a simple function.

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}, \quad (E_k \text{ measurable } \& \, m(E_k) < \infty)$$

$$\int_{\mathbb{R}^d} f := \sum_{k=1}^{N} a_k \, m(E_k).$$

(Has to be checked that this is independent of $((a_k), (E_k), N)$.)

Example. $\displaystyle\int_{\mathbb{R}} \chi_{[0,1]} = 1.$

(2) Let f be a bounded measurable function with

$$m(\operatorname{supp} f) < \infty \quad \text{where}$$

$\operatorname{supp} f = \{x : f(x) \neq 0\}$. $\longrightarrow$ will be measurable since f is

Then, $\exists \{\varphi_n\}_n$ of simple functions s.t. $\varphi_n \leq M$ and

$$\varphi_n \longrightarrow f \quad \text{a.e.}$$

(i.e., the set of points x for which $\varphi_n(x) \not\to f(x)$ is of measure zero.)

and $\operatorname{supp} \varphi_n \subseteq \operatorname{supp} f$.

Then,
$$\int_{\mathbb{R}^d} f := \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n .$$

this defined by (1)

(Again, independent of this $\{\varphi_n\}_n$.)

(3) Assume $f \geq 0$.

$$\int_{\mathbb{R}^d} f := \sup \left\{ \int_{\mathbb{R}^n} g : 0 \leq g \leq f,\ g \text{ is bounded, measurable with } m(\operatorname{supp} g) < \infty \right\}$$

$$\int_E f := \int_{\mathbb{R}^d} f \cdot \chi_E \qquad (E \subseteq \mathbb{R}^d \text{ is measurable})$$

this is defined earlier

note $f\cdot\chi_E$ is measurable and ≥ 0.

Def^n. $f \geq 0$ is integrable if $\int_{\mathbb{R}^d} f < \infty$. (Integrable)

Now, if $f:\mathbb{R}^d \to \mathbb{R}$ is any function, we can write

$$f(x) = f^+(x) - f^-(x)$$

$$f^+(x) := \max\{f(x), 0\}, \qquad f^-(x) := \max\{-f(x), 0\}.$$

Note that $f^+, f^- \geq 0$.

Def^n. f is integrable if $\int_{\mathbb{R}^d} |f| < \infty$ and

$$\int_{\mathbb{R}^d} f := \int_{\mathbb{R}^d} f^+ - \int_{\mathbb{R}^d} f^-$$

Can be extended to $f:\mathbb{R}^d \to \mathbb{C}$ componentwise

EXAMPLE. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1 \; ; & x \in \mathbb{Q} \cap [0,1] \\ 0 \; ; & x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

Then, f is not Riemann integrable on $[0,1]$.

However,

$$\int_{[0,1]} f = \int_{\mathbb{Q}\cap[0,1]} f + \int_{[0,1]\setminus\mathbb{Q}} f$$

$= 0$

Thm. Let f be Riemann integrable on $[a, b]$. Then, f is measurable and both the integrals (Riemann & Lebesgue) coincide.

Lecture 2 (08-01-2021)

08 January 2021 09:24

Recap. $f \geq 0$

1. $f = \sum a_i \chi_{E_i}$, then $\int_{\mathbb{R}^d} f := \sum a_i \, m(E_i)$

2. $m(\text{supp } f) < \infty$, then $\exists \{\varphi_n\}$ simple s.t. $\varphi_n \to f$ a.e.
(f bounded)

$$\int_{\mathbb{R}^d} f := \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n$$

3. $\int_{\mathbb{R}^d} f \, dx := \sup \left\{ \int_{\mathbb{R}^d} g \; : \; \underset{g \text{ bounded}}{0 \leq g \leq f} \;\&\; m(\text{supp } g) < \infty \right\}$

Properties.

1. $\int (af + bg) = a\int f + b\int g \quad \forall a, b \in \mathbb{C}$

2. $E \cap F = \emptyset$ and E, F measurable, then

$$\int_{E \cup F} f = \int_E f + \int_F f$$

3. $\left| \int f \right| \leq \int |f|$

4. $f \geq 0$ and $\int_{\mathbb{R}^d} f = 0 \Rightarrow f = 0$ a.e.

If $f = 0$ a.e., then $\int f = 0$.

5. $\int_{\mathbb{R}^d} |f| < \infty \Rightarrow |f| < \infty$ a.e.

Suppose $f_n \to f$ pointwise.

$$\lim_{n\to\infty} \int f_n \overset{?}{=} \int f$$

(We know above is true if uniform conv. & f_n Riemann integ.)

Thm. (Monotone Convergence Theorem) (Monotone Convergence Theorem)

Let $\{f_n\}_n$ be a sequence of non-negative measurable functions, converging pointwise to f and $f_n \le f_{n+1}$.
Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.$$

Thm. (Dominated Convergence Theorem) (Dominated Convergence Theorem) (DCT)

Let $\{f_n\}_n$ be a sequence of measurable functions such that

$$f_n \longrightarrow f \quad \text{a.e.}$$

Assume further that $\exists$ an integrable function g s.t

$$|f_n(x)| \le g(x).$$

Then,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.$$

Thm.
- $\displaystyle\int_{\mathbb{R}^d} f(x-h)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$
- $\displaystyle\int_{\mathbb{R}^d} f(-x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$
- $\displaystyle\int_{\mathbb{R}^d} f(cx)\,dx = \frac{1}{c^d}\int_{\mathbb{R}^d} f(x)\,dx \quad ; \quad c>0$

Thm. (Fubini's Theorem)

(Fubini's Theorem)

(a) Let f be a non-negative measurable function on

$$\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^{d_1+d_2}.$$

Then,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy \right) dx$$

$$\| \qquad\qquad\qquad //$$

$$\int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,d(x,y)$$

(b) Let f be integrable on $\mathbb{R}^{d_1+d_2}$ (i.e., $\int_{\mathbb{R}^{d_1+d_2}} |f| < \infty$).

Then,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy \right) dx$$

$$\| \qquad\qquad\qquad //$$

$$\int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,d(x,y)$$

To use (b), we need to check if $\int_{\mathbb{R}^{d_1+d_2}} |f| < \infty$. However,

since $|f| \geq 0$, we can compute the above integral using (a).

Def^n. $\cdot\ \mathcal{L}^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} \mid f \text{ is meas. and } \int_{\mathbb{R}^d} |f|^p < \infty \right\}$.

$1 \leq p < \infty$

(L^p spaces, Lp spaces)

· Normed linear space: $(X, \|\cdot\|)$ (Normed linear space)

(NLS)

$X \to$ vector space over $\mathbb{R}$ or $\mathbb{C}$

and $\|\cdot\| : X \to [0, \infty)$ s.t.

(i) $\|x\| = 0 \iff x = 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in \mathbb{F}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

Any nls is a metric space with $d_X(x, y) = \|x - y\|$.

· $\mathcal{L}^p(\mathbb{R}^d)$ is a vector space, easy to see.
(linear space)

Moreover, defining $\|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \right)^{1/p}$

· $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

· $\|\alpha f\|_p = |\alpha| \, \|f\|_p$

· $\|f\|_p = 0 \Rightarrow \int |f|^p = 0 \to |f|^p = 0$ a.e.

- $\|f\|_p = 0 \Rightarrow \int_{\mathbb{R}^d} |f|^p = 0 \rightarrow |f|^p = 0$ a.e.

$$\Downarrow$$

$f = 0$ a.e

not necessarily 0

In fact, $\mathcal{L}^p(\Omega)$ is actually classes of functions where $f \sim g \Leftrightarrow f = g$ a.e.

Then, $\mathcal{L}^p(\mathbb{R}^d)$ is an NLS.

- $\mathcal{L}^\infty(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C}$ meas. f^n which are bounded a.e.$\}$

$$\|f\|_\infty := \text{ess sup } |f|$$

$\therefore |f(x)| \le \|f\|_\infty$ a.e.

- An NLS $(X, \|\cdot\|)$ is called a Banach space if X is complete as a metric space.

- $\mathcal{L}^p(\mathbb{R}^d)$ is a Banach space for $1 \le p \le \infty$.

Thm. (Hölder's Theorem) (Hölder's Theorem, Holder's Theorem)

$\|fg\|_1 \le \|f\|_p \|g\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

$\left(p \ge 1, \quad p = \infty \Rightarrow \frac{1}{p} = 0 \right)$

Result. (Using Hahn-Banach Theorem)

$$\|f\|_p = \sup \left\{ \left| \int_{\mathbb{R}^d} fg \right| : \|g\|_q < 1 \right\}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Convolution (Convolution)

Defn. Let f, g be integrable functions on $\mathbb{R}^d$ ($f, g \in \mathcal{L}^1(\mathbb{R}^d)$).
Then, convolution of f and g is defined as

$$(f*g)(x) := \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy.$$

Q. Does RHS exist? Yes, for almost every $x \in \mathbb{R}^d$.

Proof. Note

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(y)|\,|g(x-y)|\,dy\right)dx$$

Fubini (a)

$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(y)|\,|g(x-y)|\,dx\right)dy$$

$$= \int_{\mathbb{R}^d} |f(y)|\left(\int_{\mathbb{R}^d} |g(x-y)|\,dx\right)dy$$

translation

$$= \int_{\mathbb{R}^d} |f(y)|\Big(\underbrace{\int_{\mathbb{R}^d} |g(x)|\,dx}_{\text{constant}}\Big)dy$$

$$= \left[\int_{\mathbb{R}^d} |f|\right]\left[\int_{\mathbb{R}^d} |g|\right] < \infty \quad \text{since } f, g \in \mathcal{L}^1$$

$$\Rightarrow x \mapsto \int_{\mathbb{R}^d} |f(x)|\,|g(x-y)|\,dy \quad \text{is finite a.e.} \quad \blacksquare$$

Thus, $(f*g)(x)$ exists for almost every x.

Also, $\|f*g\|_1 \le \|f\|_1 \|g\|_1$, by the above.

Thm. Let $p \in [1, \infty)$. If $f \in \mathcal{L}^p(\mathbb{R}^d)$ and $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$f*g \in \mathcal{L}^p(\mathbb{R}^d)$$

and

$$\|f*g\| \le \|f\|_p \|g\|_1.$$

- $$(f*g)(x) = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy \qquad y \mapsto x-z$$

$$= \int_{\mathbb{R}^d} f(x-z)\, g(z)\, dz$$

$$= (g*f)(x)$$

- Convolution can be defined on any measurable group $(G, \cdot)$.

  $f, g \in \mathcal{L}^1(G)$, then

$$(f*g)(x) = \int_G f(y)\, g(xy^{-1})\, dy$$

Can define convolution on $S^1 = \mathbb{T} \cong [0, 2\pi]/\sim$.

- $f*(g*h) = (f*g)*h$

- $(f+g)*h = f*h + g*h$

Now, $\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$.

Thm. Let $C_c(\mathbb{R}^d)$ be the set of continuous func $f : \mathbb{R}^d \to \mathbb{C}$ with compact support. (C_c(\Bbb R^d))

Obs. $C_c(\mathbb{R}^d) \subseteq \mathcal{L}^p(\mathbb{R}^d)$. (well, technically $\mathcal{L}^p$ is equiv. classes but note that if cont. f^n are equal a.e., then they are equal.)

Proof. $f \in C_c(\mathbb{R}^d)$

$$\Rightarrow \int_{\mathbb{R}^d} \|f\|^p = \int_{\operatorname{supp} f} \|f\|^p \leq \|f\|_\infty \int_{\operatorname{supp}} 1 = \|f\|_\infty \, m(\operatorname{supp} f) < \infty.$$

Thm. 1. $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$. Here, $1 \leq p \lneq \infty$.

2. $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$. Here, $1 \leq p \lneq \infty$.
↳ inf. differentiable

Def^n. (Approximate identity in $\mathcal{L}^1(\mathbb{R}^d)$) (Approximate identity)
A sequence $\{k_n\}_n$ in $\mathcal{L}^1(\mathbb{R}^d)$ is called approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$ if

(1) $k_n \geq 0 \quad \forall n \in \mathbb{N}$

(2) $\int_{\mathbb{R}^d} k_n = 1 \quad \forall n \in \mathbb{N}$

(3) For any $\delta > 0$,

$$\lim_{n \to \infty} \int_{\|x\| > \delta} k_n = 0.$$

Thm. Let $\{k_n\}_n$ be an approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$. Let $f \in \mathcal{L}^1(\mathbb{R}^d)$.

Then,

$$f * K_n \longrightarrow f \quad \text{in} \quad \mathcal{L}^1 \quad \text{as} \quad n \to \infty.$$

That is,

$$\lim_{n \to \infty} \| f * K_n - f \| = 0.$$

Remark. $(\mathcal{L}^1(\mathbb{R}^d), *)$ does not have an identity.

That is, $\nexists\, g \in \mathcal{L}^1(\mathbb{R}^d)\ \forall f \in \mathcal{L}^1(\mathbb{R}^d)\ (f * g = f.)$

We prove the theorem in the next class. Before that, we have the following lemma.

(Lemma) Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then, the map $y \mapsto T_y f$ is a continuous function on $\mathbb{R}^d \to \mathcal{L}^1(\mathbb{R}^d)$, where

$$T_y f(x) := f(x - y).$$

That is, for any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|y_1 - y_2\|_2 < \delta \Rightarrow \|T_{y_1} f - T_{y_2} f\|_1 < \epsilon$.

Proof. Let $g \in C_c(\mathbb{R}^d)$. Then

$$\|T_{y_1} g - T_{y_2} g\|_1 = \int_{\mathbb{R}^d} |T_{y_1} g(x) - T_{y_2} g(x)|\, dx$$

$$= \int_{\mathbb{R}^d} |g(x - y_1) - g(x - y_2)|\, dx$$

$$= \int_{\mathbb{R}^d} |g(x + y_2 - y_1) - g(x)|\, dx$$

let $K = \operatorname{supp} g$ (compact)

$$= \int_{K \cup (K + y_2 - y_1)} |g(x + y_2 - y_1) - g(x)|\, dx$$

$\because g$ is continuous, can choose $\delta > 0$ s.t.

$\|y_1 - y_2\| < \delta \Rightarrow |g(x + y_2 - y_1) - g(x)| < \epsilon / m(K \cup (K + y_2 - y_1))$

$$\|y_1 - y_2\| < \delta \Rightarrow |g(x + y_2 - y_1) - g(x)| < \epsilon / m(K \cup (K + y_2 - y_1))$$

$$< \epsilon \quad \text{if} \quad \|y_1 - y_2\| < \delta.$$

Now, use the fact that $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^1(\mathbb{R}^d)$.
Note the following to conclude:

$$\|T_{y_1} f - T_{y_2} f\|_1 \leq \|T_{y_1} f - T_{y_1} g\|_1 + \|T_{y_1} g - T_{y_2} g\|_1 + \|T_{y_2} g - T_{y_2} f\|_1$$

$$= \|T_{y_1}(f-g)\| + \|T_{y_1} g - T_{y_2} g\| + \|T_{y_2}(g-f)\|$$

$$= \|f-g\| + \|T_{y_1} g - T_{y_2} g\| + \|f-g\|$$

can be made $< \epsilon$.

Lecture 3 (13-01-2021)

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Recall:

A sequence $\{k_n\}_n$ in $\mathcal{L}^1(\mathbb{R}^d)$ is called approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$ if

(1) $k_n \geq 0 \quad \forall n \in \mathbb{N}$

(2) $\int_{\mathbb{R}^d} k_n = 1 \quad \forall n \in \mathbb{N}$

(3) For any $\delta > 0$,

$$\lim_{n \to \infty} \int_{\|x\| > \delta} k_n = 0.$$

Thm. Let $\{k_n\}_n$ be an approximate identity for $\mathcal{L}^1(\mathbb{R}^d)$. Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then,

$$f * k_n \to f \quad \text{in } \mathcal{L}^1 \quad \text{as } n \to \infty.$$

That is,

$$\lim_{n \to \infty} \| f * k_n - f \| = 0.$$

To prove that, we had seen the following lemma.

Lemma. Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then, the map $y \mapsto T_y f$ is a continuous function $\mathbb{R}^d \to \mathcal{L}^1(\mathbb{R}^d)$, where

$$T_y f(x) := f(x - y).$$

Remark. In fact, if $f \in \mathcal{L}^p(\mathbb{R}^d)$, $1 < p < \infty$, then

$y \mapsto T_y f$ is continuous as a f^n $\mathbb{R}^d \to \mathcal{L}^p(\mathbb{R}^d)$.

Proof (of Thm). $f \in \mathcal{L}^1(\mathbb{R}^d)$ and $\{K_n\}_n$ is approximate identity in $\mathcal{L}^1$.

$$\|f * K_n - f\|_1 = \int_{\mathbb{R}^d} |(f * K_n)(x) - f(x)|\,dx$$

$$= \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} f(x-y) K_n(y)\,dy - \underbrace{\int_{\mathbb{R}^d} f(x) K_n(y)\,dy}_{(\because \int_{\mathbb{R}^d} K_n = 1)} \Big|\,dx$$

$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y) - f(x)|\, K_n(y)\,dy\,dx \qquad (K_n \geq 0)$$

interchanging, we'll finally show it's finite (Fubini)

$$= \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} |T_y f(x) - f(x)|\,dx \right\} K_n(y)\,dy$$

$$= \int_{\mathbb{R}^d} \|T_y f - f\|_1 \; K_n(y)\,dy$$

$\because$ $y \mapsto T_y f$ is continuous, for every $\epsilon > 0$, $\exists \delta > 0$ s.t. $\|T_y f - f\|_1 < \epsilon$ if $\|y\| < \delta$.

$$= \int_{|y|<\delta} \|T_y f - f\|_1 K_n(y)\,dy + \int_{|y|\geq\delta} \|T_y f - f\|_1 K_n(y)\,dy$$

$\|T_y f\| = \|f\|$

$$\leq \frac{\epsilon}{2} \int_{|y|<\delta} K_n(y)\,dy + 2\|f\|_1 \int_{|y|\geq\delta} K_n(y)\,dy$$

By def^n of approx. id.

$\int_{\mathbb{R}^d} K_n = 1$

By our approx. id.

$\int_{|y|\geq\delta} k_n \to 0$

$\int_{\mathbb{R}^d} k_n = 1$

$$\leq \frac{\varepsilon}{2}\cdot 1 + 2\|f\|_1 \frac{\varepsilon}{4\|f\|_1} \qquad \forall n \geq N$$

$$= \varepsilon$$

$\therefore\ f * k_n \to f$ in $\mathcal{L}^1$.

Remark. We shall see later that $(\mathcal{L}^1(\mathbb{R}^d), *)$ does not have an identity but it has (many) approximate identities.

Construction of an Approximate Identity

Let $\varphi \geq 0$ be an integrable function.

(That is, $\int_{\mathbb{R}^d} \varphi < \infty$. That is $\varphi \in \mathcal{L}^1(\mathbb{R}^d)$)

Suppose $\int_{\mathbb{R}^d} \varphi = 1$.

For each $n \in \mathbb{N}$, let $\varphi_n(t) := n^d \varphi(nt)$, $t \in \mathbb{R}^d$.

(nt is the usual scalar multiplication in the space $\mathbb{R}^d$)

Then, $\{\varphi_n\}_n$ is an approximate identity in $\mathcal{L}^1(\mathbb{R}^d)$.

Check. (i) $\varphi_n \geq 0 \quad \forall n \in \mathbb{N}$ is obvious.

(ii) $\int_{\mathbb{R}^d} \varphi_n = n^d \int_{\mathbb{R}^d} \varphi(nt)\, dt$

$$= n^d \int_{\mathbb{R}^d} \varphi(y) \frac{dy}{n^d} = \int_{\mathbb{R}^d} \varphi(y)\, dy = 1.$$

(iii) Fix $\delta > 0$.

$$\int_{|t| \geq \delta} \varphi_n = n^d \int_{|t| \geq \delta} \varphi(nt)\, dt$$

$$= \int_{|y| \geq n\delta} \varphi(y)\, dy \xrightarrow{n \to \infty} 0$$

EXAMPLE. For $d = 1$, $\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ works.

Obser. $f \in \mathcal{L}^1,\ g \in \mathcal{L}^p \Rightarrow f * g \in \mathcal{L}^p$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Q. Let $g \in \mathcal{L}^p$ and $\{k_n\}_n$ an approx. id. in $\mathcal{L}^1$. $(1 \leq p < \infty)$

Is true that

$$\|g * k_n - g\|_p \to 0\ ?$$

(That is, $\lim_{n \to \infty} g * k_n = g$?)

Yes! Will show later.

Thm. (Minkowski's integral inequality)

Given two measure spaces (X, μ) and (Y, ν) with σ-finite

Given two measure spaces (X, μ) and (Y, ν) with σ-finite measures :

$$\left(\int_X \left| \int_Y f(x, y)\, d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |f(x,y)|^p \, d\mu(x) \right)^{1/p} d\nu(y)$$

Thm. Let $\{K_n\}_n$ be an approximate identity in $\mathcal{L}^1(\mathbb{R}^d)$.
Then, $f \in \mathcal{L}^p(\mathbb{R}^d)$. Then,

$$\| f * K_n - f \|_p \longrightarrow 0.$$

Proof.

$$\| f * K_n - f \|_p = \left(\int_{\mathbb{R}^d} |(f * K_n)(x) - f(x)|^p \, dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [(f(x-y) - f(x)) K_n(y)] dy \right|^p dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [(T_y f(x) - f(x)) K_n(y)] dy \right|^p dx \right)^{1/p}$$

Minkowski

$$\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |T_y f(x) - f(x)|^p \cdot |K_n(y)|^p \, dx \right)^{1/p} dy$$

$$= \int_{\mathbb{R}^d} K_n(y) \left\{ \int_{\mathbb{R}^d} |T_y f(x) - f(x)|^p dx \right\}^{1/p} dy$$

$$= \int_{\mathbb{R}^d} \| T_y f - f \|_p \, K_n(y) \, dy$$

Now we are in the same position as earlier. ☐

($y \mapsto T_y f$ is continuous as $\mathbb{R}^d \to \mathcal{L}^p(\mathbb{R}^d)$ since $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$, for $1 \leq p < \infty$.)

Thm. Let f be a continuous function with compact support. (Then, $f \in \mathcal{L}^\infty$.) Let $\{K_n\}_n$ be an approximate identity in $\mathcal{L}^1$.
Then,

$$\| f * K_n - f \|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

(Need to prove that for $f \in C_c(\mathbb{R}^d)$, $y \mapsto T_y f$ is cts as $\mathbb{R}^d \to \mathcal{L}^\infty(\mathbb{R}^d)$)

- Let $\mathbb{T} := S^1 = \mathbb{R}/\mathbb{Z}$. $\qquad x \mapsto e^{2\pi i x}$.

  We will do analysis on $\mathbb{T}$. (Torus)
  Then, we identify f with a function on $\mathbb{R}$ which is periodic with period 1.

- $\mathcal{L}^p(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} \text{ measurable} \mid \int_0^1 |f|^p < \infty \}$.

$$\|f\|_{\mathcal{L}^p(\mathbb{T})} := \left(\int_0^1 |f|^p \right)^{1/p}.$$

for emphasis. Sometimes we will simply write $\|f\|_p$.

- $\mathcal{L}^p(\mathbb{T}) \subseteq \mathcal{L}^q(\mathbb{T})$ if $q \leq p$
  (note the reversal)

  $(\mathcal{L}^1(\mathbb{T}) \supseteq \mathcal{L}^2(\mathbb{T}) \supseteq \dots)$

  ↳ true for any finite measure space.
  Not true in $\mathbb{R}$.

Proof. We show for 1 and 2 using Cauchy Schwarz.

$$\int_0^1 |f(t)|\, dt = \int_0^1 |f(t)| \cdot 1\, dt$$

$$\leq \left(\int_0^1 |f(t)|^2 \right)^{1/2} \left(\int_0^1 1\, dt \right)^{1/2}$$

$$= \|f\|_2 < \infty$$

Thus $f \in \mathcal{L}^1$ and $\|f\|_1 \leq \|f\|_2$.

· In general, we use Hölder.

$$\int_0^1 |f|^q = \int_0^1 \left(|f(t)|^p \right)^{q/p} \cdot 1\, dt$$

$\alpha = p/q$

$$\int fg \leq \left(\int f^\alpha\right)^{1/\alpha} \left(\int g^\beta\right)^{1/\beta}$$

(Hölder) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\leq \left(\int_0^1 \left(\left(|f(t)|^p \right)^{q/p} \right)^{p/q} \right)^{q/p} \cdot 1$$

$$= \left(\int_0^1 |f|^p \right)^{q/p}$$

$$\Rightarrow \left(\int_0^1 |f|^q \right)^{1/q} \leq \left(\int_0^1 |f|^p \right)^{1/p}$$

$\therefore f \in \mathcal{L}^q$ $\blacksquare$

$$\mathcal{L}^1(\mathbb{T}) \supseteq \mathcal{L}^p(\mathbb{T}) \quad \forall\, p \geq 1.$$

Thus, we only do Fourier Analysis for $\mathcal{L}^1$, which takes care of all.

Lecture 4 (15-01-2021)

15 January 2021 09:30

Let f be a function on an interval (in $\mathbb{R}$).

Q. When can f be expressed in terms of sin and cos?
That is,

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \sin(2\pi k x) + b_k \cos(2\pi k x)$$

for some complex sequences $\{a_k\}_{k=-\infty}^{\infty}$ and $\{b_k\}_{k=-\infty}^{\infty}$.

Obs.: the RHS is periodic with period 1. Thus, the LHS must also have period 1. (not necessarily fundamental)

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$$

for some complex sequence $\{c_k\}_{k=-\infty}^{\infty}$

Assume convergence unif. & abs.

Then,

$$c_k = \int_0^1 f(x) e^{-2\pi i k x} \, dx.$$

$$\mathcal{L}^1(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} \text{ measurable} : \int_0^1 |f| < \infty \}.$$

Def Let $f \in \mathcal{L}^1(\mathbb{T})$. Then the Fourier coefficient of f is defined by

$$\hat{f}(k) := \int_0^1 f(x) e^{-2\pi i k x} \, dx \qquad \text{for } k \in \mathbb{Z}.$$

Also, the series $\sum_{k=-\infty}^{\infty} \hat{f}(x) e^{2\pi i k x}$ is called the

Fourier series of f.

Q. When and in which sense does the Fourier series of f converge to f?

Thm. There exists a continuous function f such that its Fourier series diverges at a point.

Thm. (Dini's theorem)

Let $f \in \mathcal{L}^1(\mathbb{T})$. If, for some x_0, $\exists \delta > 0$ s.t.

$$\int_{|t|<\delta} \left| \frac{f(x_0+t) - f(x_0)}{t} \right| dt < \infty.$$

Then,

$$f(x_0) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x_0} = \lim_{N\to\infty} \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x_0}$$

Note that it is only at that point.

Lemma (Riemann Lebesgue Lemma)

Let $f \in \mathcal{L}^1(\mathbb{T})$. Then,

$$\lim_{|k|\to\infty} \hat{f}(k) = 0.$$

Proof. Let g be a continuous function on $\mathbb{T}$.

Then,

$$\hat{g}(k) = \int_0^1 g(x) e^{-2\pi i k x} dx \quad \text{— (1)}$$

$$y = x + \tfrac{1}{2k} \qquad = -\int_0^1 g(x) e^{-2\pi i k \left(x + \frac{1}{2k}\right)} dx$$

$y = x + \frac{1}{2k}$

$$= -\int_0^1 g(x)\, e^{-2\pi i k\left(x + \frac{1}{2k}\right)}\, dx$$

$$= -\int_{1/2k}^{1+1/2k} g\left(x - \frac{1}{2k}\right) e^{-2\pi i k x}\, dx$$

periodic continuous function.

$$= -\int_0^1 g\left(x - \frac{1}{2k}\right) e^{-2\pi i k x}\, dx \quad \text{—(2)}$$

Using (1) and (2), we get

$$\hat{g}(k) = \frac{1}{2}\int_0^1 \left(g(x) - g\left(x - \frac{1}{2k}\right)\right) e^{-2\pi i k x}\, dx$$

$$\Rightarrow |\hat{g}(k)| \leq \frac{1}{2}\int_0^1 \left|g(x) - g\left(x - \frac{1}{2k}\right)\right| dx$$

$$\Rightarrow |\hat{g}(k)| \leq \frac{1}{2} \max_{x \in [0,1]} \left|g(x) - g\left(x - \frac{1}{2k}\right)\right|$$

But g is uniformly continuous. Thus, by choosing $|k|$ large, the RHS can be arbitrarily small.

Thus, the lemma is true for continuous functions.
Now, we use the fact that $\mathcal{C}(\mathbb{T})$ is dense in $\mathcal{L}^1(\mathbb{T})$.

($\mathbb{T}$ is compact, not need to write $\mathcal{C}_c$)

Take $f \in \mathcal{L}^1(\mathbb{T})$. Let $\epsilon > 0$ be given.
Fix $g \in \mathcal{C}(\mathbb{T})$ s.t.

$$\|g - f\|_1 < \epsilon/2.$$

(g exists, by density)

Also, fix N s.t.

$$\|\hat{g}(k)\| < \epsilon/2 \qquad \forall\ |k| \geq N$$

(Note $|\hat{f}(k)| = \left| \int_0^1 f(x)\, e^{-2\pi i k x}\, dx \right| \leq \int_0^1 |f| = \|f\|_1$.)

Thus, for $|k| \geq N$, we have

$$\begin{aligned} |\hat{f}(k)| &\leq |\hat{f}(k) - \hat{g}(k)| + |\hat{g}(k)| \\ &\leq |\widehat{f-g}(k)| + |\hat{g}(k)| \\ &\leq \|f - g\|_1 + |\hat{g}(k)| \\ &< \epsilon. \end{aligned}$$

∎

We shall use the above to prove Dini's Theorem.

(Dirichlet kernel)

Def^n. The N-th partial sum of the Fourier series of f is defined as

$$\begin{aligned} S_N f(x) &:= \sum_{k=-N}^{N} \hat{f}(k)\, e^{2\pi i k x} \\ &= \sum_{k=-N}^{N} \left\{ \int_0^1 f(t)\, e^{-2\pi i k t}\, dt \right\} e^{2\pi i k x} \\ &= \int_0^1 f(t) \cdot \sum_{k=-N}^{N} e^{2\pi i k (x-t)}\, dt \end{aligned}$$

Let $D_N(t) := \sum_{k=-N}^{N} e^{2\pi i k t}$ be the Dirichlet kernel.

Then,

$$S_N f(x) = \int_0^1 f(t)\, D_N(x - t)\, dt.$$

$$= (f * D_N)(x).$$

Some properties

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t}$$

$$= e^{-2\pi i N t} + e^{-2\pi i (N-1)t} + \cdots + e^{2\pi i N t}$$

$$= e^{-2\pi i N t}\left(\frac{1-\left(e^{2\pi i t}\right)^{2N+1}}{1-e^{2\pi i t}}\right)$$ (assuming $e^{2\pi i t} \neq 1$)

$$= \frac{e^{-2\pi i N t} - e^{(4N+2)\pi i t - (2N)\pi i t}}{1-e^{2\pi i t}}$$

$$= \frac{e^{-2\pi i N t} - e^{2\pi i t(N+1)}}{1-e^{2\pi i t}} \cdot \frac{e^{-i\pi t}}{e^{-i\pi t}}$$

$$= \frac{e^{-\pi i (2N+1)t} - e^{\pi i (2N+1)t}}{e^{-\pi i t} - e^{\pi i t}}$$

$$= \frac{\sin (2N+1)\pi t}{\sin \pi t}$$

(Note that if $e^{2\pi i t} = 1$, then the sum is $2N+1$, works as limit.)

From the summation def^n, we get

$$\int_0^1 D_N(t)\,dt = 1$$

$$\therefore \quad S_N f(x) = \int_0^1 f(t)\, D_N(x-t)\,dt,$$

(Note that D_n is also periodic with period 1)

$$D_N(t) = \frac{\sin (2N+1)\pi t}{\sin \pi t}$$

$\frac{1}{2}$

is ... with ... period

$$\Rightarrow S_N f(x) = \int_{-1/2}^{1/2} f(t)\, D_N(x-t)\, dt$$

$$= \int_{x-1/2}^{x+1/2} f(x-t)\, D_N(t)\, dt$$

$$= \int_{-1/2}^{1/2} f(x-t)\, D_N(t)\, dt$$

$$= \int_{-1/2}^{1/2} f(x-t) \frac{\sin (2N+1)\pi t}{\sin \pi t}\, dt$$

Proof of Dini's Theorem. Let $f \in L^1(\mathbb{T})$ and fix x s.t. $\exists \delta > 0$

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty.$$

We wish to show $S_N f(x) \xrightarrow[N\to\infty]{} f(x)$. (Note x is fixed.)

Note

$$S_N f(x) - f(x) = \int_{-1/2}^{1/2} f(x-t)\, D_N(t)\, dt - f(x)$$

$$= \int_{-1/2}^{1/2} f(x-t)\, D_N(t)\, dt - \int_{-1/2}^{1/2} f(x)\, D_N(t)\, dt$$

$$= \int_{-1/2}^{1/2} \left[f(x-t) - f(x) \right] D_N(t)\, dt$$

Choose $\delta_1 < \delta$ s.t.

$$\sin \pi t \asymp \pi t \quad \text{if } |t| < \delta_1$$

That is, $\exists\, C_1, C_2 > 0$ s.t.

$$C_1 \pi t \le \sin \pi t \le C_2 \pi t \quad \forall |t| < \delta_1$$

$$\Rightarrow S_N f(x) - f(x) = \underbrace{\int_{|t|<\delta_1} [f(x-t) - f(x)] \frac{\sin(2N+1)\pi t}{\sin \pi t}\, dt}_{=: I_1}$$

$$+ \underbrace{\int_{\frac{1}{2} \ge |t| \ge \delta_1} [\quad] \frac{\sin(\quad)}{\sin(\quad)}\, dt}_{=: I_2}$$

$$I_1 = \int_{-1/2}^{1/2} \underbrace{\frac{f(x-t) - f(x)}{\sin \pi t} \cdot \chi_{|t|<\delta_1}(t)}_{\text{want to show this is in } \mathcal{L}^1} \sin(2N+1)\pi t\, dt$$

Then Riemann-Lebesgue shows that $I_1 \longrightarrow 0$.

(Since I_1 is the imaginary part of an appropriate Fourier co-eff.)

Let
$$g_1(t) := \frac{f(x-t) - f(x)}{\sin \pi t} \cdot \chi_{|t|<\delta_1}(t)$$

By hypothesis in the Thm,

$$\int_0^1 |g_1| \le \int_{-1/2}^{1/2} \left| \frac{f(x-t) - f(x)}{\sin \pi t} \right| \chi_{(\,)}\, dt$$

$$\le C_1 \int_{-1/2}^{1/2} \left| \frac{f(x-t) - f(x)}{t} \right| \chi\, dt < \infty$$

For I_2 : define
$$g_2(t) = \frac{f(x-t) - f(x)}{\sin \pi t} \chi_{t \ge |\delta_1|}(t)$$

Again, we show $g_2 \in \mathcal{L}_1$. Note $|\sin \pi t| \ge \sin \pi \delta_1$

Again, we show $g_2 \in \mathcal{L}_1$. Note $|\sin \pi t| \geq \sin \pi\delta_1$ for $\frac{1}{2} \geq |t| \geq \delta_1$.

$$\int_{-1/2}^{1/2} |g_2(t)|\,dt \leq \frac{1}{|\sin \pi \delta_1|} \int_{-1/2}^{1/2} |f(x-t) - f(x)|\, \chi_{|t| \geq \delta_1}(t)\,dt$$

$$< \infty \quad \text{since } f \text{ is in } \mathcal{L}^1$$

Thus, $g_2 \in \mathcal{L}^1(\mathbb{T})$ and $I_2 \to 0$.

Thus, we have $S_N f(x) - f(x) = I_1 + I_2 \to 0$

Hence, $\lim\limits_{N \to \infty} S_N f(x) = f(x)$. ∎

(For the fixed x which satisfies Dini's condition)

Cor. Suppose $f \in \mathcal{L}^1(\mathbb{T})$ and f satisfies Lipschitz condition in a neighbourhood of some $x \in \mathbb{T}$. That is,

$$|f(x+t) - f(x)| \leq c \cdot |t|^a$$

for some $c, a > 0$ and $\forall\, |t| < \delta$ for some $\delta > 0$.

Then, $\lim\limits_{N \to \infty} S_N f(x) = f(x)$.

Proof. Just need to check Dini's condition. Let δ be as in the Lips. condition.

Then,
$$\int_{|t|<\delta} \left|\frac{f(x+t) - f(x)}{t}\right| dt \leq c \int_{|t|<\delta} \frac{1}{|t|^{1-a}} < \infty$$

$(a > 0 \Rightarrow 1-a < 1)$

Now we will prove the first theorem. First we recall the following from Functional Analysis.

- Let X, Y be normed linear spaces.

Let $T : X \to Y$ be a linear map.
T is bounded (linear map) if $\exists c > 0$ s.t.

$$\|Tx\|_Y \leq c \cdot \|x\|_X .$$

Since T is linear, we have
boundedness $\Leftrightarrow$ T is continuous

$$\underbrace{\|T\|_{op}}_{\text{norm on all (bounded) linear maps from } X \text{ to } Y} = \inf \{ c > 0 : \|Tx\|_Y \leq c \|x\|_X \ \forall x \}$$

$$= \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

(1) any linear $T : \mathbb{R}^n \to \mathbb{R}^n$ is always bounded.

(2) derivative $D : \mathcal{C}'([0,1]) \to \mathcal{C}([0,1])$

$f \mapsto f'$

in sup norm

is a linear map which is not bounded.

→ Proof mainly uses Baire Category Theorem

(Uniform Boundedness principle)

X — Banach space
Y — normed linear space

Let $\{T_\alpha\}_{\alpha \in \Lambda}$ be a collection of linear maps
$T_\alpha : X \to Y$ s.t.

for each $x \in X$,

$$\sup \left\{ \|T_\alpha x\|_Y : \alpha \in \Lambda \right\} < \infty.$$

Then, $\sup_{\alpha \in \Lambda} \|T_\alpha\|_{op} < \infty.$

Lecture 5 (20-01-2021)

20 January 2021 09:26

Recall that we were asking the question:

When does the Fourier series of f converge to f?

Dini's Theorem had given one condition for convergence at a point.

We now give a continuous example for which this does not happen.

To show the existence of such a f, we use the uniform boundedness principle.

Thm. There exists a continuous function f whose Fourier series diverges at a point.

Proof. $X = \mathcal{C}(\mathbb{T})$ (space of continuous function on $\mathbb{T}$) with sup norm $\leftarrow$ Banach space

$Y = \mathbb{R}$ $\leftarrow$ with $|\cdot|$ norm

$\leftarrow$ NLS

Define, for each $N \in \mathbb{N}$,

$T_N : X \to Y$ by $T_N(f) = S_N f(0)$.

$$\left(\text{Recall : } S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x} = \int_{-1/2}^{1/2} f(t) D_N(x-t)\, dt \right)$$

$$\therefore T_N(f) = \int_{-1/2}^{1/2} f(t) D_N(t)\, dt \qquad \left(D_N(-x) = D_N(x) \right)$$

We first prove that $\{T_N\}_{N\in\mathbb{N}}$ is a family of bounded operators.

Note
$$|T_N(f)| = \left|\int_{-1/2}^{1/2} f(t)\, D_N(t)\, dt\right|$$
$$\leq \|f\|_\infty \int_{-1/2}^{1/2} |D_N(t)|\, dt$$
$$\Rightarrow \quad \|T_N\|_{op} \leq \int_{-1/2}^{1/2} |D_N(t)|\, dt \quad \leftarrow \text{finite}$$

Claim. (a) $\|T_N\|_{op} = \int_{-1/2}^{1/2} |D_N(t)|\, dt$

(b) $\lim_{N\to\infty} \int_{-1/2}^{1/2} |D_N(t)|\, dt = \infty$

Supposing the Claim is true for now, we complete the proof.

$\{T_N\}_{N\in\mathbb{N}}$ - family of bdd. operators from X to Y.

Also, $\|T_N\|_{op} = \int_{-1/2}^{1/2} |D_N(t)|\, dt \longrightarrow \infty$.

By UBP, $\exists\, f \in C(\mathbb{T})$ s.t.
$$\sup_{n\in\mathbb{N}} |T_N f| = \infty.$$

This means that the Fourier series of f at 0 diverges.

Proof of the Claim:

(a) Need to prove $\geq$.

Fix N. Define
$$g(t) = \begin{cases} 1\ ; & \text{if } D_N(t) \geq 0 \\ -1\ ; & \text{if } D_N(t) < 0 \end{cases}$$

Step function with finitely many discontinuities

Then, $\exists \{f_j\}_j$ continuous s.t. $f_j \to g$ (pt-wise, etc.)

with $-1 \le f_j \le 1 \quad \forall j$.

Then, $T_N(f_j) = S_N f_j(0) = \int_{-1/2}^{1/2} f_j(t) D_N(t)\, dt$

$$\lim_{j\to\infty} T_N(f_j) = \lim_{j\to\infty} \int_{-1/2}^{1/2} f_j(t) D_N(t)\, dt$$

DCT $|f_j| \le 1$

$$= \int_{-1/2}^{1/2} \lim_{j\to\infty} f_j(t) D_N(t)\, dt$$

$$= \int_{-1/2}^{1/2} g(t) D_N(t)\, dt$$

def^n of g

$$\lim_{j\to\infty} T_N(f_j) = \int_{-1/2}^{1/2} |D_N(t)|\, dt$$

$$\Rightarrow \|T_N\|_{op} = \sup_{f \ne 0} \frac{\|T_N(f)\|}{\|f\|}$$

$$\ge \sup_{f_j \ne 0} \frac{\|T_N(f_j)\|}{\|f_j\|} \ge \sup_{f_j \ne 0} \|T_N(f_j)\| \ge \int_{-1/2}^{1/2} |D_N|$$

$\|f_j\| \le 1$

Thus, $\|T_N\|_{op} \ge \int_{-1/2}^{1/2} |D_N|$, as desired.

(b) $$\int_{-1/2}^{1/2} |D_N(t)|\, dt = \int_{-1/2}^{1/2} \left| \frac{\sin(2N+1)\pi t}{\sin \pi t} \right| dt$$

$$= 2 \int_0^{1/2} \left| \frac{\sin(2N+1)\pi t}{\sin \pi t} \right| dt$$

$$\ge \frac{2}{\pi} \int_0^{1/2} \frac{\sin(2N+1)\pi t}{t}\, dt$$

$(2N+1)t = s$

$\int^{\frac{2N+1}{2}}$

$$= \frac{2}{\pi} \frac{1}{(2N+1)} \int_0^{\frac{2N+1}{2}} \frac{|\sin \pi s|}{s} \cdot (2N+1)\, ds$$

$$= \frac{2}{\pi} \int_0^{N+\frac{1}{2}} \frac{|\sin \pi s|}{s}\, ds$$

$$\geq \frac{2}{\pi} \sum_{k=0}^{N-1} \int_k^{k+1} \frac{|\sin \pi s|}{s}\, ds$$

$$\geq \frac{2}{\pi} \sum_{k=0}^{n-1} \int_k^{k+1} \frac{|\sin \pi s|}{k+1}\, ds$$

$$= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \int_k^{k+1} |\sin \pi s|\, ds$$

$$= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \int_0^{1} \sin \pi s\, ds$$

$$= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \longrightarrow \infty$$

harmonic series.

- Cesaro summability

Def^n. $\sum_{n=1}^{\infty} a_n$ is Cesaro summable to A if

$$\left\{ \frac{S_1 + \cdots + S_N}{N} \right\}_N \longrightarrow A.$$

$$\left(S_N = \sum_{k=1}^{N} a_k. \right)$$

Prop^n. If $\sum_{n=1}^{\infty} a_n$ converges, then it is Cesaro summable to the same sum.

Prop. If $\sum_{n=1}$ ... converges, then it is Cesaro summable to the same sum.

Proof. Omitted. ∎

Prop^n. There exists a series which does not converge but it is Cesaro summable.

Proof. Let $a_n = (-1)^{n+1}$. Then $\sum a_n$ does not converge.

Note
$$S_n = \begin{cases} 1 & ; n \text{ odd} \\ 0 & ; n \text{ even} \end{cases}$$

$$\Rightarrow \quad \frac{S_1 + \dots + S_N}{N} = \begin{cases} \frac{N/2}{N} = \frac{1}{2} & ; N \text{ even} \\ \frac{(N+1)/2}{N} = \frac{1}{2} + \frac{1}{N} & ; N \text{ odd} \end{cases} \xrightarrow{N \to \infty} \frac{1}{2}$$
∎

Thus, Cesaro summable is more relaxed. We now wish to do so with our Fourier series.

- Cesaro Summability of Fourier Series

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x}.$$

Define
$$\sigma_N f(x) := \frac{1}{N+1} \left(\sum_{k=0}^{N} S_k f(x) \right).$$

We now wish to see if $\sigma_N f(x)$ converges.

$$\sigma_N f(x) = \frac{1}{N+1} \left(\sum_{k=0}^{N} \int_0^1 f(t) D_k(x-t)\, dt \right)$$

$$= \int^1 f(t) \cdot \left(\frac{1}{\dots} \sum^N D_k(x-t) \right) dt$$

$$= \int_0^1 f(t) \cdot \left(\frac{1}{N+1} \sum_{k=0}^{N} D_k(x-t) \right) dt$$

$$= \int_0^1 f(t)\, F_N(x-t)\, dt = (f * F_N)(x).$$

where $F_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t)$ - Fejer kernel

(Fejer kernel)

$$= \frac{1}{N+1} \sum_{k=0}^{N} \frac{\sin(2k+1)\pi t}{\sin \pi t}$$

$$= \frac{1}{\sin \pi t} \cdot \frac{1}{N+1} \cdot \sum_{k=0}^{N} \sin(2k+1)\pi t$$

sin in AP

$$= \frac{1}{(N+1)\sin \pi t} \cdot \frac{\sin^2 (N+1)\pi t}{\sin \pi t}$$

$$= \frac{1}{N+1} \cdot \left(\frac{\sin (N+1)\pi t}{\sin \pi t} \right)^2$$

(i) $F_N(t) \geq 0 \quad \forall t \quad \forall N$

(ii) $\int_{-1/2}^{1/2} F_N(t)\, dt = 1 \quad \forall N$ (from the Σ def", use $\int_{-1/2}^{1/2} D_N = 1$)

(iii) Fix $\delta > 0$, $\int_{\frac{1}{2} \geq |t| > \delta} F_N(t)\, dt \longrightarrow 0$ as $N \to \infty$

$$\int_{\delta < |t| \leq \frac{1}{2}} F_N(t) dt \leq \frac{1}{\sin^2 \pi\delta} \cdot \frac{1}{N+1} \int_{\delta < |t| \leq \frac{1}{2}} \underbrace{(\sin(N+1)\pi t)^2}_{\leq 1} dt$$

$$\leq \left(\frac{2\left(\frac{1}{2} - \delta\right)}{\sin^2 \pi\delta} \right) \frac{1}{N+1} \longrightarrow 0.$$

Cor. $\{F_N\}_N$ is an approximate identity in $\mathcal{L}^1$

Thm. (1) If $f \in \mathcal{L}^p(\mathbb{T})$, $1 \le p < \infty$, then

$$\| f - f * F_N \|_p \to 0 \quad \text{as } N \to \infty.$$

(That is, $\| \sigma_N f - f \|_p \to 0$ as $N \to \infty$.)

(2) Let f be a continuous function. Then,

$$(f - f * F_N) \to 0 \quad \text{uniformly.}$$

That is, $\sigma_N f \to f$ uniformly.
($\| \cdot \|_\infty$ is sup norm.)

This proves that the Fourier series of f is Cesaro summable to f, if f is continuous.

Ex. $$F_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t)$$

(Recall $D_k(t) = \sum_{m=-k}^{k} e^{-2\pi i m t}$)

$$= \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j t}$$

$$\therefore \; f * F_N(t) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \underbrace{\left(f * e^{2\pi i j t}\right)}_{?}$$

$$\left(f * e^{2\pi i j t}\right)(x) = \int_0^1 f(t)\, e^{2\pi i j (x-t)} \, dt$$

$$= e^{2\pi i j x} \, \hat{f}(j)$$

$$\therefore \quad f * F_N(x) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)\, e^{2\pi i j x}$$

Def^n A trigonometric polynomial of degree n is of the form

$$\sum_{j=-N}^{N} a_k \, e^{2\pi i j x}.$$

Cor. (1) The trigonometric polynomials are dense in $\mathcal{L}^p(\mathbb{T})$ for $1 \le p < \infty$.

(2) The trigonometric functions are dense in $\mathcal{C}(\mathbb{T})$ with sup norm.

Cor. Suppose $f \in \mathcal{L}^1(\mathbb{T})$ and $\hat{f}(k) = 0 \quad \forall k \in \mathbb{Z}$.
Then, $f = 0$ in $\mathcal{L}^1$.
(If f is continuous, then $f \equiv 0$.)

$$\left(\begin{aligned} \hat{f}(k) = 0 \ \forall k \quad &\Rightarrow f * F_N = 0 \\ &\Rightarrow \|f - f * F_N\|_1 = \|f\|_1 \to 0 \\ &\Rightarrow \|f\|_1 = 0. \end{aligned} \right)$$

Def^n. Hilbert space

X with $\langle\, ,\, \rangle$ s.t. $(X, \|\cdot\|)$ is complete (Banach).

wrt $\langle\, ,\, \rangle$

ℓ_p / $\mathcal{L}^p$ for $p = 2$ only is Hilbert space. (can't define $\langle\, ,\, \rangle$ else)

$\mathbb{C}^n$, $\mathbb{R}^n$ are Hilbert spaces.

Def. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.
A sequence $\{e_n\}$ in H is • orthogonal if $\langle e_i, e_j \rangle = 0$ $\forall i \neq j$.
• orthonormal if orthogonal and $\langle e_i, e_i \rangle = 1$ $\forall i$.

We say $\{e_n\}$ is complete orthonormal (or orthonormal basis or o.n.b.) if
(i) $\{e_n\}$ are orthonormal
(ii) $\langle f, e_n \rangle = 0 \ \forall n \Rightarrow f = 0$

Thm. Let $(H, \langle \cdot, \cdot \rangle)$ be an Hilbert space and $\{e_n\}_n$ be an o.n.b. Then, for $f \in H$,

(i) $f = \sum_n \langle f, e_n \rangle e_n$,

(ii) $\|f\|^2 = \sum_n |\langle f, e_n \rangle|^2$.

Consider $(\mathcal{L}^2(\mathbb{T}), \langle \cdot, \cdot \rangle)$ where

$$\langle f, g \rangle := \int_0^1 f(t) \overline{g(t)} \, dt$$

$$\langle f, f \rangle = \|f\|^2_{\mathcal{L}^2(\mathbb{T})}$$

$\therefore$ $\mathcal{L}^2(\mathbb{T})$ is a Hilbert space.

Thm $\{t \mapsto e^{2\pi i n t}\}_{n \in \mathbb{Z}}$ is an onb for $\mathcal{L}^2(\mathbb{T})$.

Proof. Put $e_n(t) := e^{2\pi i n t}$.

Easy to see that $\langle e_n, e_m \rangle = \begin{cases} 0 & ; n \neq m \\ 1 & ; n = m \end{cases}$

Now, suppose $\langle f, e_n \rangle = 0 \ \forall n$ for $f \in L^2(\mathbb{T})$.

To show: $f = 0$

$\int_0^1 f(t)\, e^{-2\pi i n t} = 0 \quad \forall n$

$\Rightarrow \hat{f}(n) = 0 \ \forall n$

$\Rightarrow f = 0$ in L^2. ∎

Lecture 6 (22-01-2021)

22 January 2021 09:33

Recall:

$$f_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t)$$

(1) Trig. polynomials are dense in $\mathcal{L}^p(\mathbb{T})$, $1 \le p < \infty$ and also in $\mathcal{C}(\mathbb{T})$.

(2) If $f \in \mathcal{L}^1(\mathbb{T})$ and $\hat{f}(k) = 0 \;\; \forall k \in \mathbb{Z}$, then $f = 0$ in $\mathcal{L}^1$.

$\mathcal{L}^2(\mathbb{T})$: $\langle f, g \rangle := \int_0^1 f(t)\,\overline{g(t)}\,dt$

↳ inner product

$$\|f\|_2^2 = \langle f, f \rangle.$$

Let $e_n(t) := e^{2\pi i n t}$, $t \in [0,1]$

Then, $e_n \in \mathcal{L}^2(\mathbb{T})$.

1. $\langle e_n, e_m \rangle = \int_0^1 e_n \overline{e_m}$

$$= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

$\therefore$ $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal set in $\mathcal{L}^2(\mathbb{T})$.

2. <u>Claim</u>. $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $\mathcal{L}^2(\mathbb{T})$.

That is, $\langle f, e_n \rangle = 0 \;\; \forall n \in \mathbb{Z} \Rightarrow f = 0$ in $\mathcal{L}^2$.

Note that $f \in \mathcal{L}^2(\mathbb{T}) \subseteq \mathcal{L}^1(\mathbb{T})$ and

$$0 = \langle f, e_n \rangle = \int_0^1 f \, \bar{e}_n = \hat{f}(n) \quad \forall n \in \mathbb{Z}$$

$$\Rightarrow f = 0 \text{ in } \mathcal{L}^2$$

Thus, $\{e_n\}_{n\in\mathbb{Z}}$ is an o.n.b. for $\mathcal{L}^2(\mathbb{T})$

Then (by the Hilbert space theorem):

(1) $$f = \sum_{n\in\mathbb{Z}} \langle f, e_n \rangle e_n, \quad \text{i.e.,}$$

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) \, e_n, \quad \text{i.e.,}$$

$$f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \, e^{2\pi i n t}.$$

Equality in $\mathcal{L}^2(\mathbb{T})$

(That is, $\left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 \to 0$ as $N \to \infty$)

(Parseval's Theorem)

(2) $$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\langle f, e_n \rangle|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

Parseval's Theorem

This means that the map

$$\mathcal{F} : \mathcal{L}^2(\mathbb{T}) \to \ell_2(\mathbb{Z})$$

$$\mathcal{F}(f) = \{\hat{f}(n)\}_{n\in\mathbb{Z}}$$

is an isometry.

In particular, it will be one-one.

Is it onto? (Yes. Needs an argument)

Summary So Far:

(Q) Does the Fourier series of f converge to f?

1. We showed that if f satisfies Dini's theorem, then $S_N f \to f$ at that point.
2. $\exists$ a continuous function f whose Fourier series does not converge to f at 0.
3. We looked at Cesaro summation. Then,

$$\sigma_N f \to f \text{ in } \mathcal{L}^p(\mathbb{T}), \quad 1 \leq p < \infty$$

and

$$\sigma_N f \longrightarrow f \text{ uniformly if } f \in \mathcal{C}(\mathbb{T}).$$

4. (Q) Does $\|S_N f - f\|_p \to 0$ as $N \to \infty$? $1 \leq p < \infty$

($\mathcal{L}^p$ Convergence of Fourier series)

That is,

$$S_N f \longrightarrow f \text{ in } \mathcal{L}^p \text{ as } N \to \infty?$$

(Ans) Not true if $p = 1$. True if $1 < p < \infty$.

Going to prove this

Remark. Note that for $p = 2$, we have shown it above as $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$.
(The Hilbert space theory.)

Lemma 1. Fix $p \in [1, \infty]$

$$S_N f \longrightarrow f \text{ in } \mathcal{L}^p(\mathbb{T}) \quad \forall f \in \mathcal{L}^p$$

$\Leftrightarrow$ $\exists\, C_p > 0$ (independent of N) s.t. $\forall f \in \mathcal{L}^p$

$$\|S_N f\|_p \leq C_p \|f_p\|_p \quad \forall N$$

$$\left(\text{i.e., } \|S_N\|_{op} \leq C_p\right)$$

Proof. ($\Rightarrow$) Suppose $S_N f \to f$ in $\mathcal{L}^p$, $1 \leq p < \infty$.

That is, $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t.

$$\|S_N f - f\|_p < \varepsilon \qquad \forall N \geq N_0$$

Hence, $\|S_N f\|_p = \|S_N f - f + f\|_p < \varepsilon + \|f\|_p$

$\forall N \geq N_0$

$\therefore$ For a fixed $f \in L^p$:

$$\sup_N \|S_N f\|_p < \infty$$

$\therefore$ By UBP, $\sup_N \|S_N\|_{op}$ is finite.

$C_p = \sup_N \|S_N\|_{op}$ works.

$(\Leftarrow)$ Suppose $\exists C_p > 0$ s.t.

$$\|S_N f\|_p \leq C_p \|f\|_p \qquad \forall N.$$

We will prove $S_N f \to f$ in L^p.

Let g be a trig poly of degree M, i.e.,

$$g(t) = \sum_{k=-M}^{M} a_M(k)\, e^{2\pi i k t}$$

$$S_N g(t) = \sum_{l=-N}^{N} \hat{g}(l)\, e^{2\pi i l t}$$

$$= \sum_{l=-M}^{M} a_M(l)\, e^{2\pi i l t} \qquad \text{if } N \geq M$$

$$= g(t)$$

$$\hat{g}(l) = \int_0^1 g(t) e^{-2\pi i l t}\, dt$$

$$= \sum_{k=-M}^{M} \int_0^1 a_M(k) e^{2\pi i (k-l)}\, dt$$

$$= \begin{cases} a_M(l) & M \geq |l| \\ 0 & M < |l| \end{cases}$$

$\therefore \quad S_N g = g \quad \text{if} \quad N \geq \deg g.$

Now we use the fact that trigonometric polynomials are dense in $\mathcal{L}^p$. Given $f \in \mathcal{L}^p$. That is, for $\varepsilon > 0$,

$\exists g \to$ trig polynomial s.t.

$$\|f - g\|_p < \varepsilon.$$

Let $N \geq \deg g$. Then,

$$\|S_N f - f\|_p = \|S_N f - S_N g + \underbrace{S_N g - g}_{=0} + g - f\|_p$$

$$\leq \|S_N f - S_N g\|_p + \|g - f\|_p$$

$$\leq \|S_N (f-g)\|_p + \|f - g\|_p$$

$$\leq C_p \|f - g\|_p + \|f - g\|_p$$

$$< (1 + C_p)\,\varepsilon.$$

Thus, $S_N f \to f$ in $\mathcal{L}^p$.

- For $p = 1$, we will see that $\|S_N\|_{\mathcal{L}^1 \to \mathcal{L}^1} = \int_{-1/2}^{1/2} |D_N|$.

  We had seen that RHS $\to \infty$ as $N \to \infty$.
  This will show $S_N f \to f$ in $\mathcal{L}^1$ is not true for all f, by Lemma 1.
  (That is, $\exists f \in \mathcal{L}^1$ s.t. $S_N f \not\to f$ in $\mathcal{L}_1$.)

- Harmonic function

$f : \Omega \to \mathbb{R}$, f continuous and $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ exist.

Let $\Delta f := \dfrac{\partial^2}{\partial x^2} f + \dfrac{\partial^2}{\partial x^2} f.$

Let $\Delta f := \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f.$

If $\Delta f = 0$, then f is called harmonic.

• Dirichlet problem

$\mathbb{D} := \{ z \in \mathbb{C} : z < 1 \}.$

Then, $\bar{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and

$\partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}.$

Given $f : \partial \mathbb{D} \to \mathbb{R}$ continuous function,

consider $\Delta u = 0$ on $\mathbb{D}$ and
$u = f$ on $\partial \mathbb{D}$. } harmonic extension

Then, does u exist? What is u?

We have

$$(P_r f(\theta) :=) \; u(re^{2\pi i\theta}) = \int_0^1 P_r(\theta - t) f(e^{2\pi i t}) \, dt$$

(Poisson integral)

$$(0 \leq r \leq 1, \quad 0 \leq \theta \leq 1) \qquad = (P_r * f)(\theta)$$

– where P_r is called Poisson kernel

$$P_r(\theta) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n\theta}.$$

$$= \operatorname{Re}\left(\frac{1+z}{1-z}\right), \quad \text{where } z = re^{2\pi i\theta}$$

$$= \frac{1-r^2}{1 - r\cos(2\pi\theta) + r^2} = \frac{1-r^2}{|1-re^{2\pi i\theta}|^2}$$

• Properties of Poisson kernel

(a) $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$

(b) $P_r(\theta) > 0 \quad \forall \theta \quad \forall 0 \le r < 1$

(c) $P_r(\theta) = P_r(-\theta)$

(d) $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \le \pi$.

(e) For each $\delta > 0$, $\lim_{r \to 1^-} P_r(\theta) = 0$

uniformly in θ for $\delta \le |\theta| \le \pi$.

(f) $\lim_{r \to 1^-} \int_{\delta \le |\theta| \le \pi} P_r(\theta) d\theta = 0$.

- Hence $\{P_r\}_r$ is an approximate identity in $L^1(\mathbb{T})$

and

$$\|f * P_r - f\|_p \to 0 \quad \text{as} \quad r \to 1^-.$$

(Not a sequence, a continuous family.)

Poisson integral

For $f \in C(\mathbb{T})$

$$(P_r f)(t) = (P_r * f)(t)$$

$$= \left(\sum_{n \in \mathbb{Z}} r^{|n|} e^{2\pi i n t} \right) * f(t)$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \left(e^{2\pi i n t} * f \right)(t)$$

$$P_r f(t) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{2\pi i n t}.$$

defined on $\mathbb{T}$, identify t with $e^{2\pi i t}$

will extend to $\mathbb{D}$

Conjugate Poisson integral

$f \in \mathcal{C}(\mathbb{T})$.

$$Q_r f(t) := (-i) \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\, \hat{f}(n)\, r^{|n|} e^{2\pi i n t}$$

$$= (q_r * f)(t) \qquad \operatorname{sgn}(n) = \begin{cases} -1 & ; n<0 \\ 0 & ; n=0 \\ 1 & ; n>0 \end{cases}$$

Where

$$q_r(t) = -i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\, r^{|n|} e^{2\pi i n t}$$

(Conjugate Poisson kernel)

Note $\displaystyle\int_0^1 q_r(t)\,dt = 0$

- $$Pf(t) = \sum_{k=0}^{\infty} \hat{f}(k)\, e^{2\pi i k t} \qquad \text{for } f \in \mathcal{C}(\mathbb{T})$$

  Extend P to $\mathcal{L}^2$ functions

  Can do so since $$\|Pf\|_2^2 \leq \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \leq \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 = \|f\|_2^2$$

- Aim: $S_N f \to f$ in $\mathcal{L}^p$ for all f $(1<p<\infty)$

  Have proven:

  Lemma 1. $S_N f \to f \quad \forall f \in \mathcal{L}^p$

  $$\Leftrightarrow \|S_N\|_{op} \leq C_p \quad \forall N$$

Will show: Lemma 2. S_N is uniformly bounded on $\mathcal{L}^p$

$\Leftrightarrow P : \mathcal{L}^p(\mathbb{T}) \to \mathcal{L}^p(\mathbb{T})$ is a bounded linear operator

Observe:

$$P_r f(t) + i\, Q_r f(t) = \sum_{k=1}^{\infty} 2 \hat{f}(k)\, r^k e^{2\pi i k t} + \hat{f}(0)$$

$$= 2 Pf(r e^{2\pi i t}) - \hat{f}(0)$$

(extend Pf to $\mathbb{D}$.)

Lemma 3. $P : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is bounded linear

$\Leftrightarrow$ P_r, Q_r are uniformly bounded on $L^p(\mathbb{T})$ $\forall\ 0 < r < 1$.

(i.e., $\|P_r f\|_p \le C \|f\|_p$ $\forall\ 0 < r < 1$,
$\|Q_r f\|_p \le C' \|f\|_p$ C, C' independent of r)

Lemma 4. $\|P_r f\|_p \le \|f\|_p$ $\forall\ 0<r<1$, $1 \le p < \infty$

Lemma 5. $\|Q_r f\|_p \le C \|f\|_p$ $\forall\ 0<r<1$, $1 \cancel{\le} p < \infty$

Lecture 7 (27-01-2021)

27 January 2021 09:29

Recall.

Lem 1. $S_N f \to f \quad \forall f$ in $L_p \Leftrightarrow \exists C_p > 0$ s.t. $\|S_N\|_{op} < C_p \quad \forall N$ $\boxtimes$

Lem 2. S_N is uniformly bounded on $L^p(\mathbb{T})$
$\Leftrightarrow P : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ extends to a bounded linear operator

$e_n(t) := e^{2\pi i n t}$; $\quad S_N f(t) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k t}$

$$\hat{f}(k-N) = \int_0^1 f(t) e^{-2\pi i (k-N) t} \, dt \qquad = \sum_{k=0}^{2N} \hat{f}(k-N) e^{2\pi i (k-N) t}$$

$$= \int_0^1 \left(f(t) e^{2\pi i N t}\right) e^{-2\pi i k t} \, dt \qquad = e_{-N}(t) \sum_{k=0}^{2N} \widehat{e_N f}(k) e^{2\pi i k t}$$

$$= \widehat{e_N f}(k)$$

$$\Rightarrow e_N(t) S_N f(t) = \sum_{k=0}^{2N} \widehat{f \cdot e_N}(k) e_k(t)$$

$$= P_{2N}(f \cdot e_{-N})(t)$$

where $\quad P_N f(t) = \sum_{k=0}^{N} \hat{f}(k-N) e_k(t) = \sum_{k=0}^{N} \widehat{f \cdot e_N}(k) e_k(t)$

$\therefore \quad e_N \cdot S_N f = P_{2N}(f \cdot e_{-N})$

$\therefore \quad S_N$ is uniformly bounded $\quad L^p(\mathbb{T}) \to L^p(\mathbb{T})$
$\Leftrightarrow P_{2N}$ is ———— " ————

Proof of lem2 ($\Rightarrow$) Suppose S_N is uniformly bounded on $\mathcal{L}^p(\mathbb{T})$.

$\therefore$ P_{2N} is —"— on $\mathcal{L}^p(\mathbb{T})$

$\therefore$ $\|P_{2N} f\|_p \le c\|f\|_p \quad \forall f \in \mathcal{L}^p$

c indep. of N

Let f be to trigonometric polynomial of deg. m, i.e.,

$$f = \sum_{k=-m}^{m} c_k e_k$$

$\hat{f}(p) = 0$ if $|p| > m$

$$\Rightarrow P_{2N}(f \cdot e_{-2N})(t) = \sum_{k=0}^{2N} \hat{f}(k)\, e_k(t)$$

$$= \sum_{k=0}^{\infty} \hat{f}(k)\, e_k(t) \quad \text{if } N > \deg f$$

$$= Pf$$

$\therefore$ $P_{2N}(f \cdot e_{-2N}) = Pf \quad \forall$ trig poly f with $\deg f < N$

$$\Rightarrow \|Pf\|_p = \|P_{2N}(f \cdot e_{-2N})\|_p$$

$$\le c \cdot \|f \cdot e_{-2N}\|_p$$

$$= c \cdot \|f\|_p \quad \leftarrow \text{indep of } N$$

Thus, $\|Pf\|_p \le c \cdot \|f\|_p \quad \forall$ trig polynomials f

Use density of trig poly to conclude the above is true for all f.

($\Leftarrow$) Suppose $\|Pf\|_p \le c\|f\|_p \quad \forall f \in \mathcal{L}^p$

Let f be a trig poly. Then

$$P_{2N}(f \cdot e_{-2N}) = Pf \quad \text{if } N > \deg f$$

or $$P_{2N}(f) = P(f \cdot e_{2N}) \quad \text{if } N > \deg f.$$

Fix an f with $\deg f < N$.

Then,

$$\|P_{2N}(f)\|_p = \|P(f \cdot e_{2N})\| \leq \|P\|_{op} \, \|f \cdot e_{2N}\|_p$$
$$= \|P\|_{op} \, \|f\|_p$$

Use the denseness of trig poly in L^p to get

$$\sup_N \|P_{2N}(f)\| \leq C_f < \infty$$

so is S_N. ∎

$t \in [0,1]$ identified with $\mathbb{T}$

Extension of Pf: $$Pf(t) = \sum_{k=0}^{\infty} \hat{f}(k) \, e^{2\pi i k t} \; ; \; f \in C(\mathbb{T})$$

Let $z = re^{2\pi i t}$, $0 \leq r < 1$, $t \in [0,1]$

$$Pf(z) = Pf(re^{2\pi i t}) := \sum_{k=0}^{\infty} \hat{f}(k) \, r^k \, e^{2\pi i k t}$$
$$= \sum_{k=0}^{\infty} \hat{f}(k) \, z^k.$$

Thus, Pf is holo on $\mathbb{D}$. Moreover, the convergence is uniform on compact subsets.

Recall: $$P_r f(t) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) \, e_k(t)$$

$$Q_r f(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) \, r^{|k|} \, \hat{f}(k) \, e_k(t)$$

$$P_r f(t) + i Q_r f(t) = 2Pf(re^{2\pi i t}) - \hat{f}(0). \qquad \text{— (1)}$$

Lem 3. If P_r and Q_r are uniformly bounded on L^p $\forall\ 0 \le r < 1$, then $P : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is bounded.

Proof.
$$\|P_r f\| \le c \cdot \|f\|_p \qquad \forall r \in [0,1)$$
$$\|Q_r f\| \le c' \cdot \|f\|_p \qquad \forall r \in [0,1)$$

Fix an r. Then, $t \mapsto re^{2\pi i t}$ is a function

— by (1)

$$\Rightarrow \quad \|Pf(re^{2\pi i(\cdot)})\|_p \le c \cdot \|f\|_p \qquad \forall r$$

Take $r \to 1^-$. (Justify exchange using DCT.)

$$\Rightarrow \quad \|Pf\|_p \le C\|f\|_p$$

Lem 4. $\|P_r f\| \le \|f\|_p \quad \forall r \quad \forall\ 1 \le p < \infty$

Proof. We know $P_r f = p_r * f$ and that $\{p_r\}$ is an approximate identity.

$$\therefore \|P_r f\|_p = \|f * p_r\|_p \le \|p_r\|_1 \|f_p\| = \|f_p\| \qquad (\|fg\|_p \le \|f\|_1 \|g\|_p)$$

$$\Rightarrow \|P_r f\|_p \le \|f_p\|$$

Lemma 5. $\|Q_r f\|_p \le C_p \|f\|_p \quad \forall r \quad \forall\ 1 < p < \infty$

Proof. $Q_r f = f * q_r$ where $q_r \to$ conjugate Poisson kernel

and

Proof. $Q_r f = f * q_r$ where $q_r \to$ conjugate Poisson kernel

and

$$q_r(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k)\, r^{|k|}\, e_N(t)$$

$$= \operatorname{Im}\left(\frac{1}{1-re^{2\pi i t}}\right)$$

$$= \frac{2r \sin 2\pi t}{1+r^2-2r\cos 2\pi t}$$

$$\int_0^1 q_r(t)\,dt = 0$$

- Let $f \geq 0$ and $1 < p < 2$:

$$F(z) = F(re^{2\pi i t}) := P_r f(t) + i\, Q_r f(t)$$ ← holomorphic on $\mathbb{D}$

$$= f * (P_r + i\, q_r)$$

$$\therefore \operatorname{Re} F(z) = (f * P_r)(t) > 0$$

Then, let $G(z) := F(z)^p = e^{p \log F(z)}$

↪ well defined since F is non-vanishing and $\mathbb{D}$ is simply connected

Note that

$G(0) = (F(0))^p$

$= \hat{f}(0)$

Let $\gamma < \pi/2$ and $\frac{\pi}{2} < p\gamma < \pi$.

(Possible since $1 < p < 2$.)

Let $A_\gamma = \{ t \in [0,1) : \arg F(re^{2\pi i t}) < \gamma \}$

and $B_\gamma = [0,1) \setminus A_\gamma$.

We prove : $\displaystyle\int_0^1 |F(re^{2\pi i t})|^p\,dt \leq c \cdot \|f\|_p^p$

This would give $\displaystyle\int_0^1 |Q_r f(re^{2\pi i t})|^p\,dt \leq c \|f\|_p^p$

This would give $\int_0^1 |Q_r f(re^{2\pi i t})|^p \, dt \le c \cdot \|f\|_p^p$

$$\overset{\shortparallel}{\int_0^1 |Q_r f(t)|^p \, dt}$$

- $F(re^{2\pi i t}) = |F(re^{2\pi i t})| \cdot e^{i \arg F(re^{2\pi i t})}$

$\therefore \ \mathrm{Re}\, F(re^{2\pi i t}) = |F(re^{2\pi i t})| \cos\left(\arg\left(F(re^{2\pi i t})\right)\right)$

For $t \in A_\gamma$, $\qquad \ge |F(re^{2\pi i t})| \cos\gamma$

Hence, $|F(re^{2\pi i t})| \le \dfrac{1}{\cos\gamma} \mathrm{Re}\, F(re^{2\pi i t})$ for $t \in A_\gamma$.

$$\therefore \int_{A_\gamma} |F(re^{2\pi i t})|^p \, dt \le (\cos\gamma)^{-p} \int_{A_\gamma} \mathrm{Re}\, F(re^{2\pi i t})^p \, dt$$

$$= (\cos\gamma)^{-p} \int_{A_\gamma} (P_r f(t))^p \, dt$$

$$\le (\cos\gamma)^{-p} \int_0^1 (P_r f(t))^p \, dt$$

$P_r f \ge 0$ since $f \ge 0$ and $A_\gamma \subset [0,1]$

$$\le (\cos\gamma)^{-p} \|f\|_p^p$$

$\leftarrow$ independent of r

- $\mathrm{Re}\, G(re^{2\pi i t})$ — is a harmonic function and hence, it satisfies the mean value property, i.e.,

$$\int_0^1 \mathrm{Re}\, G(re^{2\pi i t}) \, dt = \mathrm{Re}\, G(0) = \hat{f}(0)$$

- $\mathrm{Re}\, G(re^{2\pi i t}) = |F(re^{2\pi i t})|^p \cos\left(p \cdot \arg F(re^{2\pi i t})\right)$

$$\therefore \int_{A_\gamma} |\mathrm{Re}\, G(re^{2\pi i t})|\, dt = \int_{A_\gamma} |F(re^{2\pi i t})|^p \, |\cos(p \cdot \arg(F re^{2\pi i t}))|\, dt$$

$$\le C \cdot \int_{A_\gamma} | F(re^{2\pi i t})|^p \, dt$$

$\because \pi/2 < p\gamma < \pi$

$$\left| \int_{B_\gamma} \mathrm{Re}\; G(re^{2\pi i t})\, dt \right| \le \hat{f}(0) + \int_{A_\gamma} |\mathrm{Re}\; G(re^{2\pi i t})|\, dt$$

$$\le C \cdot \int_{A_\gamma} | F(re^{2\pi i t})|^p \, dt$$

$$|G(re^{2\pi i t})| = \mathrm{Re}\; G(re^{2\pi i t}) \cdot [\cos(p \arg F(\;))]^{-1}$$

If $t \in B_\gamma$, $\arg F(\;) \geq \gamma$

$\therefore p \arg F(\;) \geq p\gamma > \pi/2$

$\mathrm{Re}\, F > 0 \Rightarrow |\arg F(\;)| < \pi/2$

$$\therefore \frac{\pi}{2} < p \arg F(\;) < \frac{\pi}{2} p < \pi$$

$$\therefore |G(re^{2\pi i t})| = |\mathrm{Re}\; G(\;)| \cdot |\cos(p \arg F(\;))|^{-1}$$
$$\le |\cos p\gamma|^{-1} \cdot (-\mathrm{Re}\; G(\;))$$

$$\therefore \int_{B_\gamma} |G(re^{2\pi i t})|\, dt \le |\cos p\gamma|^{-1} \left| \int_{B_\gamma} \mathrm{Re}\; G \right|$$

$$\le C \cdot |\cos p\gamma|^{-1} \int_{A_\gamma} |F|^p$$

$$\le C \cdot \|f\|_p^p$$

$$\therefore \int_0^1 |f(re^{2\pi i t})|^p \, dt \leq C \cdot \|f\|_p^p \qquad \forall r$$

Thus, $$\int_0^1 |Q_r f|^p \, dt \leq C\|f\|_p^p \qquad \forall r$$

Can extend this for all f by

$$f = f^+ - f^- + i(f_i^+ - f_i^-)$$

- $p = 2$ works by Hilbert space theory
- $p > 2$ works by duality.

Lecture 8 (29-01-2021)

29 January 2021 09:26

For $p > 2$: $\quad Q_r f(t) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) \hat{f}(k) e^{2\pi i k t}$

adjoint

$\langle Q_r^* f, g \rangle := \langle f, Q_r g \rangle \qquad$ (in L^2)

$\frac{1}{p} + \frac{1}{p'} = 1$

$p > 2 \Rightarrow p' < 2$

Then, $Q_r^* = -Q_r$.

$$\|Q_p f\| = \sup_{\|g\|_{p'} \le 1} \left| \int Q_r f \cdot \bar{g} \right| \qquad (g \in L^{p'})$$

$$= \sup_{\|g\|_{p'} \le 1} \left| \int f \cdot \overline{Q_r^* g} \right|$$

$Q_r^* = -Q_r$

0

$$= \sup_{\|g\|_{p'} \le 1} \left| \int f \cdot \overline{Q_r g} \right|$$

$$\le \sup_{\|g\|_{p'} \le 1} \|f\|_p \|Q_r g\|_{p'}$$

since $p' < 2$, we earlier part

$$\le \sup_{\|g\|_{p'} \le 1} \|f\|_p \, c \cdot \|g\|_{p'}$$

$\sup = 1$

$$\|Q_p f\| \le c \cdot \|f\|_p$$

Thus, we are done for all $p \in (1,2) \cup (2, \infty)$. ∎

For $p = 2$, we already knew convergence of partial.

For $p = 1$?

We show $\|S_N\|_{op}$ is not uniformly bounded.

($S_N : L^1 \to L^1$)

$$S_N f(t) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k t}$$

$$= (f * D_N)(t)$$

$$\therefore \|S_N f\|_1 = \|f * D_N\|_1 \leq \|f\|_1 \cdot \|D_N\|_1$$

$$\therefore \|S_N\|_{op} \leq \|D_N\|_1$$

$$\|S_N\|_{op} = \sup_{\|f\|_1 \neq 0} \frac{\|S_N f\|_1}{\|f\|_1}$$

$$\geq \sup_M \frac{\|S_N F_M\|_1}{\|F_M\|_1}$$

$$= \|D_N\|_1$$

(F_Ms are Fejer kernel)

($S_N F_M = F_M * D_N \to D_N$ in $\mathcal{L}^1$)

$$\therefore \|S_N\|_{op} = \|D_N\|_1$$

For $\|D_N\|_1$, we have already shown divergence.
Thus, we are done.

(For $p>1$, the argument does not hold since $\|D_N\|_p \not\to \infty$.)

Conclusion

For $p \in (1, \infty)$: $S_N f \to f$ in $\mathcal{L}^p$ $\forall f \in \mathcal{L}^p$

For $p = 1$: $\exists f \in \mathcal{L}^1$ s.t. $S_N f \not\to f$ in $\mathcal{L}^1$

For $p \in [1, \infty)$: $\sigma_N f \to f$ in $\mathcal{L}^p$ $\forall f \in \mathcal{L}^p$

For $p = \infty$: $\sigma_N f \to f$ in $\mathcal{L}^\infty$ if f continuous

Isoperimetric problem

Given a string of length $L > 0$, find the maximum area enclosed by the string.
What is the position?

Ans. Position: circle, area $= \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$.

Defn. A parameterized curve is a function $\gamma : [a,b] \to \mathbb{R}^2$ which is continuous.

γ is closed if $\gamma(a) = \gamma(b)$.

γ is simple if $\gamma|[a,b)$ is injective.

We will take simple closed γ which is C^1. We assume $\gamma'(t) \neq 0 \ \forall t \in [a,b]$.

The length $l(\gamma)$ of a curve γ is defined as

$$l(\gamma) := \int_a^b |\gamma'(t)| \, dt$$

$\gamma = (\gamma_1, \gamma_2)$

$$= \int_a^b \left[(\gamma_1'(t))^2 + (\gamma_2'(t))^2 \right]^{1/2} dt$$

The area $A(\gamma)$ enclosed by γ:

$$A(\gamma) := \frac{1}{2} \left| \int_\gamma x\,dy - y\,dx \right|$$

$$= \frac{1}{2} \left| \int_a^b \gamma_1 \gamma_2' - \gamma_1' \gamma_2 \right|$$

- The length and area above is independent of parametrisation.
- We assume the orientation is such that it moves at a constant speed, i.e., $|\gamma'|$ is constant. (Can show such a parameterisation exists.)
- We will assume $\gamma : [0, 1] \to \mathbb{R}^2$.

Write $\gamma(t) = (x(t), y(t))$ and the speed as u, i.e.,

$$(x'(t))^2 + (y'(t))^2 = u^2 \qquad \forall\, t \in [0,1].$$

· Let the length of curve $l(\gamma)$ be L.
We have $\gamma(0) = \gamma(1)$. Thus, we can extend it periodically.

Note that
$$L = l(\gamma) = \int_0^1 |\gamma'(t)|\,dt = \int_0^1 u\,dt = u.$$

Thus,
$$|x'(t)|^2 + |y'(t)|^2 = L^2. \qquad \text{— (1)}$$

$t \mapsto x(t)$ is periodic, $x(0) = x(1)$

$t \mapsto y(t)$ ——"——, $y(0) = y(1)$

x, y are both C^1

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t} \quad \forall t \in [0,1]$$

($x \in C^1$ and thus, it satisfies Lipschitz condition for Dini's)

Also,
$$y(t) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n t} \quad \forall t \in [0,1]$$

The convergence is uniform and one can show

$$x'(t) = \sum_{n=-\infty}^{\infty} (2\pi i)\, a_n n e^{2\pi i n t}$$

$$y'(t) = \sum_{n=-\infty}^{\infty} (2\pi i)\, b_n n e^{2\pi i n t}$$

$$\int_0^1 |x'(t)|^2\,dt = \sum_{n=-\infty}^{\infty} 4\pi^2 |a_n|^2 n^2 \qquad \text{(by Parseval)}$$

$$\text{lly} \quad \int_0^1 |y'(t)|^2\,dt = \sum_{n=-\infty}^{\infty} 4\pi^2 |b_n|^2 n^2$$

$$\therefore \int_0^1 L^2 dt = L^2 = 4\pi^2 \cdot \sum_{n=-\infty}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2 \right) \qquad \text{[by (1)]}$$

$$\text{or} \quad \sum_{n=-\infty}^{\infty} \left(|a_n|^2 + |b_n|^2 \right) = \frac{L^2}{4\pi^2}. \qquad \text{— (2)}$$

$$\therefore A = \frac{1}{2} \left| \int_0^1 x(t)\, y'(t)\, dt - \int_0^1 y(t)\, x'(t)\, dt \right| \qquad \text{Parseval}$$

$$= \frac{1}{2} \left| \sum_{n=-\infty}^{\infty} a_n \overline{2\pi i n b_n} - \sum_{n=-\infty}^{\infty} b_n \overline{2\pi i n a_n} \right|$$

$$= \pi \left| \sum_{n=-\infty}^{\infty} n \left(a_n \bar{b}_n - \bar{a}_n b_n \right) \right|$$

Now,

$$L^2 - 4\pi A = 4\pi^2 \left[\sum_{n=-\infty}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2 \right) - \left| \sum_{n=-\infty}^{\infty} n \left(a_n \bar{b}_n - b_n \bar{a}_n \right) \right| \right]$$

$$|n| \, |a_n \bar{b}_n - b_n \bar{a}_n| \le 2 |a_n| \, |b_n| \, |n| \le 2n^2 |a_n| \, |b_n|$$

$$\therefore L^2 - 4\pi A \geq 4\pi^2 \left[\sum_{n=-\infty}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2 - 2 |a_n| |b_n| \right) \right]$$

$$= 4\pi^2 \left[\sum_{n=-\infty}^{\infty} n^2 \left[|a_n| - |b_n| \right]^2 \right]$$

$$\therefore L^2 - 4\pi A \geq 0 \quad \text{or} \quad A \le \frac{L^2}{4\pi}.$$

Moreover, the equality holds as can be seen by
$\gamma : [0,1] \to \mathbb{R}^2$ given by $t \mapsto \left(\frac{L}{2\pi} \cos 2\pi t, \frac{L}{2\pi} \sin 2\pi t \right)$.

When else?

Note that if $|n| > 2$, then $n \geq 2$, $|n| < n^2$.

Thus, $L^2 = 4\pi A \implies a_n = b_n = 0$.

Thus, $x(t) = a_{-1} e^{-2\pi i t} + a_0 + a_1 e^{2\pi i t}$

$y(t) = b_{-1} e^{-2\pi i t} + b_0 + b_1 e^{2\pi i t}$

$$y(t) = b_{-1} e^{-2\pi i t} + b_0 + b_1 e^{2\pi i t}$$

Note $x(t) = \overline{x(t)}$ and $y(t) = \overline{y(t)}$ $\forall t \in [0,1]$.

$\therefore$ $\overline{a_{-1}} = a_1$ and $\overline{b_{-1}} = b_1$

Also $L^2 = 4\pi^2 \sum_{n=-1}^{1} n^2(|a_n|^2 + |b_n|^2)$

$$= 4\pi^2 \, 2(|a_1|^2 + |b_1|^2) = 8\pi^2(|a_1|^2 + |b_1|^2)$$

$$\Rightarrow |a_1|^2 + |b_1|^2 = \frac{L^2}{8\pi^2} \quad \text{— (3)}$$

Also, $L^2 - 4\pi A = 0$ and thus

$$|a_1|^2 + |b_1|^2 - |a_1\bar{b}_1 - b_1\bar{a}_1| = 0 \quad \text{— (4)}$$

$$\Rightarrow 0 \geq (|a_1| - |b_1|)^2$$

Thus, $|a_1| = |b_1|$ and hence, (3) tells us that

$$|a_1|^2 = \frac{L^2}{16\pi^2} \quad \text{or} \quad |a_1| = |b_1| = \frac{L}{4\pi}$$

$$\therefore a_1 = \frac{L}{4\pi} e^{i\theta}, \quad b_1 = \frac{L}{4\pi} e^{i\varphi}$$

for some $\theta, \varphi \in [0, 2\pi]$.

Put the above in (4) to get

$$\frac{L^2}{8\pi^2} = |a_1\bar{b}_1 - b_1\bar{a}_1| = \frac{L^2}{16\pi^2}\left|e^{i(\theta-\varphi)} - e^{i(\varphi-\theta)}\right|$$

$$\Rightarrow \left|e^{i(\theta-\varphi)} - e^{i(\varphi-\theta)}\right| = 2$$

$$\Rightarrow |2\sin(\theta-\varphi)| = 2$$

$$\Rightarrow |\sin(\theta-\varphi)| = 1 \quad \text{or} \quad \theta - \varphi = (2k+1)\frac{\pi}{2}$$

2

$k \in \mathbb{Z}$

$$\therefore \quad a_1 = \frac{L}{4\pi} e^{i\varphi + (2k+1)\frac{\pi}{2}} \quad \text{and} \quad b_1 = \frac{L}{4\pi} e^{i\varphi}$$

$$\therefore \quad a_1 = \pm i \frac{L}{4\pi} e^{i\varphi} \quad \text{and} \quad b_1 = \frac{L}{4\pi} e^{i\varphi}$$

Thus,
$$\begin{aligned} x(t) &= a_0 + a_1 e^{2\pi i t} + \bar{a}_1 e^{-2\pi i t} \\ &= a_0 \pm i \left(\frac{L}{4\pi} e^{i\varphi + 2\pi i t} - \frac{L}{4\pi} e^{-i\varphi - 2\pi i t} \right) \\ &= a_0 \pm i \frac{L}{4\pi} \left(2 i \sin(\varphi + 2\pi t) \right) \\ &= a_0 \mp \frac{L}{2\pi} \sin(\varphi + 2\pi t) \end{aligned}$$

Similarly,
$$y(t) = b_0 + \frac{L}{2\pi} \cos(\varphi + 2\pi t)$$

Thus, we only get a circle, at the end, of radius $\frac{L}{2\pi}$.

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Lecture 9 (05-02-2021)

05 February 2021 09:31

Fourier Transform

$L^1(\mathbb{T}) \supseteq L^p(\mathbb{T})$ for $p \geq 1$.

No relation as above if we replace $\mathbb{T}$ with $\mathbb{R}^n$.

- We will work in $L^p(\mathbb{R}^n)$, $p \geq 1$.

- Fourier transform:

$$L^1(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \text{ measurable} \mid \int_{\mathbb{R}^n} |f| < \infty \}$$

$$\|f\|_1 = \|f\|_{L^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |f| \, dx.$$

Similarly, $$\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p \right)^{1/p}.$$

- $C_c(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ are dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Defn. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) \, e^{-2\pi i \langle x, \xi \rangle} \, dx$$

for all $\xi \in \mathbb{R}^n$.

Note $\langle x, \xi \rangle = x_1 \xi_1 + \cdots + x_n \xi_n$ where $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$.

Since $f \in L^1$, the above integral does exist since the integrand is absolutely integrable.

Thus, for $f \in \mathcal{L}^1(\mathbb{R}^n)$, we have $\hat{f} : \mathbb{R}^n \to \mathbb{C}$.

Q. Is $\hat{f}$ continuous?

Note. (i) $\widehat{(\alpha f + \beta g)}(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi)$ $\quad \forall f, g \in \mathcal{L}^1(\mathbb{R}^n)$,
$\forall \alpha, \beta \in \mathbb{C}$, $\forall \xi \in \mathbb{R}^n$

(ii) $$|\hat{f}(\xi)| \le \left| \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx \right| \le \int_{\mathbb{R}^d} |f| \, dx = \|f\|_1.$$

$\therefore \|\hat{f}\|_\infty \le \|f\|_1$

$\Rightarrow \hat{f}$ is a bounded function

(iii) $$\lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0 \quad \text{for } f \in \mathcal{L}^1(\mathbb{R}^n).$$

(Riemann-Lebesgue Lemma)

Proof. $n = 1$:
$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx \qquad -(1)$$
$$= -\int_{\mathbb{R}} f(x) e^{-2\pi i \left(x + \frac{1}{2\xi}\right)\xi} dx$$

$$= -\int_{\mathbb{R}} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i x \xi} dx \qquad -(2)$$

By (1) and (2):

$$\hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}} \left(f(x) - f\left(x - \frac{1}{2\xi}\right) \right) e^{-2\pi i x \xi} dx$$

$$\Rightarrow |\hat{f}(\xi)| \le \frac{1}{2} \int_{\mathbb{R}} \left| f(x) - f\left(x - \frac{1}{2\xi}\right) \right| dx$$

(Want to take $\lim\limits_{|\xi|\to\infty}$ inside. Not sure if possible.)

• Let $f \in \mathcal{C}_c(\mathbb{R})$. Let $\operatorname{supp} f \subseteq [-M, M]$.

$$\left(\begin{array}{l} \Rightarrow f(x) = 0 \text{ if } x > M \\ \Rightarrow f\left(x - \frac{1}{2\xi}\right) = 0 \text{ if } x > M + \frac{1}{2\xi} \end{array}\right)$$

Then, $$|\hat{f}(\xi)| \le \frac{1}{2} \int_{\mathbb{R}} \left|f(x) - f\left(x - \tfrac{1}{2\xi}\right)\right| dx$$

$$\le \frac{1}{2} \int_{-M-\left|\frac{1}{2\xi}\right|}^{M+\left|\frac{1}{2\xi}\right|} \left| f(x) - f\left(x - \tfrac{1}{2\xi}\right)\right| dx$$

if $|\xi| > 1$ $$\le \frac{1}{2} \int_{-M-1}^{M+1} \left| f(x) - f\left(x - \tfrac{1}{2\xi}\right)\right| dx$$

$$\therefore \lim_{|\xi|\to\infty} |\hat{f}(\xi)| \le \frac{1}{2} \lim_{|\xi|\to\infty} \int_{-M-1}^{M+1} \left|f(x) - f\left(x - \tfrac{1}{2\xi}\right)\right| dx$$

DCT compact domain

$$= \frac{1}{2} \int_{-M-1}^{M+1} \lim_{|\xi|\to\infty} \left|f(x) - f\left(x - \tfrac{1}{2\xi}\right)\right| dx$$

f is continuous

$$= 0$$

$\therefore$ if $f \in \mathcal{C}(\mathbb{R})$, then $\lim\limits_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$.

Let $f \in \mathcal{L}^1(\mathbb{R})$, for any $\varepsilon > 0$, $\exists\, g \in \mathcal{C}_c(\mathbb{R})$ s.t.

$\|g - f\|_1 < \varepsilon/2$. ($\mathcal{C}_c(\mathbb{R})$ is dense in $\mathcal{L}^1(\mathbb{R})$)

$\therefore |\hat{f}(\xi)| \le |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)|$

$$\le |\widehat{f-g}(\xi)| + |\hat{g}(\xi)|$$
$$\le \|f-g\|_1 + |\hat{g}(\xi)|$$
$$|\hat{f}(\xi)| \le \varepsilon/2 + |\hat{g}(\xi)|$$

Choose $M > 0$ s.t. $|\xi| > M \Rightarrow |\hat{g}(\xi)| < \varepsilon/2$.

Then,
$$|\hat{f}(\xi)| < \varepsilon \qquad \forall\, \xi \text{ s.t. } |\xi| > M. \qquad \boxtimes$$

Ex. Try for $\mathbb{R}^n$?

(iv) $f, g \in \mathcal{L}^1(\mathbb{R}^n) \Rightarrow f * g \in \mathcal{L}^1(\mathbb{R}^n)$

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-t)\, g(t)\, dt$$

Then, $\widehat{f*g}$ makes sense and we have

$$\widehat{(f*g)}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi) \quad \text{for } \xi \in \mathbb{R}^n. \qquad \text{(Ex.)}$$

(v) $f \in \mathcal{L}^1(\mathbb{R}^n)$. Translation of f by $h \in \mathbb{R}^n$:

$$(T_h f)(x) := f(x+h)$$

$f \in \mathcal{L}^1(\mathbb{R}^n) \Rightarrow (T_h f) \in \mathcal{L}^1(\mathbb{R}^n)$

In fact, $\|f\|_1 = \|T_h f\|_1$

$$\widehat{T_h f}(\xi) = e^{2\pi i \xi \cdot h}\, \hat{f}(\xi) \qquad (\xi \cdot h := \langle \xi, h \rangle)$$

(Do a change of variable.)

(vi) Fix $h \in \mathbb{R}^n$.

Given $f \in \mathcal{L}^1(\mathbb{R}^n)$, define $g(x) := f(x)\, e^{2\pi i h \cdot x}$.

Then, $\hat{g}(\xi) = \hat{f}(\xi - h)$

$$= \tau_{-h}(\hat{f})(\xi) \qquad \left(\text{Not } \widehat{\tau_{-h} f}(\xi)!\right)$$

(vii) $n=1$. Suppose $f \in C_c^\infty$ - function.

Then,

$$\widehat{f'}(\xi) = (2\pi i \xi)\, \hat{f}(\xi).$$

Proof. $$\widehat{f'}(\xi) = \int_{\mathbb{R}} f'(x)\, e^{-2\pi i x \xi}\, dx$$

$$= e^{-2\pi i x \xi} f(x)\Big|_{-\infty}^{\infty} + 2\pi i \xi \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx$$

$f \in C_c(\mathbb{R})$

$$= 0 + 2\pi i \xi\, \hat{f}(\xi)$$

Similarly, $$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = (2\pi i \xi_j)\, \hat{f}(\xi).$$

(viii) Let $f \in C_c^\infty(\mathbb{R}^n)$.

Then, $$\left(\widehat{-2\pi i x f}\right)(\xi) = (\hat{f})'(\xi).$$

$$(\hat{f})'(\xi) = \lim_{h \to 0} \frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h}$$

$$\overset{?}{=} \lim_{h \to 0} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i x \xi}}{h}\right] dx$$

$$= \int_{-\infty}^{\infty} f(x) \lim_{h \to 0} \left[\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i x \xi}}{h}\right] dx$$

$$= \int_{-\infty}^{\infty} f(x)(-2\pi i x)\, e^{-2\pi i \xi x}\, dx = \widehat{(-2\pi i x f(x))}(\xi)$$

Justification of taking lim inside: we DCT

Take $$g_h(x) = \left(\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i x \xi}}{h}\right) f(x)$$

$-2\pi i h$ has finite limit

Then, $|g_n(x)| = |f(x)| \left| \frac{e^{-2\pi i h} - 1}{h} \right| \leq M \cdot |f(x)|$ has finite limit

$\hookrightarrow \in \mathcal{L}^1$.

- Let $f(x) := e^{-\pi t |x|^2}$ for $x \in \mathbb{R}^n$. ($t > 0$ fixed)

Is $f \in \mathcal{L}^1(\mathbb{R}^n)$?

Any $x \in \mathbb{R}^n \setminus \{0\}$ can be written (uniquely) as $x = r\omega$, $0 < r < \infty$ and $\omega \in S^{n-1}$.

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \left(\int_{S^{n-1}} f(r\omega)\,d\omega \right) r^{n-1}\,dr$$

$$\therefore \int_{\mathbb{R}^n} e^{-\pi t |x|^2}\,dx = \int_0^\infty \int_{S^{n-1}} e^{-\pi t r^2}\,d\omega\; r^{n-1}\,dr$$

$$= \underbrace{\left(\int_{S^{n-1}} 1 \cdot d\omega \right)}_{<\infty} \underbrace{\int_0^\infty e^{-\pi t r^2} r^{n-1}\,dr}_{<\infty}$$

$$\Big(\underbrace{r^{n-1} e^{-\frac{\pi t r^2}{2}}}_{< C \text{ after } r > M} \Big) e^{-\pi t r^2/2}$$

- $n = 1$. $f(x) = e^{-\pi t x^2}$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi t x^2} \cdot e^{-2\pi i x \xi}\,dx$$

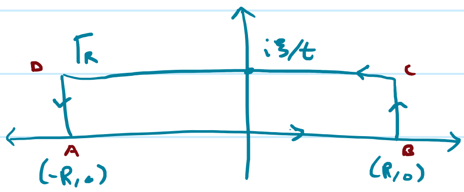
$$= \int_{\mathbb{R}} e^{-\pi t \left(x^2 + \frac{2 i \xi}{t} \cdot x \right)}\,dx$$

$$= \int_{\mathbb{R}} \exp\left(-\pi t \left[\left(x + \frac{i\xi}{t} \right)^2 - \left(\frac{i\xi}{t} \right)^2 \right] \right) dx$$

$$\cdots \int \; e^{-\pi t \left(x + \frac{i\xi}{t} \right)^2}$$

$$= \exp\left(-\frac{\pi \xi^2}{t}\right) \int_{\mathbb{R}} e^{-\pi t \left(x + \frac{i\xi}{t}\right)^2} dx$$

Put $g(z) = e^{-\pi t z^2}$. $\leftarrow$ entire



(assuming $\xi > 0$)

$$\therefore \int_{\Gamma_R} g = 0.$$

$$\therefore \int_A^B g + \int_B^C g + \int_C^D g + \int_D^A g = 0$$

$$\Rightarrow \int_{-R}^{R} e^{-\pi t x^2} dx + \int_0^{\xi/t} e^{-\pi t (R + iy)^2} dy + \int_R^{-R} e^{-\pi t \left(x + \frac{i\xi}{t}\right)^2} dx$$

$$+ \int_{\xi/t}^{0} e^{-\pi t (-R + iy)^2} dy = 0$$

0 as $R \to \infty$

Thus, $$\int_{\mathbb{R}} e^{-\pi t x^2} dx = \int_{\mathbb{R}} e^{-\pi t \left(x + \frac{i\xi}{t}\right)^2} dx$$

$$\therefore \hat{f}(\xi) = \exp\left(-\frac{\pi \xi^2}{t}\right) \int_{-\infty}^{\infty} e^{-\pi t x^2} dx$$

$$= \exp\left(-\frac{\pi \xi^2}{t}\right) \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{\sqrt{\pi t}} dy \qquad x = \frac{1}{\sqrt{\pi t}} y$$

$$= \exp\left(-\frac{\pi \xi^2}{t}\right) \frac{\sqrt{\pi}}{\sqrt{\pi t}}$$

$$= \frac{1}{\sqrt{t}} \exp\left(-\frac{\pi \xi^2}{t}\right)$$

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For $\mathbb{R}^n$: $\quad f(x) = e^{-\pi t \|x\|^2}$ $\quad t>0$ fixed

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\pi t (x_1^2 + \dots + x_n^2)} e^{-2\pi i (x_1\xi_1 + \dots + x_n \xi_n)} \, d(x_1, \dots, x_n)$$

$$= \left(\int_{\mathbb{R}} e^{-\pi t x_1^2 - 2\pi i x_1 \xi_1} dx_1 \right) \cdots \left(\int_{\mathbb{R}} e^{-\pi t x_n^2 - 2\pi i x_n \xi_n} dx_n \right)$$

$$= \frac{1}{(\sqrt{t})^n} e^{-\frac{\pi}{t}|\xi|^2}$$

$C_c^\infty(\mathbb{R}^n)$: collection of C^∞ compactly supported functions

$L^p(\mathbb{R}^n) \supseteq C_c^\infty(\mathbb{R}^n)$ dense

Schwartz class functions

$\underline{n=1}$:

$$S(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \ : \ f \in C^\infty \text{ and } \underbrace{\sup_{x \in \mathbb{R}} (1+|x|)^m |f^{(n)}(x)|}_{\rho_{m,n}(f)} < \infty \ \ \forall\, m, n \in \mathbb{N} \cup \{0\} \right\}$$

Clearly $C_c^\infty(\mathbb{R}) \subseteq S(\mathbb{R})$.

Moreover, $S(\mathbb{R}) \subset L^p(\mathbb{R}) \quad \forall\, p \geq 1$

Proof.

Take $n = 0$ and $m > 2p$.

Then,

$$\sup_{x \in \mathbb{R}} (1+|x|)^m |f(x)| = M < \infty$$

$$\therefore \quad |f(x)|^p \leq \frac{M^p}{(1+|x|)^{mp}}$$

$$(1+|x|)^{mp}$$

$$\Rightarrow \int |f|^p < \infty. \qquad \blacksquare$$

$\therefore \quad C_c^\infty(\mathbb{R}) \subseteq S(\mathbb{R}) \subseteq L^p(\mathbb{R})$

$\because \quad C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, so is $S(\mathbb{R})$.

- $$\rho_{m,n}(f) = \sup_{x\in\mathbb{R}} (1+|x|)^m |f^{(n)}(x)|$$

is a seminorm on $S(\mathbb{R})$

(Topology on $S(\mathbb{R})$ is generated by the seminorms $\rho_{m,n}$. These seminorms are countable and hence, $S(\mathbb{R})$ is metrisable space.)

Convergence on $S(\mathbb{R})$:

Let $(f_j)_j$ be a sequence in $S(\mathbb{R})$ and $f \in S(\mathbb{R})$.

$$f_j \to f \text{ in } S(\mathbb{R}) \iff \rho_{m,n}(f_j - f) \to 0 \text{ as } j \to 0$$
$$\forall m,n \in \mathbb{N}\cup\{0\}$$

($(x \mapsto e^{-|x|}) \notin S(\mathbb{R})$ since not in C^∞)

- General n: If $x = (x_1, \dots, x_n)^{\in\mathbb{R}^n}$ and $\alpha = (\alpha_1, \dots, \alpha_n)^{\in(\mathbb{N}_0)^n}$,

then $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and

$$D^\alpha f(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x).$$

$$S(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{C} \;:\; f \in C^\infty \text{ and } \underbrace{\sup_{x\in\mathbb{R}^n} |x^\alpha||D^\beta f(x)|}_{``\rho_{\alpha,\beta}(f)"} < \infty \quad \forall \alpha, \beta \in (\mathbb{N}_0)^n \right\}.$$

As before, $C_c^\infty(\mathbb{R}^n) \subseteq S(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n) \quad \forall p \geq 1.$

As before, $C_c^\infty(\mathbb{R}^n) \subseteq \underbrace{S(\mathbb{R}^n) \subsetneq}_{\text{dense}} L^p(\mathbb{R}^n) \quad \forall p \geq 1.$

We defined Fourier Transform for $L^1 \supseteq S$.

What if $f \in S(\mathbb{R}^n)$? What more can we say?

- $f \in L^1 \Rightarrow \hat{f} \in C_0(\mathbb{R}^n) = \{ g \text{ continuous and } \lim_{|x| \to \infty} g(x) = 0 \}$.

had seen this

Prop^n. If $f \in S(\mathbb{R})$, then $\hat{f} \in S(\mathbb{R})$.

Proof.

$$\sup_{\xi \in \mathbb{R}} (1 + |\xi|)^m \left| \frac{d^n}{d\xi^n} \hat{f}(\xi) \right| \qquad \left(\frac{d}{d\xi} \hat{f}(\xi) = C \cdot \widehat{\xi f}(\xi) \right)$$

$$= C \cdot \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^m \left| \widehat{x^n f}(\xi) \right|$$

$$\leq C \cdot \sup_{\xi \in \mathbb{R}} \widehat{(x^n \cdot f)^{(m)}}(\xi)$$

If $(x^n f)^{(m)} \in L^1$, then we are done, by Riemann-Lebesgue.

$$\int_{-\infty}^{\infty} |(x^n f)^{(m)}| \, dx \leq \sum_{m_1 + m_2 = m} \int_{-\infty}^{\infty} |(x^n)^{(m_1)} f^{(m_2)}(x)| \, dx \underbrace{<}_{f \in S(\mathbb{R})} \infty$$

- Fourier Series: If $f \in C^1(\mathbb{T})$, then

$$f(x) = \sum \hat{f}(n) e^{2\pi i n x}.$$

inversion formula

- Fourier inversion for Schwartz class functions.

- Step 1 : $f, g \in S(\mathbb{R})$, then

$$\int f \hat{g} = \int \hat{f} \cdot g.$$

(Note: $f, g \in S(\mathbb{R}) \Rightarrow fg \in S(\mathbb{R})$)

Proof. $\displaystyle\int_{\mathbb{R}} f(x)\, \hat{g}(x)\, dx$

$$= \int_{x\,\mathbb{R}} f(x) \int_{t\,\mathbb{R}} g(t)\, e^{-2\pi i t x}\, dt\, dx$$

(*) Fubini

$$= \int_{t\,\mathbb{R}} g(t) \int_{x\,\mathbb{R}} f(x)\, e^{-2\pi i t x}\, dx\, dt$$

$$= \int_{\mathbb{R}} g(t)\, \hat{f}(t)\, dt.$$

(*) Fubini justification:

Note

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left| f(x)\, g(t)\, e^{-2\pi i t x} \right| dt\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||g(t)|\, dt\, dx$$

$$= \int_{\mathbb{R}} |f(x)|\, dx \int_{\mathbb{R}} |g(t)| dt < \infty.$$

$\therefore$ Fubini is indeed applicable. $\boxtimes$

- Step 2. $\displaystyle\int_{\mathbb{R}} f\left(\frac{x}{\lambda}\right) \hat{g}(x)\, dx = \int_{\mathbb{R}} \hat{f}(x)\, g\left(\frac{x}{\lambda}\right) dx$ (Ex.)

Take $\lambda \to \infty$ on both sides.

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} f\left(\frac{x}{\lambda}\right) \hat{g}(x)\, dx = \lim_{\lambda\to\infty} \int_{\mathbb{R}} \hat{f}(x)\, g\left(\frac{x}{\lambda}\right) dx$$

use DCT

$$\Rightarrow \int_{\mathbb{R}} \lim_{\lambda\to\infty} f\left(\tfrac{x}{\lambda}\right)\hat{g}(x)\,dx = \int_{\mathbb{R}} \lim_{\lambda\to\infty} \hat{f}(x)\, g\left(\tfrac{x}{\lambda}\right) dx$$

use DCT

$f, g \in S(\mathbb{R})$ and hence, continuous

$$\Rightarrow \int_{\mathbb{R}} f(0)\,\hat{g}(x)\,dx = \int_{\mathbb{R}} \hat{f}(x)\,g(0)\,dx$$

$$\Rightarrow f(0)\int_{\mathbb{R}} \hat{g}(x)\,dx = g(0)\int_{\mathbb{R}} \hat{f}(x)\,dx$$

The above is true for all $f, g \in S(\mathbb{R})$.

Take $g(x) = e^{-\pi x^2}$. $\hat{g}(\xi) = e^{-\pi \xi^2}$.

$$\therefore f(0)\underbrace{\int_{\mathbb{R}} e^{-\pi x^2}\,dx}_{=1} = \int_{\mathbb{R}} \hat{f}(x)\,dx$$

$$\therefore f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\,d\xi. \qquad \text{—— (1)}$$

• Step 3. Fix $x \in \mathbb{R}$. Let $\tau_x f(y) := f(x+y)$.
$f \in S(\mathbb{R}) \Rightarrow \tau_x f \in S(\mathbb{R})$

Use (1) for $\tau_x f$.

$$\tau_x f(0) = \int_{\mathbb{R}} \widehat{\tau_x f}(\xi)\,d\xi$$

$$\Rightarrow f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\,d\xi \qquad \text{for all } f \in S(\mathbb{R}).$$

(Inversion formula for S(R))

$$\left.\begin{aligned} (\hat{\hat{f}})(-x) &= f(x) \\ (\hat{\hat{\hat{\hat{f}}}})(x) &= f(x) \end{aligned}\right\} f \in S(\mathbb{R})$$

The above calculations go through for $\mathbb{R}^n$ as well.

· Inversion formula for L^1 function:

Thm. Let $f \in L^1(\mathbb{R})$ be such that $\hat{f} \in L^1(\mathbb{R})$.
Then,
$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \qquad \text{a.e. } x.$$

Proof. $h_t(x) = \widehat{e^{-\pi t x^2}} = \dfrac{1}{t^{1/2}}\, e^{-\frac{\pi}{t}x^2}$.

heat kernel

Steps. 1. $(f * h_t)(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{-\pi t \xi^2}\, e^{2\pi i x \xi}\, d\xi.$

2. $(h_t)_{t>0}$ is an approximate identity in $L^1(\mathbb{R})$.

3. $\|f * h_t - f\|_1 \longrightarrow 0$ as $t \longrightarrow 0$

4. Then, $\exists$ a subsequence $(t_n)_n$ s.t.
$(t_n \to 0)$

$$f * h_{t_n}(x) \longrightarrow f(x) \qquad \text{a.e. } x.$$

from measure theory

$$\| $$
$$\int \hat{f}(\xi)\, e^{-\pi t_n |\xi|^2}\, e^{2\pi i x \xi}\, d\xi \longrightarrow \int \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

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Thm. Fourier inversion formula for L^1

Let $f \in L^1(\mathbb{R}^n)$ be such that $\hat{f} \in L^1(\mathbb{R}^n)$. Then,

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} dx \qquad \text{a.e. } x.$$

Fact: If $\{f_n\}_n$ is in L^p and $f \in L^p$ with $\|f_n - f\|_p \to 0$, then $\exists$ a subsequence $\{f_{n_k}\}_k$ s.t.

$$f_{n_k}(x) \to f(x) \qquad \text{a.e. } x.$$

Proof (of thm). Define $h_t(x) := \dfrac{1}{t^{n/2}} e^{-\frac{\pi}{t}|x|^2}$; $t > 0$

heat kernel

Note $\hat{h}_t(\xi) = e^{-\pi|\xi|^2}$.

Step 1. Claim: $\displaystyle\int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t|\xi|^2} e^{2\pi i x\cdot\xi} d\xi = (f * h_t)(x).$

Proof.

$$\int_{\xi\,\mathbb{R}^n}\left(\int_{y\,\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi} dy\right) e^{-\pi t|\xi|^2} e^{2\pi i x\cdot\xi} d\xi$$

(*) Fubini

$$= \int_{y\,\mathbb{R}^n} f(y) \int_{\xi\,\mathbb{R}^n} e^{-\pi t|\xi|^2} e^{-2\pi i (y-x)\cdot\xi} d\xi\, dy$$

$$= \int_{y\,\mathbb{R}^n} f(y) \left(\widehat{\xi \mapsto e^{-\pi t|\xi|^2}}\right)(y-x)\, dy$$

$$= \int_{y\,\mathbb{R}^n} f(y) \frac{1}{t^{n/2}} e^{-\frac{\pi}{t}|y-x|^2} dy$$

$$= \int_{\mathbb{R}^n} f(y) h_t(x-y)\, dy = (f * h_t)(x). \qquad \blacksquare$$

Justification of Fubini:

Justification of Fubini:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| f(y)\, e^{-\pi t|\xi|^2} e^{-2\pi i (y-x)\cdot\xi} \right| d\xi\, dy$$

$$= \underbrace{\left(\int_{\mathbb{R}^n} f(y)\,dy\right)}_{f\in L^1}\underbrace{\left(\int_{\mathbb{R}^n} e^{-\pi t|\xi|^2}\right)}_{\text{very nice function}} < \infty$$

Step 2 **Claim**: $\{h_t\}_{t>0}$ is an approximate identity in $L^1(\mathbb{R}^n)$.

Proof.

(1) $h_t(x) \geq 0 \quad \forall x$ ✓

(2)
$$\int_{\mathbb{R}^n} h_t(x)\,dx = \frac{1}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\pi/t\,|x|^2}\,dx$$

$$= \frac{1}{t^{n/2}}\, \widehat{e^{-\pi/t|x|^2}}(0)$$

$$= \frac{1}{t^{n/2}} \cdot t^{n/2} \cdot e^{-\pi t(0)^2} = 1.$$

(3) Fix $\delta > 0$.

$$\lim_{t\to 0} \int_{|x|>\delta} h_t(x)\,dx = \lim_{t\to 0} \frac{1}{t^{n/2}} \int_{|x|>\delta} e^{-\pi/t\,|x|^2}\,dx$$

$$= \lim_{t\to 0} \frac{1}{t^{n/2}} \int_{|x|>\delta} \exp\left(-\frac{\pi}{2t}|x|^2\right)\exp\left(-\frac{\pi}{2t}|x|^2\right) dx$$

$$\leq \lim_{t\to 0} \frac{1}{t^{n/2}} \exp\left(-\frac{\pi}{2t}\delta^2\right) \int_{|x|>\delta} \exp\left(-\frac{\pi}{2t}|x|^2\right) dx$$

for $t \leq 1$

$$\leq \lim_{t\to 0} \underbrace{\frac{1}{t^{n/2}} \exp\left(-\frac{\pi}{2t}\delta^2\right)}_{\to\, 0} \underbrace{\int_{|x|>\delta} \exp\left(-\frac{\pi}{2}|x|^2\right) dx}_{\text{finite constant, indep. of } t}$$

$$\therefore \lim_{t\to 0} \int_{|x|>\delta} h_t(x)\,dx = 0.$$

Hence, $\lim_{t\to 0} \|f * h_t - f\|_1 = 0.$

Step 3. $\exists$ a subsequence $\{t_k\}$ s.t. $f * h_{t_k}(x) \to f(x)$ a.e. x.

By Step 1, $(f * h_{t_k})(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t_k |\xi|^2} e^{2\pi i x\cdot\xi} d\xi$

Then,

$$\lim_{t_k \to 0} [f * h_{t_k}](x) = \underset{\text{DCT}}{\lim_{t_k\to 0} \int_{\mathbb{R}^n}} \hat{f}(\xi) e^{-\pi t_k |\xi|^2} e^{2\pi i x\cdot\xi} d\xi$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

[DCT: Put $g_k(\xi) = \hat{f}(\xi) e^{-\pi t_k |\xi|^2} e^{-2\pi i x \xi}$

$|g_k(\xi)| \le |\hat{f}(\xi)|$ but $\hat{f} \in L^1$, as given.

Thus, can apply DCT.]

Hence, $\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi x} dx = f(x)$ a.e. x. ⊠

Cor. If $f \in L^1(\mathbb{R}^n)$ and $\hat{f}(\xi) = 0$ a.e. x,

then $f(x) = 0$ a.e. x.

- Heat kernel

Let $u: \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ nice function.

$f \in L^1(\mathbb{R}^n)$ given.

$$\left.\begin{aligned} &\Delta u(x,t) - \frac{\partial}{\partial t} u(x,t) = 0 \\ &\& \quad u(x,0) = f(x) \end{aligned}\right\} \text{heat equation}$$

$\left(\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \quad \text{Laplacian in } \mathbb{R}^n.\right)$

Let us solve using Fourier transform :

$\cdot n=1$

$$\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial t} u(x,t) = 0 \quad (*) \; ; x \in \mathbb{R}, t>0$$

$$u(x,0) = f(x)$$

In $(*)$, take $\hat{\ }$ on both sides (in variable x) after fixing t.

$$\widehat{\left(\frac{\partial^2}{\partial x^2} u(x,t)\right)}(\xi) - \widehat{\left(\frac{\partial}{\partial t} u(x,t)\right)}(\xi) = 0$$

$\int \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \int$

$$\Rightarrow (2\pi i \xi)^2 \hat{u}(\xi, t) - \frac{\partial}{\partial t}\left(\hat{u}(\xi,t)\right) = 0$$

$$\Rightarrow -4\pi^2 \xi^2 \hat{u}(\xi,t) - \frac{\partial}{\partial t}\left(\hat{u}(\xi,t)\right) = 0$$

$\hat{u}(\xi,t)$ is the Fourier transform of $x \mapsto u(x,t)$ at ξ. (t fixed)

$$\Rightarrow \frac{\partial}{\partial t}\left(\hat{u}(\xi,t)\right) = -4\pi \xi^2 \hat{u}(\xi,t)$$

$$\Rightarrow \hat{u}(\xi,t) = c \cdot e^{-4\pi^2 \xi^2 t}$$

Take $t \to 0$.

$$\begin{aligned} c &= \hat{u}(\xi, 0) \\ &= \int_{\mathbb{R}} u(x,0)\, e^{-2\pi i x \cdot \xi}\, dx \\ &= \int f(x)\, e^{-2\pi i x \cdot \xi}\, dx \\ &= \hat{f}(\xi). \end{aligned}$$

$$\therefore \hat{u}(\xi,t) = \hat{f}(\xi)\, e^{-4\pi^2 \xi^2 t}$$

$$= \hat{f}(\xi) \underbrace{\widehat{\left(\frac{1}{(4\pi t)^{1/2}} e^{-\frac{1}{4t}|x|^2}\right)}}_{\text{call the inner part } h_t(x)}(\xi)$$

$$= \widehat{(f * h_t)}(\xi)$$

$$\therefore u(x,t) = (f * h_t)(x)$$

Let $f \in S(\mathbb{R}^n)$. $\therefore$ $f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi$.

$$\|f\|_2^2 = \int_{\mathbb{R}^n} f(x) \overline{f(x)} \, dx$$

$$= \int_{\mathbb{R}^n} f(x) \overline{\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi} \, dx$$

$$= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} e^{-2\pi i \xi \cdot x} \, d\xi \, dx$$

Fubini ($\hat{f} \in S(\mathbb{R}^n) \subset L^1$)

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx \, d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{f}(\xi) \, d\xi = \|\hat{f}\|_2^2$$

Thus, for $f \in S(\mathbb{R}^n)$, $\|f\|_2 = \|\hat{f}\|_2$.

Aim: Extend the Fourier Transform to L^2 functions

Let $f \in L^2(\mathbb{R}^n)$. Then, $\exists \{f_k\}_k \subset S(\mathbb{R}^n)$ s.t.

$$\|f_k - f\|_2 \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

(S is dense in L^p for all $1 \leq p < \infty$.)

$f_k - f_m \in S$

Thus, $\|f_k - f_m\| \to 0$ as $m, k \to \infty$

$\Rightarrow$ $\|\widehat{f_k - f_m}\| \to 0$ as $m, k \to \infty$

$\Rightarrow$ $\|\hat{f}_k - \hat{f}_m\| \to 0$ as $m, k \to \infty$

$\Rightarrow$ $\{\hat{f}_k\}_k$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.

But $L^2(\mathbb{R}^n)$ is complete. Thus, $\exists g \in L^2(\mathbb{R}^n)$ s.t.

$$\hat{f}_k \longrightarrow g \quad \text{in } L^2.$$

By definition, we define

$$\hat{f} := g = \lim_{k\to\infty} \hat{f}_k.$$

↳ limit in L^2, not pointwise!

Q Is the above well-defined?

Suppose $\exists \{f_k\}$ and $\{F_k\} \subset S(\mathbb{R}^n)$ s.t.

$$\|f_k - f\|_2 \to 0 \quad \& \quad \|F_k - f\|_2 \to 0.$$

Then, both $\{\hat{f}_k\}$ and $\{\hat{F}_k\}$ are Cauchy, as earlier.

Let $g = \lim \hat{f}_k$ and $G = \lim \hat{F}_k$ (in L^2).

$$\|G - g\|_2 = \lim_k \|\hat{F}_k - \hat{f}_k\|_2$$

$$= \lim_k \|F_k - f_k\|_2 = \|f - f\|_2 = 0.$$

$\therefore G = g$ in L^2. ∎

- If $f \in S(\mathbb{R}^n)$, $\|f\|_2 = \|\hat{f}\|_2$.

Now if $f \in L^2$, $\|f\| = \|\hat{f}\|_2$. (Plancherel Theorem)

↳ $\|\hat{f}\| = \lim \|\hat{f}_k\| = \lim \|f_k\| = \|f\|$.

- $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

$$f \longmapsto \hat{f}$$

is an isometry and onto.

↳ fourier inversion

In fact, $\mathcal{F} : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ is also 1-1, onto, iso.

- $\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$.

Is $\mathcal{F}$ onto? No!

next class

- Paley Wiener Theorem: what is image of $C_c^\infty(\mathbb{R}^n)$ under $\mathcal{F}$

No characterisation of $\mathcal{F}(L^1(\mathbb{R}^n))$ so far!

Lecture 12 (17-02-2021)

17 February 2021 09:32

Paley-Wiener Theorem - We will characterise the image of $C_c^\infty(\mathbb{R}^n)$ under the Fourier transform.

· Let $n = 1$: Fix $R > 0$.

$$PW_R(\mathbb{C}) = \left\{ h : \mathbb{C} \to \mathbb{C} \text{ entire} : \sup_{\lambda \in \mathbb{C}} (1+|\lambda|)^m |h(\lambda)| e^{-2\pi R \cdot |\operatorname{Im} \lambda|} < \infty \quad \forall m \in \mathbb{N} \cup \{0\} \right\}$$

If $h \in PW_R(\mathbb{C})$, $|h(\lambda)| \le \dfrac{C_m}{(1+|\lambda|)^m} e^{2\pi R |\operatorname{Im} \lambda|} \quad \forall \lambda \in \mathbb{C}$

note this means that $h|_{\mathbb{R}} \in \mathcal{S}(\mathbb{R})$

$$PW(\mathbb{C}) := \bigcup_{R>0} PW_R(\mathbb{C}).$$

Paley - Wiener Space

Thm. $\mathcal{F} : C_c^\infty(\mathbb{R}) \to PW(\mathbb{C})$ is an isomorphism.

Proof. Let $f \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp} f \subseteq [-R, R]$

$\left(\operatorname{supp} h = \overline{\{x : h(x) \neq 0\}}.\right)$

· First, we show that $\hat{f} \in PW_R(\mathbb{C}) \subset PW(\mathbb{C})$.

- We show $\hat{f}$ is entire.

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \lambda} \, dx$$

Note that $\hat{f}(\lambda)$ exists for $\lambda \in \mathbb{C}$ since f is compactly supported.

To show $\hat{f}$ is entire, we use Morera's theorem.

Let Γ be a triangle in $\mathbb{C}$. Then,

$\int_\Gamma \hat{f} \ldots = \int \int f(\ldots) e^{-2\pi i \lambda x} \ldots$

$$\int_\Gamma \hat{f}(\lambda)\,d\lambda = \int_\Gamma\int_{\mathbb{R}} f(x)e^{-2\pi i\lambda x}\,dx\,d\lambda$$

$$= \int_{\mathbb{R}} f(x)\int_\Gamma e^{-2\pi i\lambda x}\,d\lambda\,dx$$

$\lambda \mapsto e^{-2\pi i\lambda x}$ is holomorphic

$$= \int_{\mathbb{R}} f(x)\cdot 0\,dx = 0.$$

$\therefore\ \hat{f}$ is entire.

- $$|\hat{f}(\lambda)|(1+|\lambda|)^m = |\hat{f}(\lambda)|\left(1+C_1|\lambda|+\cdots+C_m|\lambda|^m\right)$$

$$\le |\hat{f}(\lambda)| + C\cdot\left[|\lambda||\hat{f}(\lambda)|+\cdots+|\lambda|^m|\hat{f}(\lambda)|\right]$$

Note $\quad |\lambda^k||\hat{f}(\lambda)| = C\cdot|\widehat{f^{(k)}}(\lambda)|$

$$= C\cdot\left|\int_{\mathbb{R}} f^{(k)}(x)e^{-2\pi i x\lambda}\,dx\right|$$

$|e^z| = e^{\operatorname{Re} z}$

$\operatorname{supp} f \subseteq [-R,R]$

$\Rightarrow \operatorname{supp} f^{(k)} \subseteq [-R,R]$

$$\le C\cdot\int_{\mathbb{R}} |f^{(k)}(x)|\,e^{2\pi|x||\operatorname{Im}\lambda|}\,dx$$

$$= C\cdot\int_{-R}^{R} |f^{(k)}(x)|\,e^{2\pi|x||\operatorname{Im}\lambda|}\,dx$$

$$\le \underbrace{C\cdot\left(\int_{-R}^{R}|f^{(k)}(x)|\,dx\right)}_{C_k} e^{2\pi R|\operatorname{Im}\lambda|}$$

$$\therefore\ |\lambda^k|\cdot|\hat{f}(\lambda)| \le C_m e^{2\pi R|\operatorname{Im}\lambda|}$$

$$\Rightarrow \sup_\lambda |\lambda^k||\hat{f}(\lambda)|e^{-2\pi R|\operatorname{Im}\lambda|} < \infty$$

$$\therefore\ \sup_\lambda (1+|\lambda|)^m|\hat{f}(\lambda)|e^{-2\pi R|\operatorname{Im}\lambda|} < \infty$$

$\therefore \sup_{\lambda} (1+|\lambda|)^m |\hat{f}(\lambda)| e^{-2\pi R |\mathrm{Im}\, \lambda|} < \infty$

Thus, $\hat{f} \in PW_R(\mathbb{C})$.

- Conversely, let $h \in PW_R(\mathbb{C})$. We show that $\exists f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq [-R, R]$ s.t. $\hat{f} = h$.

Define $f(x) = \int_{\mathbb{R}} h(\lambda) e^{2\pi i x \lambda} \, d\lambda$ for $x \in \mathbb{R}$.

(The above integral exists since $|h(\lambda) e^{2\pi i x \lambda}| = |h(\lambda)| \le \frac{c}{(1+|\lambda|)^2}$.)

- Need to prove: $f \in C_c^\infty(\mathbb{R})$.

Easy to prove that f is smooth.

$$f'(x) = \lim_{\xi \to 0} \frac{f(x+\xi) - f(x)}{\xi}$$

$H_\xi(\lambda) = h(\lambda) \frac{e^{2\pi i (x+\xi)\lambda} - e^{2\pi i x \lambda}}{\xi}$

$$= \lim_{\xi \to 0} \int_{\mathbb{R}} h(\lambda) \frac{e^{2\pi i (x+\xi)\lambda} - e^{2\pi i x \lambda}}{\xi} \, d\lambda$$

$|H_\xi(\lambda)| \le c(1+|\lambda|)|h(\lambda)|$

DCT

$$= \int_{\mathbb{R}} \lim_{\xi \to 0} h(\lambda) e^{2\pi i x \lambda} \cdot \frac{e^{2\pi i \xi \lambda} - 1}{\xi} \, d\lambda$$

$$= \int_{\mathbb{R}} h(\lambda) \, 2\pi i \lambda \, e^{2\pi i x \lambda} \, d\lambda$$

$$= 2\pi i \int_{\mathbb{R}} \lambda h(\lambda) e^{2\pi i x \lambda} \, d\lambda < \infty$$

$h|_{\mathbb{R}} \in S(\mathbb{R})$

Similarly, for $f^{(k)}$, we get $\int_{\mathbb{R}} \lambda^k h(\lambda) e^{2\pi i x \lambda} \, d\lambda$.

- We now prove $\operatorname{supp} f \subseteq [-R, R]$.

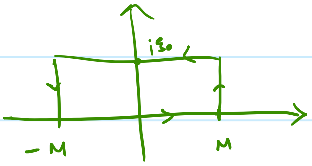
Fix $\xi_0 > 0$. We have

$$\int \quad 2\pi i x \lambda$$

$$f(x) = \int_{\mathbb{R}} h(\lambda)\, e^{2\pi i x \lambda}\, d\lambda$$

Claim $$= \int_{\mathbb{R}} h(\lambda + i\xi_0)\, e^{2\pi i x(\lambda + i\xi_0)}\, d\lambda$$

$f(\lambda) = h(\lambda) e^{2\pi i x \lambda}$



Just need to show that integrals along vertical sides go to zero as $M \to \infty$. (Function has no poles.) Is entire.

Right arm: $$R_M = \int_0^{\xi} h(M + iy)\, e^{2\pi i x (M+iy)}\, i\,dy$$

$$\Rightarrow R_M \leq \int_0^{\xi_0} |h(M+iy)|\, e^{-2\pi x y}\, dy$$

$$\leq \int_0^{\xi_0} \frac{C_m}{(1+|M+iy|)^2}\, e^{2\pi R y}\, e^{-2\pi x y}\, dy$$

$$\leq \frac{C_2}{(1+|M|)^2} \underbrace{\int_0^{\xi_0} e^{2\pi y (R-x)}\, dy}_{\text{fixed}} \longrightarrow 0 \text{ as } M \to \infty.$$

Similarly, for left $\to 0$ as $M \to \infty$.

Thus $$f(x) = \int_{\mathbb{R}} h(\lambda + i\xi_0)\, e^{2\pi i x(\lambda + i\xi_0)}\, d\lambda$$ for all $\xi_0 > 0$.

$$\therefore |f(x)| \leq \int_{\mathbb{R}} |h(\lambda + i\xi_0)|\, e^{-2\pi \xi_0 \cdot x}\, d\lambda$$

$$\leq C_3 \int_{\mathbb{R}} \frac{e^{2\pi R \xi_0}}{(1+|\lambda + i\xi_0|)^3} \cdot e^{-2\pi \xi_0 \cdot x}\, d\lambda$$

$$= C_3 \cdot e^{2\pi \xi_0 (R-x)} \int \frac{1}{(1+|\lambda + i\xi_0|)^3} \cdot d\lambda$$

$$= C_3 \cdot e^{2\pi \xi_0 (R-x)} \int_{\mathbb{R}} \frac{1}{(1+|\lambda + i\xi_0|)^3} \cdot d\lambda \quad \text{—} \ (*)$$

Now, if $x > R$, then $R - x < 0$.

However, $(*)$ holds for all $\xi_0 > 0$. Taking $\xi_0 \to \infty$ gives $|f(x)| \le 0$ and thus, f vanishes on (R, ∞).

Similarly, the above can be proven for $\xi_0 < 0$. This gives that f vanishes on $(-\infty, -R)$ as well.

Thus, $\operatorname{supp} f \subset [-R, R]$.

$\therefore \ f \in C_c^\infty(\mathbb{R})$. By Fourier inversion, $\hat{f} = h$. ∎

- Similar result for $n > 1$. One needs to know what entire for $h: \mathbb{C}^n \to \mathbb{C}^n$ means.

- Fourier transform of an L^p function, $1 < p < 2$.

If $f \in L^p$, $1 < p < 2$, then we can write

$$f = f_1 + f_2 \quad \text{for some } f_1 \in L^1 \text{ and } f_2 \in L^2.$$

Proof. Let $A = \{x : |f(x)| > 1\}$ and $B = \mathbb{R}^n \setminus A$.

Put $f_1 = f \cdot \chi_A$ and $f_2 = f \cdot \chi_B$.

$$\int_{\mathbb{R}^n} |f_1| = \int_A |f| \le \int_A |f|^p < \infty \quad \text{and thus, } f_1 \in L^1.$$

$$\int_{\mathbb{R}^n} |f_2|^2 = \int_B |f|^2 \le \int_B |f|^p < \infty \quad \text{and thus, } f_2 \in L^2.$$

$\mathbb{R}^n$ $\quad\quad$ B

Now, we define $\hat{f} := \hat{f}_1 + \hat{f}_2$.

defined pointwise $\quad$ normwise

Q. Is it well-defined?

Suppose $f = f_1 + f_2 = g_1 + g_2$ for $f_1, g_1 \in L^1$ & $f_2, g_2 \in L^2$.

$$\therefore \ f_1 - g_1 = g_2 - f_2 \in L^1 \cap L^2$$

and hence, the fourier definitions coincide and

$$\hat{f}_1 - \hat{g}_1 = \hat{g}_2 - \hat{f}_2$$

$$\Rightarrow \hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2.$$

Thm. $\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C \cdot \|f\|_{L^p(\mathbb{R}^n)}$ $\quad$ $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

$Tf = \hat{f}$ is linear.

$T: L^1 \longrightarrow L^\infty$

$\|\hat{f}\|_\infty \leq \|f\|_1$

$T: L^2 \longrightarrow L^2$

$\|\hat{f}\|_2 = \|f\|_2$

$\|Tf\|_\infty \leq 1 \cdot \|f\|_1$ $\quad$ $p_0 = 1, \ q_0 = \infty$

$\|Tf\|_2 \leq 1 \cdot \|f\|_2$ $\quad$ $p_1 = 2, \ q_1 = 2$

$$\frac{1}{p_\theta} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2}$$

$$\frac{1}{q_\theta} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \theta/2$$

$\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1$

By Riesz - Thorin interpolation theorem,

$$\|Tf\|_{q_\theta} \leq \|f\|_{p_\theta}$$

As θ varies from 0 to 1, p_θ varies from 1 to 2.
This proves the theorem. ⊠

Lecture 13 (19-02-2021)

19 February 2021 09:33

"Recall" Riesz - Thorin. $T : L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n) + L^{q_1}(\mathbb{R}^n)$

sublinear

T is strong (p_0, q_0) $\quad \|Tf\|_{q_0} \le M_0 \|f\|_{p_0}$,

— " — (p_1, q_1) $\quad \|Tf\|_{q_1} \le M_1 \|f\|_{p_1}$.

Then,

strong (p_θ, q_θ) $\quad \|Tf\|_{q_\theta} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{p_\theta}$ where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

- Young's Inequality: $\|\hat{f}\|_{p'} \le \|f\|_p \quad \forall\ 1 \le p \le 2$, $\frac{1}{p'} + \frac{1}{p} = 1$

Proof. Consider $Tf = \hat{f}$.

T is strong $(1, \infty)$ and $(2, 2)$.

Then, T is strong (p, p'). ☐

- Hausdorff Young Inequality

$f \in L^p(\mathbb{R}^n)$, $\quad g \in L^q(\mathbb{R}^n)$.

Then,

$$f * g \in L^r(\mathbb{R}^n), \quad \text{where} \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

More precisely,

$$\|f * g\|_r \le c \|f\|_p \|g\|_q$$

Recall Minkowski's integral inequality:

$$\left[\int_t \left(\int_x f(x,t)\, d\nu(x) \right)^p d\mu(t) \right]^{1/p} \le \int_x \left[\int_t f(x,t)^p\, d\mu(t) \right]^{1/p} d\nu(x)$$

$$\left[\int_{t,\mathbb{R}^n}\left(\int_{x,\mathbb{R}^n} f(x,t)\,d\nu(x)\right)^{r} d\mu(t)\right] \le \int_{x,\mathbb{R}^n}\left[\int_{t,\mathbb{R}^n} f(x,t)^{p}\,d\mu(t)\right]^{1/r} d\nu(x)$$

Proof. (Haus- Young)

Step 1. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ and $1 \le p \le \infty$.

$$\|f * g\|_p = \left(\int |f*g(x)|^p dx\right)^{1/p}$$

$$= \left(\int \left|\int f(x-y)g(y)\,dy\right|^p dx\right)^{1/p}$$

$$\le \left[\int \left(\int |f(x-y)||g(y)|\,dy\right)^p dx\right]^{1/p}$$

Minkowski

$$\le \int_y \left[\int_x |f(x-y)|^p |g(y)|^p\,dx\right]^{1/p} dy$$

$$= \int_y |g(y)| \left[\int_x |f(x-y)|^p dx\right]^{1/p} dy$$

$$= \int_y |g(y)|\, \|f\|_p\, dy = \|g\|_1 \|f\|_p$$

$$\therefore \quad \|f * g\|_p \le \|f\|_p \|g\|_1 \quad \text{— } (1)$$

Step 2. $\|f * g\|_\infty \le \|f\|_p \|g\|_{p'}$ $\qquad \frac{1}{p} + \frac{1}{p'} = 1$

$$|(f*g)(x)| = \left|\int f(x-y)\,g(y)\,dy\right|$$

$$\le \int_{\mathbb{R}^n} |f(x-y)||g(y)|\,dy$$

Hölder's inequality

$$= \int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy$$

Hölder's inequality

$$\le \left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dy\right)^{1/p}\left(\int_{\mathbb{R}^n} |g(y)|^{p'}\,dy\right)^{1/p'}$$

$$= \|f\|_p \,\|g\|_{p'}$$

$$\therefore\ |f*g(x)| \le \|f\|_p\,\|g\|_{p'} \quad \forall x$$

$$\Rightarrow \|f*g\|_\infty \le \|f\|_p\,\|g\|_{p'} \quad \text{—— ②}$$

Step 3. So far, we have

$$\|f*g\|_p \le \|f\|_p\,\|g\|_1,$$

$$\|f*g\|_\infty \le \|f\|_p\,\|g\|_{p'}.$$

Now, for $f \in L^p$ and define

$$T_f(g) := f*g.$$

$p_0=1,\ q_0=p$ $\quad\therefore\ \|T_f(g)\|_p \le M_0\cdot\|g\|_1, \qquad M_0 = \|f\|_p,$

$p_1=p',\ q_1=\infty$ $\quad \|T_f(g)\|_\infty \le M_1\cdot\|g\|_{p'}, \qquad M_1 = \|f\|_p.$

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = 1-\theta + \frac{\theta}{p'} = 1-\theta+\theta\left(1-\frac{1}{p}\right)$$

$$= 1-\frac{\theta}{p}$$

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{p}$$

By Riesz Thorin,

$$\|T_f g\|_{q_\theta} \le \underbrace{M_0^{1-\theta} M_1^{\theta}}_{=\|f\|_p} \|g\|_{p_\theta}$$

$$\Rightarrow \|T_f g\|_{q_\theta} \le \|f\|_p\,\|g\|_{p_\theta}$$

where $\frac{1}{p_\theta} = 1 - \frac{\theta}{p}$, $\frac{1}{q_\theta} = \frac{1-\theta}{p}$.

Note $1 + \frac{1}{q_\theta} = 1 + \frac{1-\theta}{p} = 1 + \frac{1}{p} - \frac{\theta}{p} = \frac{1}{p_\theta} + \frac{1}{p}$.

Take θ so that $\frac{1}{p_\theta} = \frac{1}{q}$. Then, $\frac{1}{q_\theta} = \frac{1}{r}$.

$\therefore \quad \|T_f g\|_r \leq \|f\|_p \|g\|_q$

or $\quad \|f * g\|_r \leq \|f\|_p \|g\|_q$, as desired. ∎

Lecture 14 (03-03-2021)

03 March 2021 09:29

Suppose $f \in \mathcal{L}^1(\mathbb{R})$. Fix $a \in \mathbb{R}$

Define

$$F(x) := \int_a^x f \qquad \text{for } x \in \mathbb{R}.$$

Is F differentiable?

If f is cont. at x_0, then F is diff at x_0.

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f$$

Thus, the question is whether $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f$ exists at x?

$$= \lim_{\substack{|I| \to 0 \\ x \in I}} \frac{1}{|I|} \int_I f.$$

(limit over all intervals I containing x.)

$|I|$ = length of I

On $\mathbb{R}^n$: Q: Whether

$$\lim_{\substack{|B| \to 0 \\ x \in B}} \frac{1}{|B|} \int_B f \quad \text{exists?}$$

($B \to$ ball; limit over all $B \ni x$)

$|B|$ = Lebesgue measure of B

Let f be continuous at $x \in \mathbb{R}^n$.

Claim: $\lim_{\substack{|B| \to 0 \\ B \ni x}} \frac{1}{|B|} \int_B f = f(x).$

Proof. $\left| \frac{1}{|B|} \int f - f(x) \right| = \left| \frac{1}{|B|} \int f(y)\,dy - \frac{1}{|B|} \int f(x)\,dy \right|$

Proof.
$$\left| \frac{1}{|B|} \int_B f - f(x) \right| = \left| \frac{1}{|B|} \int_B f(y)\,dy - \frac{1}{|B|} \int_B f(x)\,dy \right|$$

$$= \left| \frac{1}{|B|} \int_B [f(y) - f(x)]\,dy \right|$$

$$\leq \frac{1}{|B|} \int_B |f(y) - f(x)|\,dy$$

Since f is continuous at x, $\forall \epsilon > 0$, $\exists B_0 \ni x$ s.t.
$|f(y) - f(x)| < \epsilon \quad \forall y \in B_0$.

$$\therefore \frac{1}{|B_0|} \int_{B_0} |f(y) - f(x)|\,dy < \epsilon.$$ ☒

Q: What happens if we drop continuity?

Hardy-Littlewood maximal operator

Given $f \in L^1(\mathbb{R}^n)$, we define

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|.$$ (Uncentered maximal function.)

Example. Consider $f \in L^1(\mathbb{R})$ given as $f = \chi_{[a,b)}$.

$$Mf(x) = \begin{cases} \dfrac{b-a}{2|x-b|} & ; x \leq a, \\ 1 & ; a < x < b, \\ \dfrac{b-a}{2|x-a|} & ; b \leq x. \end{cases}$$

Observation: If $f \in L^\infty(\mathbb{R}^n)$, then $Mf \in L^\infty(\mathbb{R}^n)$.

$$Mf(x) = |Mf(x)| = \frac{1}{|B|}\int_B |f| \leq \frac{1}{|B|}\|f\|_\infty |B| = \|f\|_\infty$$

$$\Rightarrow \quad \|Mf\|_\infty \leq \|f\|_\infty$$

Thus, M is strong type (∞, ∞).

Q. Why is Mf measurable?

$$\frac{1}{|B|}\int_B |f(y)|\,dy = \frac{1}{|B|}\int_{\mathbb{R}^n} |f(y)|\,\chi_B(y)\,dy = \frac{1}{|B|}\int_{\mathbb{R}^n} |f(y)|\,\chi_{B'+x}(y)\,dy$$

$y \in B'+x \Leftrightarrow y-x \in B'$

$$B' = \{b - x : b \in B\} \ni 0 \qquad \left(= \frac{1}{|B|}\int_{\mathbb{R}^n} |f(y)|\,\chi_{B'}(y-x)\,dy\right)$$

$B' = B - x$

$$= \frac{1}{|B|}\int_{\mathbb{R}^n} |f(y)|\,\check{\chi}_{B'}(x-y)\,dy$$

$$= \frac{1}{|B|}\,|f| * \check{\chi}_{B'}.$$

- If $f \in L^1(\mathbb{R}^n)$, is $Mf \in L^1(\mathbb{R}^n)$? No!

Proof. Let $0 \neq f \in L^1(\mathbb{R}^n)$.

$$\therefore \exists R > 0 \text{ s.t. } \int_{B(0,R)} |f| \geq \delta > 0.$$

Let $|x| > R$. Then, $B(x, 2|x|) \supseteq B(0, R)$

Then, $$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\,dy \geq \frac{1}{|B(x, 2|x|)|}\int_{B(x,2|x|)} |f(y)|\,dy$$

$$\geq \frac{\delta}{}$$

$$\left(|B(0,r)| = \frac{1}{C_n} \cdot r^n \right) \qquad = C_n \cdot \frac{\delta}{|x|^n}$$

$\overline{|B(x, 2|x|)|}$

Thus, for $x \in \mathbb{R}^n$ with $|x| > R$, we have

$$Mf(x) \geq \frac{C}{|x|^n} \quad \hookrightarrow \text{not integrable!}$$

$$\Rightarrow Mf \notin L^1(\mathbb{R}^n). \qquad \blacksquare$$

$\therefore$ M is not strong type $(1,1)$.

Thm. M is weak type $(1,1)$. That is,

$$|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \leq c \cdot \frac{\|f\|_1}{\alpha}$$

Cor. (by Marcinkiewicz interpolation) M is strong type (p,p) for $1 < p \leq \infty$.

Thm. Let $f \in L^1(\mathbb{R}^n)$. Then,

$$\lim_{\substack{|B| \to 0 \\ B \ni x}} \frac{1}{|B|} \int_B f = f(x) \qquad \text{for a.e. } x$$

Proof. For $\alpha > 0$, let

$$E_\alpha = \left\{ x \in \mathbb{R}^n : \limsup_{\substack{|B| \to 0 \\ B \ni x}} \left| \frac{1}{|B|} \int f(y)\,dy - f(x) \right| > 2\alpha \right\}.$$

<u>Claim</u>: $|E_\alpha| = 0 \quad \forall \alpha > 0$

Then, we are done because $\bigcup_{n \in \mathbb{N}} E_{1/n}$ is the set of points for which

$\lim (\quad) \neq f(x)$

points for which $\lim_{n\in\mathbb{N}} (\ \ ) \neq f(x)$.

(Either lim DNE or lim exists and $\neq f(x)$.)

Proof (of claim). We will use that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Given $\epsilon>0$, $\exists g \in C_c^\infty(\mathbb{R}^n)$ s.t. $\|f-g\|_1 < \epsilon$.

$$\frac{1}{|B|}\int_B f(y)\,dy - f(x) = \frac{1}{|B|}\int_B (f-g) + \frac{1}{|B|}\int g(y)\,dy - g(x) + g(x) - f(x)$$

$$\Rightarrow \limsup_{\substack{|B|\to 0\\ B\ni x}} \left| \frac{1}{|B|}\left(\int_B f\right) - f(x)\right| \le \limsup_{\substack{|B|\to 0\\ B\ni x}} \frac{1}{|B|}\int_B |f-g| + \left|\limsup \left(\ \ \right) \frac{1}{|B|}\int g - g(x)\right)\Big| + |g(x)-f(x)|$$

$=0$, g is cts.

$$\le M(f-g)(x) + |g(x)-f(x)|$$

Let $F_\alpha = \{x\in\mathbb{R}^n : M(f-g)(x) > \alpha\}$ and

$G_\alpha = \{x\in\mathbb{R}^n : |f(x)-g(x)| > \alpha\}$.

Then, $E_\alpha = F_\alpha \cup G_\alpha$.

$$|F_\alpha| \le C\cdot\frac{\|f-g\|_1}{\alpha} \qquad (\because M \text{ is weak } (1,1))$$

$$|G_\alpha| = \int_{\mathbb{R}^n} \chi_{G_\alpha} \le \int_{\mathbb{R}^n} \frac{|f(x)-g(x)|}{\alpha}\,dx = \frac{\|f-g\|_1}{\alpha} < \frac{\epsilon}{\alpha}$$

$$\therefore |F_\alpha| + |G_\alpha| < \frac{(C+1)}{\alpha}\epsilon$$

Since ϵ was arbitrary, $|F_\alpha| = |G_\alpha| = 0$ and hence,
$|E_\alpha| = 0$.

This finishes the proof. ∎

Lecture 15 (05-03-2021)

05 March 2021 09:32

Hardy - Littlewood's Maximal Operator

$$(Mf)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|.$$

M is strong type (∞, ∞).

Thm 1. M is weak type $(1,1)$, i.e., $\forall \alpha > 0$ $\exists c_\alpha > 0$ s.t.

$$\{x \in \mathbb{R}^n : |f(x)| > \alpha\} \leq c_\alpha \frac{\|f\|_1}{\alpha}.$$

In fact, we can take $c_\alpha = 3^n$.

Recall. Fourier inversion for L^1

If $f, \hat{f} \in L^1$, then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\lambda) e^{2\pi i x \cdot \lambda} d\lambda \quad \text{for a.e. } x.$$

Q. For which x does it hold?

Lemma (Vitali covering lemma)

Let $\mathcal{B} = \{B_1, \dots, B_N\}$ be a finite collection of open balls in $\mathbb{R}^n$. Then, $\exists$ a disjoint subcollection $\{B_{i_1}, \dots, B_{i_k}\}$ of $\mathcal{B}$ s.t.

$$\left| \bigcup_{i=1}^N B_i \right| \leq 3^n \sum_{j=1}^k |B_{i_j}|.$$

Proof Observation: Suppose B and B' are two intersecting balls with radius $\text{rad}(B') \le \text{rad}(B)$.

Now, let $\tilde{B}$ be the ball concentric with B and $\text{rad}(\tilde{B}) = 3\,\text{rad}(B)$. Then, $\tilde{B} \supset B \cup B'$!

Given $\mathcal{B} = \{B_1, \dots, B_N\}$ a collection of balls.

Let $B_{i_1} \in \mathcal{B}$ be with maximal radius.

Now, consider all balls which intersect B_{i_1}. (Call this $\mathcal{B}_{i_1}$.) $\ni B_{i_1}$

By the observation above, all these intersecting balls will be contained in $\tilde{B}_{i_1}$.

Then, $$\bigcup_{B \in \mathcal{B}_{i_1}} \subseteq \tilde{B}_{i_1}$$

Now, consider the maximal radius ball in $\mathcal{B} \setminus \mathcal{B}_{i_1}$. Call it B_{i_2}.

Take $\mathcal{B}_{i_2}$ to be all those balls intersecting B_{i_2}.

All these are contained $\tilde{B}_{i_2}$.

Continue to get $B_{i_1}, \dots, B_{i_k}$ s.t.

$$\bigcup_{i=1}^{N} B_i \subseteq \bigcup_{i=1}^{k} \tilde{B}_{i_k}.$$

$$\Rightarrow \left| \bigcup_{i=1}^{N} B_i \right| \le \left| \bigcup_{i=1}^{k} \tilde{B}_{i_k} \right| \le \sum_{i=1}^{k} |\tilde{B}_{i_k}| = 3^n \sum_{i=1}^{k} |B_{i_k}|. \quad \blacksquare$$

Proof that M is weak $(1,1)$:

Fix $\alpha > 0$. Let $E_\alpha = \{x : Mf(x) > \alpha\}$.

We prove: $|E_\alpha| \le 3^n \dfrac{\|f\|}{\alpha}$.

Let $K \subset E_\alpha$ be a compact set in E_α.

We show: $|K| \le 3^n \dfrac{\|f\|}{\alpha}$ (Note that K is arbitrary.)

- If $x \in E_\alpha$, then $Mf(x) > \alpha$, i.e.,

$$\sup_{B \ni x} \frac{1}{|B|} \int_B |f| > \alpha.$$

$\Rightarrow$ $\exists$ a ball $B_x \ni x$ s.t.

$$\frac{1}{|B_x|} \int_{B_x} |f| > \alpha. \quad \text{———} \quad (1)$$

Note that $\{B_x\}_{x \in E_\alpha}$ is a covering of E_α and hence of K

Since K is compact, $\exists B_{x_1}, \dots, B_{x_N}$ s.t.

$$K \subseteq B_{x_1} \cup \dots \cup B_{x_N}.$$

Then, by Vitali covering lemma, $\exists$ a disjoint subcollection $\{B_{x_{i_1}}, \dots, B_{x_{i_K}}\} \subseteq \{B_{x_1}, \dots, B_{x_N}\}$ s.t.

$$\left| \bigcup_{i=1}^{N} B_{x_i} \right| \leq 3^n \sum_{j=1}^{K} |B_{x_{i_j}}|.$$

$$\therefore \quad |K| \leq \left| \bigcup_{i=1}^{N} B_{x_i} \right| \leq 3^n \sum_{j=1}^{K} |B_{x_{i_j}}|$$

by (1)

$$\leq \frac{3^n}{\alpha} \sum_{j=1}^{K} \int_{B_{x_{i_j}}} |f|$$

disjoint

$$\leq \frac{3^n}{\alpha} \int_{\bigcup_{j=1}^{K} B_{x_{i_j}}} |f| \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f|$$

$$= \frac{3^n}{\alpha} \|f\|_1.$$

Thus, $|K| \leq \frac{3^n}{\alpha} \|f\|_1$ for any K compact.

$$\Rightarrow \sup_{\substack{K \text{ compact} \\ K \subseteq E_\alpha}} |K| \leq \frac{3^n}{\alpha} \|f\|_1$$

$$\| \qquad\qquad$$

$$|E_\alpha|$$

Thus, $|E_\alpha| \leq \frac{3^n}{\alpha} \|f\|_1$ as desired. ☒

Cor. M is strong type (p,p) for $1 < p \leq \infty$.

Def^n. (Lebesgue set) Let f be a locally integrable function, i.e., $f \in L^1_{loc}(\mathbb{R}^n)$. Then, the Lebesgue set of f is defined by

$$\mathrm{Leb}(f) = \left\{ x \in \mathbb{R}^n : f(x) < \infty, \lim_{\substack{|B| \to 0 \\ B \ni x}} \frac{1}{|B|} \int_B |f(y) - f(x)| \, dy = 0 \right\}.$$

Remarks ① If f is continuous at x, then $x \in \mathrm{Leb}(f)$.

② If $x \in \mathrm{Leb}(f)$, then $\displaystyle\lim_{\substack{|B| \to 0 \\ B \ni x}} \frac{1}{|B|} \int f = f(x)$. — $(*)$

- If $f \in L^1(\mathbb{R}^n)$, then $(*)$ holds for a.e. x.
  We show that $\mathrm{Leb}(f)$ is also a full measure set.
  That is, $|\mathbb{R}^n \setminus \mathrm{Leb}(f)| = 0$. $(Ex.)$

- Let $f \in L^1$ be s.t. $\hat{f} \in L^1$.
  If $x \in \mathrm{Leb}(f)$, then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\lambda) e^{2\pi i \lambda x} \, d\lambda.$$

https://aryamanmaithani.github.io/math/ma-5106/fourier-inv-lebesgue-set.pdf

Centered maximal function

$$\mathcal{M} f(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|.$$

Then, $\mathcal{M} f(x) \leq M f(x)$ directly by defⁿ.

But also,

$$M f(x) \leq 2^n \mathcal{M} f(x).$$

Assume true

Then, $\mathcal{M}$ is also weak $(1,1)$.

The other properties also follow.

Proof

$$M f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|$$

If $x \in B$, then $B \subseteq B(x, \text{diam } B)$.

Then, $$M f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|$$

$$\leq \sup_{B \ni x} \frac{1}{|B|} \int_{B(x, \text{diam } B)} |f|$$

$$= \sup_{B \ni x} \frac{|B(x, \text{diam } B)|}{|B|} \cdot \frac{1}{B(x, \text{diam } B)} \int_{B(x, \text{diam} B)} |f|$$

$$= 2^n \mathcal{M} f(x)$$

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10 March 2021 09:27

Hardy Littlewood Maximal Functions (on $\mathbb{R}^n$)

(uncentered) $\quad Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|$

(centered) $\quad \mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f|$

- $$\mathcal{M}f(x) \leq Mf(x) \leq 2^n \mathcal{M}f(x)$$

- M is weak $(1,1)$ and strong (∞, ∞).
  Thus, M is strong (p, p) $\forall p \in (1, \infty]$.

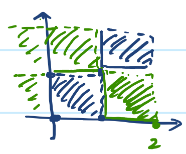
- Have proved: $\displaystyle \lim_{\substack{|B| \to 0 \\ B \ni x}} \frac{1}{|B|} \int f = f(x)$ a.e. x.

Dyadic maximal functions

Dyadic cubes

- $Q_0 \to$ collection of cubes which are congruent to $[0, 1)^n$ with vertices in the lattice $\mathbb{Z}^n$.

$n=2$



For each $k \in \mathbb{Z}$, define

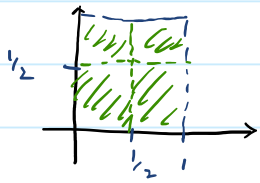
$Q_k \to$ collection of cubes in $\mathbb{R}^n$ which are dilates of cubes of Q_0 by the factor 2^{-k}.

Vertices in $\left(\frac{\mathbb{Z}}{2^k}\right)^n$.

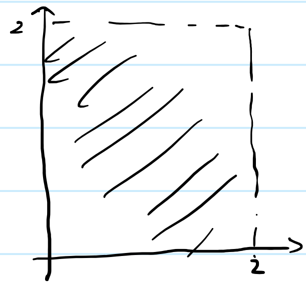
$\left(\overline{2^k}\right)$

$(n=2)$

$k=1$:



$k=-1$:



Elements of $\mathcal{Q}_k$ are called dyadic cubes.

Observation :

1. For $k \in \mathbb{Z}$, every $x \in \mathbb{R}^n$ is in a unique cube in $\mathcal{Q}_k$.

2. A cube in $\mathcal{Q}_k$ contains 2^n cubes of $\mathcal{Q}_{k+1}$ and is contained in a unique cube in $\mathcal{Q}_j$ for $j<k$.

3. Any two cubes in $\bigcup_{k \in \mathbb{Z}} \mathcal{Q}_k$ are either disjoint or comparable (w.r.t. $\subseteq$).

Def^n. Let $f \in L^1_{loc}(\mathbb{R}^n)$ (If V is bounded, then $\int_V |f| < \infty$) and for $k \in \mathbb{Z}$, we define

$$E_k f(x) = \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x).$$

(Conditional expectation of f w.r.t. \sigma-algebra generated by cubes in Q_k)

Note that k is fixed. Given any $x \in \mathbb{R}$, $\exists!\, Q_x \in \mathcal{Q}_k$ s.t. $x \in Q_x$. $\therefore E_k f(x) = \frac{1}{|Q_x|} \int_{Q_x} f < \infty$ since $f \in L^1_{loc}$.

Observation:

(1) $(E_k f)|_Q$ is constant for each $Q \in \mathcal{Q}_k$.

The constant being $\frac{1}{|Q|}\int_Q f$.

(2) Let $f \in L^1$ with $f \geq 0$. Then, for any fixed $x \in \mathbb{R}^n$,

$$E_k f(x) \longrightarrow 0 \quad \text{as} \quad k \to -\infty.$$

Let $Q_x^{(k)} \in \mathcal{Q}_k$ denote the unique cube in $\mathcal{Q}_k$ containing x.

Then,

$$E_k f(x) = \frac{1}{|Q_x|}\int_{Q_x} f \leq \frac{\|f\|_1}{|Q_x|} = 2^{n|k|}\|f\|_1 \longrightarrow 0 \text{ as } k \to -\infty.$$

3. Fix $k \in \mathbb{Z}$. Let Ω be the union of some (possibly all) cubes in $\mathcal{Q}_k$.

Then, $$\int_\Omega E_k f = \int_\Omega f.$$

$$\int_\Omega E_k f(x)\,dx = \int_\Omega \sum_{Q \in \mathcal{Q}_k}\left(\frac{1}{|Q|}\int_Q f\right)\chi_Q(x)\,dx$$

Write $\Omega = \bigsqcup_{i \in I} Q_i$

$$= \sum_{i \in I}\int_{Q_i}\sum_{Q \in \mathcal{Q}_k}\left(\frac{1}{|Q|}\int_Q f\right)\chi_Q(x)\,dx$$

$$= \sum_{i \in I}\int_{Q_i}\left(\frac{1}{|Q|}\int_{Q_i} f\right) = \sum_{i \in I}\int_{Q_i} f\,dx = \int_\Omega f.$$

Dyadic maximal function

$$M_d f(x) := \sup_{k \in \mathbb{Z}} |E_k f(x)|$$

$k \in \mathbb{Z}$

$$= \sup_{k \in \mathbb{Z}} \left| \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \cdot \chi_Q(x) \right| .$$

Thm. 1. M_d is weak $(1,1)$.

2. Let $f \in L^1_{loc}$. Then, $\lim_{k \to \infty} E_k f(x) = f(x)$ a.e. x.

Proof. 1. Step 1. Assume $f \in L^1$ and $f \geq 0$.

$\lambda > 0$.

We show $|\{x : M_d f(x) > \lambda\}| \leq \frac{\|f\|_1}{\lambda}$ $\forall \lambda > 0$.

$$\{x : M_d f(x) > \lambda\} = \bigcup_{k \in \mathbb{Z}} \Omega_k$$

– where $\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda$ and $E_j f(x) \leq \lambda,\ j < k\}$

(Note $E_k f(x) \longrightarrow 0$ as $k \to -\infty$.)

· Ω_k s are disjoint.

Step 2. Note Ω_k can be written as (some or all) cubes in $\mathcal{Q}_k$. (Use that $E_k f|_Q$ for $Q \in \mathcal{Q}_k$ is const.)

(And that $Q \in \mathcal{Q}_k$ is contained in some $Q^{(j)} \in \mathcal{Q}_j$ $\forall j < k$.)

Step 3. $|\{x : M_d f(x) > \lambda\}| = \sum_{k \in \mathbb{Z}} |\Omega_k|$

$$= \sum_{k \in \mathbb{Z}} \int_{\Omega_k} 1$$

$1 < \frac{E_k f(x)}{\lambda}$ for $x \in \Omega_k$

$$\leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{\Omega_k} E_k f(x) dx$$

$$= \frac{1}{\lambda} \sum \int f \leq \frac{\|f\|_1}{\lambda}.$$

$k \in \mathbb{Z}$ Ω_k λ

- For general f, decompose $f = f^+ - f^- + i(\tilde{f}^+ - \tilde{f}^-)$ and conclude ∎

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- Recall :
  - Dyadic cubes in $\mathbb{R}^n$.

    $\mathcal{Q}_k$: cubes with vertices at $(2^{-k}\mathbb{Z})^n$ and side length 2^{-k}.

    (negative k → bigger)

  - Conditional expectation

$$E_k f(x) = \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x)$$

$$E_k f\big|_Q \text{ constant for } Q \in \mathcal{Q}_k.$$

  - Dyadic maximal function

$$M_d f(x) = \sup_{k \in \mathbb{Z}} |E_k f(x)|$$

$$= \sup_{k \in \mathbb{Z}} \left[\sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right]$$

  - We showed M_d is weak type 1-1.

    Recall: TS : $\{x : |f(x)| > \lambda\} \le c \cdot \frac{\|f\|_1}{\lambda}$

    Assumed $f \geq 0$. Defined

$$\Omega_k = \{ x \in \mathbb{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \le \lambda \ \forall j < k \}.$$

Then,

$$\{x : f(x) > \lambda\} = \bigcup_{k \in \mathbb{Z}} \Omega_k.$$

Obs: Ω_k is union of cubes in $\mathcal{Q}_k$.

Thus,

Thus,
$$\int_{\Omega_k} E_k f = \int_{\Omega_k} f.$$

From the above, it follows that f is weak $(1,1)$.

Thm. $\lim_{k\to\infty} E_k f(x) = f(x)$ for almost every x.

Proof. Step 1. Let g be continuous. Then,
$$\lim_{k\to\infty} E_k g(x) = g(x) \quad \forall x \in \mathbb{R}^n.$$

Proof. Fix $x \in \mathbb{R}^n$.
For each $k \in \mathbb{Z}$, $\exists! Q_x^{(k)} \in \mathcal{Q}_k$ containing x.

$$|E_k g(x) - g(x)| = \left| \frac{1}{|Q_x^{(k)}|} \int_{Q_x^{(k)}} g - g(x) \right|$$

$$= \left| \frac{1}{|Q_x^{(k)}|} \int_{Q_x^{(k)}} (g(y) - g(x))\, dy \right|$$

Since g is continuous at x, the above goes to 0. ∎
(Choose $k_0 \gg 1$ s.t. $|g(y) - g(x)| < \varepsilon$ $\forall y \in Q_x^{(k)}$.
This proves step 1.

Step 2. Use that $C(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.
Given $f \in L^1(\mathbb{R}^n)$ and $\varepsilon > 0$, pick $g \in C(\mathbb{R}^n)$
s.t. $\|f - g\|_1 < \varepsilon$.

Let
$$F_\alpha = \left\{ x \in \mathbb{R}^n : \limsup_{k\to\infty} |E_k f(x) - f(x)| > \alpha \right\}.$$

We show $|F_\alpha| = 0$ $\forall \alpha > 0$.

$$|E_k f(x) - f(x)| \le |E_k(f-g)(x)| + |E_k g(x) - g(x)| + |f(x) - g(x)|$$

$$\Rightarrow \limsup_{k\to\infty} |E_k f(x) - f(x)| \le M_d(f-g)(x) + |f(x) - g(x)|$$

$$\therefore E_\alpha \subseteq \{x : M_d(f-g)(x) \geq \alpha/2\} \cup \{x : |f(x) - g(x)| \geq \alpha/2\}.$$

$$\therefore |E_\alpha| \le |\{x : M_d(f-g)(x) \geq \alpha/2\}| + |\{x : |f(x) - g(x)| \geq \alpha/2\}|$$

$$\le \frac{2c}{\alpha}\|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 = \frac{2\varepsilon}{\alpha}(1+c).$$

Calderon-Zygmund decomposition

Thm. Let $f \in L^1$ with $f \geq 0$. Given $\lambda > 0$, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that

(i) $f(x) \le \lambda$ for a.e. $x \notin \bigcup_j Q_j$,

(ii) $\left|\bigcup_j Q_j\right| \le \frac{1}{\lambda}\|f\|_1$, ($Q_j$ need not be in $\mathcal{Q}_j$.)

(iii) $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f \le 2^n \lambda \quad \forall j.$

Proof. • Construction of $\{Q_j\}_j$.

Recall Ω_k from earlier.

$$\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \le \lambda \text{ for } j < k\}.$$

We had seen $\Omega_k = \bigsqcup_{Q \in \mathcal{Q}_k} Q.$

$Q \subset \Omega_k$ countable union

Then, $\displaystyle\bigsqcup_{k \in \mathbb{Z}} \Omega_k = \bigsqcup_{k \in \mathbb{Z}} \Big(\bigsqcup_{\substack{Q \in \mathcal{Q}_k \\ Q \subset \Omega_k}} Q \Big)$

Note that Ω_ks are disjoint

again a countable union

Enumerate the above cubes as $\{Q_j\}_j$.

Then,

$$\bigcup_{k \in \mathbb{Z}} \Omega_k = \bigcup_j Q_j.$$

Required collection

(i) If $x \notin \bigcup_j Q_j$, then $x \notin \bigcup_{k \in \mathbb{Z}} \Omega_k$, then $x \notin \Omega_k \ \forall k \in \mathbb{Z}$.

Thus, $E_k f(x) \leq \lambda \quad \forall k \in \mathbb{Z}$.

let $k \to \infty$. Then, $\displaystyle\underbrace{\lim_{k \to \infty} E_k f(x)}_{= f(x)} \leq \lambda$

for a.e. x

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(ii) $$\left|\bigcup_{j\in\mathbb{Z}} Q_j\right| = \left|\bigsqcup_{k\in\mathbb{Z}} \Omega_k\right| = \left|\{x\in\mathbb{R}^n : M_d f(x) > \lambda\}\right|$$

$$\leq \frac{\|f\|_1}{\lambda}.$$

(iii) Aim: For each j, $\lambda < \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f \leq 2^n\lambda.$

Fix j. By our construction, $\exists k\in\mathbb{Z}$ s.t.

$$Q_j \subseteq \Omega_k.$$

$$\Rightarrow \frac{1}{|Q_j|}\int_{Q_j} f \underbrace{>}_{E_k f(x) > \lambda \ \forall x\in\Omega_k} \frac{1}{|Q_j|}\int_{Q_j}\lambda = \lambda.$$

Let $\tilde{Q}_j$ be a dyadic cube containing Q_j with side length twice as much.

Then, $\exists k$ s.t. $Q_j \in \mathcal{Q}_k$ and $\tilde{Q}_j \in \mathcal{Q}_{k-1}$.

$$\frac{1}{|Q_j|}\int_{Q_j} f \underbrace{\leq}_{f\geq 0} \frac{1}{|Q_j|}\int_{\tilde{Q}_j} f = \frac{|\tilde{Q}_j|}{|Q_j|}\cdot\frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} f$$

$$= 2^n \frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} f$$

$$= 2^n E_{k-1} f(x) \quad \text{for any } x\in\tilde{Q}_j$$

Choose $x\in Q_j \subset \Omega_k$. Then $E_{k-1} f(x) < \lambda$, by def^n of Ω_k.

Choose $x \in Q_j \subset \Omega_k$. Then $E_{k-1} f(x) < \lambda$, by def^n of Ω_k.
$\leq 2^n \lambda$. ∎

- $M_d f(x) = \sup_{k\in\mathbb{Z}} |E_k f(x)| = \sup_{k\in\mathbb{Z}} \left| \sum_{Q\in\mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right|$

$\hookrightarrow$ weak type (1, 1)

- strong type (∞, ∞):

Fix $x \in \mathbb{R}^n$. For each k, let $Q_x^{(k)} \in \mathcal{Q}_k$ denote the unique cube in $\mathcal{Q}_k$ containing x.

$$|M_d f(x)| \leq \sup_{k\in\mathbb{Z}} \left| \frac{1}{|Q_x^{(k)}|} \int_{Q_x^{(k)}} f \right|$$

$$\leq \sup_{k\in\mathbb{Z}} \frac{1}{|Q_x^{(k)}|} \int_{Q_x^{(k)}} |f|$$

$$\leq \|f\|_\infty. \quad \leftarrow \text{indep of } k$$

Thus, $|M_d f(x)| \leq \|f\|_\infty \quad \forall x$ and thus,
$\|M_d f\| \leq \|f\|_\infty$.

By Marcink., M_d is strong type (p, p) for all $p \in (1, \infty]$.

Hilbert transform

- Distribution.

Convergence in (topology of) $C_c^\infty(\mathbb{R}^d)$:

A sequence $\{f_j\}_j$ in $C_c^\infty(\mathbb{R}^d)$ converges to $f \in C_c^\infty(\mathbb{R}^d)$

iff $\exists$ compact $K \subseteq \mathbb{R}^d$ s.t.

$$\operatorname{supp} f_j \subseteq K \quad \forall j,$$
$$\operatorname{supp} f \subseteq K$$

and $D^\alpha f_j \xrightarrow{\text{uniformly}} D^\alpha f$ on K $\forall \alpha \in (\mathbb{N} \cup \{0\})^d$.

Convergence in compacta

(Convergence in compacta)

- Schwartz space : $\mathcal{S}(\mathbb{R}^d)$

Convergence in $\mathcal{S}(\mathbb{R}^d)$

$$f_j \to f \iff \rho_{\alpha,\beta}(f_j - f) \to 0 \text{ as } j \to \infty.$$
$$\forall \alpha, \beta \in (\mathbb{N} \cup \{0\})^d.$$

:

Recall $\rho_{\alpha,\beta}(f) := \sup |x^\alpha| \, |D^\beta f(x)|$

- Distribution $T : C_c^\infty(\mathbb{R}^d) \to \mathbb{C}$ any continuous linear map.

That is, T is linear and $f_j \to f$ in $C_c^\infty(\mathbb{R}^d)$

$$\Downarrow$$

$$T(f_j) \to T(f) \text{ in } \mathbb{C}.$$

The set of all distributions is denoted by $C_c^\infty(\mathbb{R}^d)'$.

- Observation . $C^\infty(\mathbb{R}^d) \hookrightarrow C_c^\infty(\mathbb{R}^d)'$

Given $f \in C^\infty(\mathbb{R}^d)$, define $T_f : C_c^\infty(\mathbb{R}^d) \to \mathbb{C}$ by

$$T_f(g) = \int_{\mathbb{R}^d} fg.$$

Then, $T_f \in C_c^\infty(\mathbb{R}^d)$.

- $L^p(\mathbb{R}^d) \hookrightarrow C_c^\infty(\mathbb{R}^d)'$

$f \in L^p(\mathbb{R}^d)$, then $T_f(g) := \int_{\mathbb{R}^d} fg$.

As before $T_f \in C_c^\infty(\mathbb{R}^d)'$.

- Tempered distribution: $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ continuous and linear.

That is, T is linear and $f_j \to f$ in $\mathcal{S}(\mathbb{R}^d)$

$$\Downarrow$$

$T(f_j) \to T(f)$ in $\mathbb{C}$.

· The set of tempered distributions is denoted by $\mathcal{S}(\mathbb{R}^d)'$.

· As before, $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)'$.

$$C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$$

- Note $\mathcal{S}(\mathbb{R}^d)' \subseteq C_c^\infty(\mathbb{R}^d)'$ since $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$

- Derivative of distribution

Let $T \in C_c^\infty(\mathbb{R}^d)'$. $D^\alpha T$ will be be a distribution.

Let $\alpha = (\alpha_1, \dots, \alpha_d)$. $|\alpha| := |\alpha_1| + \dots + |\alpha_d|$.

$$D^\alpha T(f) := (-1)^{|\alpha|} T(D^\alpha f) \quad \text{with } f \in C_c^\infty(\mathbb{R}^d).$$

- Question: We saw $C^\infty(\mathbb{R}^d) \subseteq C_c^\infty(\mathbb{R}^d)'$. Does the notion of derivative coincide with the usual one?

Yes! Let $f \in C^\infty(\mathbb{R}^d)$. Let T_f be as earlier.

$$\curvearrowleft C_c^\infty(\mathbb{R}^d)'$$

$$\left(T_f(g) = \int_{\mathbb{R}^d} fg.\right)$$

$$\left(\begin{array}{l} f \longleftrightarrow T_f \\ D^\alpha f \longleftrightarrow T_{D^\alpha f} \end{array}\right)$$

Is $\quad D^\alpha T_f = T_{D^\alpha f}$?

Now,

$$\begin{aligned} D^\alpha T_f(g) &= (-1)^{|\alpha|} T_f(D^\alpha g) \\ &= (-1)^{|\alpha|} \int f(x)\,(D^\alpha g)(x)\,dx \\ &\overset{\text{IBP}}{=} \int (D^\alpha f)(x)\, g(x)\, dx \\ &= T_{D^\alpha f}(g). \end{aligned}$$

$\boxtimes$

- Multiplication. $\quad T \in C_c^\infty(\mathbb{R}^d)'$ and $f \in C^\infty(\mathbb{R}^d)$.

Define : $\quad f \cdot T \in C_c^\infty(\mathbb{R}^d)'$ by

$$(f \cdot T)(g) := T(fg) \qquad \text{for } g \in C_c^\infty(\mathbb{R}^d)$$

Then, $\quad f \cdot h \longleftrightarrow f \cdot T_h = T_{f \cdot h} \qquad$ for $h \in C^\infty(\mathbb{R}^d) \hookrightarrow C_c^\infty(\mathbb{R}^d)'$

- Convolution.

$$\tau_h f(x) := f(x-h).$$

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^d} \tau_y f(x)\, g(y)\, dy$$

Let $\quad T \in C_c^\infty(\mathbb{R}^d)'$ and $f \in C_c^\infty(\mathbb{R}^d)$.

Define

$$T * f \in C^\infty(\mathbb{R}^d) \quad \text{by}$$

$$(T * f)(x) := T(\tau_x \check{f}), \qquad \text{where } \check{f}(y) = f(-y).$$

- $g \longleftrightarrow T_g$

$$(T_g * f)(x) = T_g(\tau_x \check{f})$$
$$= \int_{\mathbb{R}^d} g(y)\, \check{f}(y-x)\, dy$$
$$= \int_{\mathbb{R}^d} g(y)\, f(x-y)\, dy = (g*f)(x).$$

- Fourier transform of a tempered distribution.

Let $T \in S(\mathbb{R}^d)'$. Then, $\hat{T} \in S(\mathbb{R}^d)'$ is defined as

$$\hat{T}(f) = T(\hat{f}) \qquad \forall\, f \in S(\mathbb{R}^d).$$

- $T = T_g$ for $g \in S(\mathbb{R}^d)$. Then,

$$\hat{T_g}(f) = T_g(\hat{f}) = \int g \hat{f}$$
$$= \int \hat{g} f = T_{\hat{g}}(f).$$

$$\therefore \quad \hat{T_g} = T_{\hat{g}} \longleftrightarrow \hat{g}.$$

- Compactly supported distribution $C^\infty(\mathbb{R}^d)'$.

$$C_c^\infty(\mathbb{R}^d) \subseteq S(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d).$$

$$\therefore \quad C^\infty(\mathbb{R}^d)' \subseteq S(\mathbb{R}^d)' \subseteq C_c^\infty(\mathbb{R}^d)'.$$

If $w \in C^\infty(\mathbb{R}^d)'$, then define $\hat{w} \in C^\infty(\mathbb{R}^d)$ as

function!

$$\hat{w}(\lambda) = w(e^{-2\pi i x \lambda})$$

$$\hat{w}(\lambda) := w(x \mapsto e^{-2\pi i x \lambda}).$$

function! not distribution.

- Paley-Wiener theorem of compactly supported distributions.
- Wiener-Tauberian for L^2.
- Uncertainty principle

Lecture 19 (26-03-2021)

26 March 2021 09:31

Hilbert Transform

(on $\mathbb{R}$, not for higher $\mathbb{R}^n$).

Principal Values

Note that "$x \mapsto 1/x$" is not in L^1. Can think of a tempered distribution.

$$\left(\text{P.V. } \tfrac{1}{x}\right)(\varphi) := \lim_{t \to 0} \int_{|x| > t} \frac{\varphi(x)}{x}\, dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}).$$

Recall: if f is "good", can think of it as a tempered distribution by

$$f(\varphi) = \int_{\mathbb{R}} f(x)\, \varphi(x)\, dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

Define $\text{P.V. } \frac{1}{x} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\left(\text{P.V. } \tfrac{1}{x}\right)(\varphi) := \lim_{t \to 0} \int_{|x| > t} \frac{\varphi(x)}{x}\, dx.$$

Claim. 1. This limit exists.
2. $(\text{P.V. } \frac{1}{x})$ is a tempered distribution.

Proof 1. For $t_1 > t_2 > 0$,

$$\left| \int_{|x| > t_2} \frac{\varphi(x)}{x}\, dx - \int_{|x| > t_1} \frac{\varphi(x)}{x}\, dx \right|$$

$$= \left| \int_{t_1 > |x| > t_2} \frac{\varphi(x)}{x}\, dx \right|$$

$$\leq \left| \int_{t_1 > |x| > t_2} \frac{\varphi(x) - \varphi(0)}{x} \, dx \right| + \underbrace{\left| \int_{t_1 > |x| > t_2} \frac{\varphi(0)}{x} \, dx \right|}_{= 0, \text{ odd function}}$$

$$\leq \int_{t_1 > |x| > t_2} \frac{|\varphi(x) - \varphi(0)|}{|x|} \, dx$$

MVT

$$= \int_{t_1 > |x| > t_2} |\varphi'(\xi_x)| \, dx \leq M \, |t_1 - t_2|$$

$\varphi \in S(\mathbb{R})$

Thus, the limit as $t \to 0$ exists.

2. To show: $(P.V. \frac{1}{x}) \in S(\mathbb{R})'$. It is clearly linear.

WTS: $\varphi_n \to \varphi$ in $S(\mathbb{R}) \Rightarrow (P.V. \frac{1}{x})(\varphi_n) \to (P.V. \frac{1}{x})(\varphi)$ in $\mathbb{C}$.

$$(P.V. \tfrac{1}{x})(\varphi) = \lim_{t \to 0} \int_{|x| > t} \frac{\varphi(x)}{x} \, dx$$

odd function

$$= \lim_{t \to 0} \left\{ \int_{t < |x| < 1} \frac{\varphi(x) - \varphi(0)}{x} \, dx + \int_{|x| \geq 1} \frac{\varphi(x)}{x} \, dx \right\}$$

$$\Rightarrow \left| (P.V. \tfrac{1}{x})(\varphi) \right| \leq \left| \lim_{t \to 0} \int_{t < |x| < 1} \varphi'(\xi_x) \, dx \right| + \left| \int_{|x| \geq 1} \frac{\varphi(x)}{x} \, dx \right|$$

$$\leq \| \varphi' \|_\infty + C \cdot \| x \mapsto x\varphi(x) \|_\infty$$

$$C = \int_{|x| > 1} \frac{1}{x^2} \, dx < \infty.$$

Now, if $\varphi_n \to \varphi$ in $S(\mathbb{R})$, $\| \varphi_n' - \varphi' \|_\infty \to 0$ and

$\| \ldots \varphi \quad x \varphi_n \ldots \to 0$.

Now, if $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R})$, $\|\varphi_n' - \varphi'\|_\infty \to 0$ and $\|x\varphi_n - x\varphi\|_\infty \to 0$.

Thus, we are done. ⊠

Fourier Transform of (P.V. $\frac{1}{x}$)

$\widehat{\text{P.V.} \frac{1}{x}}$ → tempered distribution

Theorem $(\widehat{\text{P.V.} \frac{1}{x}})(\varphi) = \int_{\mathbb{R}} \pi(-i\,\text{sign}(\xi))\,\varphi(\xi)\,d\xi.$

Notationally $\widehat{\text{P.V.} \frac{1}{x}}(\xi) = \pi(-i\,\text{sign}(\xi)).$

Proof. P.V. $\frac{1}{x}$ is a tempered distribution.

$$\widehat{\text{P.V.} \tfrac{1}{x}}(\varphi) := (\text{P.V.} \tfrac{1}{x})(\hat{\varphi})$$

$$= \lim_{t \to 0} \int_{|x| > t} \frac{\hat{\varphi}(x)}{x}\,dx$$

$$= \lim_{t \to 0} \int_{|x|>t} \frac{1}{x} \int_{\mathbb{R}} \varphi(\xi)\, e^{-2\pi i \xi x}\,d\xi\,dx$$

$$= \lim_{t \to 0} \int_{\mathbb{R}} \varphi(\xi) \int_{\frac{1}{t} > |x| > t} \frac{e^{-2\pi i \xi x}}{x}\,dx\,d\xi$$

$$\int_{t<|x|<\frac{1}{t}} \frac{e^{-2\pi i \xi x}}{x}\,dx = -i \int_{t<|x|<\frac{1}{t}} \frac{\sin(2\pi \xi x)}{x}\,dx$$

$$= -2i \int_t^{1/t} \frac{\sin(2\pi\xi x)}{x}\,dx \xrightarrow[t \to 0]{} -2i\,\tfrac{\pi}{2}\,\text{sign}(\xi) = -i\pi\,\text{sign}(\xi)$$

$$= \int_{\mathbb{R}} -i\pi \, \mathrm{sign}(\xi)\, \varphi(\xi)\, d\xi, \qquad \text{as desired. } \blacksquare$$

$$= \left(\xi \mapsto -i\pi\, \mathrm{sign}(\xi)\right)(\varphi).$$

Thus, $P.V.\,i$ can be considered as a function.

Hilbert Transform:

Let $f \in S(\mathbb{R})$.

$$Hf(x) = \frac{1}{\pi}\Big(\underbrace{(\overset{\in S(\mathbb{R})'}{P.V.\,i}) * \overset{\in S(\mathbb{R})}{f}}_{L^2}\Big)(x).$$

Thus, $Hf \in L^2$.

$$\widehat{Hf}(\xi) = \frac{1}{\pi}\,\widehat{(P.V.\,i)}(\xi)\,\hat{f}(\xi)$$

$$= (-i\,\mathrm{sign}(\xi))\,\hat{f}(\xi) \qquad \forall f \in S(\mathbb{R})$$

Dirichlet Problem on $\mathbb{R}$.

Suppose f is given on $\mathbb{R}$. How do we extend f as a harmonic function to the upper half plane.

That is, Solve.
$$\begin{cases} \dfrac{\partial^2}{\partial x^2} u(x,t) + \dfrac{\partial^2}{\partial t^2} u(x,t) = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^+ \\ \text{and} \\ u(x,0) = f(x) \quad \forall x \in \mathbb{R} \end{cases}$$

By taking Fourier trans form,

$$\hat{u}(\xi, t) = \hat{f}(\xi)\, e^{-t|\xi|},$$

i.e. $u(x,t) = (f * P_t)(x)$,

where $\widehat{P_t}(\xi) = e^{-t|\xi|}$.

$$\Rightarrow \quad P_t(x) = \hat{\hat{P_t}}(-x) = \frac{t}{x^2+t^2}.$$

$$Q_t(x) = \frac{x}{x^2+t^2}. \qquad P_t + iQ_t(x) = \frac{t+ix}{t^2+x^2}$$

$z = x+it$

$$= \frac{1}{t-ix}$$

$$= -\frac{1}{i}\,\frac{1}{x+it}$$

$$= \frac{i}{z}$$ → holo on upper half plane

<u>Claim</u>. $\lim_{t\to 0} Q_t = \text{P.V. } \frac{1}{x}$. (In the sense of tempered distributions.)

That is, $\lim_{t\to 0} Q_t(\varphi) = (\text{p.v. } \frac{1}{x})(\varphi) \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R})$.

If Claim is true, then

$$Hf(x) = \frac{1}{\pi} \lim_{t\to 0} (Q_t * f)(x).$$

- Fix $p > 1$. $f \in L^p$. $z = x+it$. $x \in \mathbb{R}$, $t > 0$.

$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$.

$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}.$

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - z}\, dy$$

exists on $\mathbb{H}$.

Moreover, F is holomorphic on $\mathbb{H}$.

$$F(x+it) = \frac{1}{2\pi} \Big[(f * P_t)(x) + i (f * Q_t)(x) \Big].$$

Lecture 20 (31-03-2021)

31 March 2021 09:37

Prop^n. $\lim_{t\to 0} Q_t = P.V.\ \frac{1}{x}$, in the sense of tempered functions.

Proof. Define $\psi_t(x) := \frac{1}{x} \chi_{\{x : |x| > t\}}(x)$.

Then, $\psi_t \in S(\mathbb{R})'$.

Also, $\lim_{t\to 0} \psi_t = P.V.\ \frac{1}{x}$.

We shall prove: $\lim_{t\to 0} (Q_t - \psi_t) = 0$. (In the sense of tempered distributions.)

That is, $\lim_{t\to 0} \int_{\mathbb{R}} (Q_t(x) - \psi_t(x))\, \varphi(x)\, dx = 0 \quad \forall \varphi \in S(\mathbb{R})$.

Now, note $\int_{\mathbb{R}} (Q_t(x) - \psi_t(x))\, \varphi(x)\, dx$

$$= \int_{\mathbb{R}} \frac{x\varphi(x)}{x^2+t^2}\, dx - \int_{|x|>t} \frac{\varphi(x)}{x}\, dx$$

$$= \int_{|x|\le t} \frac{x\varphi(x)}{x^2+t^2}\, dx + \int_{|x|>t} \frac{x\varphi(x)}{x^2+t^2}\, dx - \int_{|x|>t} \frac{\varphi(x)}{x}\, dx$$

$$= \int_{|x|\le t} \frac{x\varphi(x)}{x^2+t^2}\, dx + \int_{|x|>t} \varphi(x) \left(\frac{-t^2}{x(x^2+t^2)} \right) dx$$

$\frac{x}{t} = y$

$$= \int_{|y|\le 1} \frac{y\,\varphi(ty)}{1+y^2}\, dy - \int_{|y|>1} \frac{\varphi(ty)}{y(1+y^2)}\, dy$$

$|y|\le 1$ $|y|>1$

Let $t \to 0$ and use DCT to get

$$\lim_{t\to 0} (Q_t - \Psi_t)(\varphi) = \int_{|y|\le 1} \frac{y \cdot \varphi(0)}{1+y^2} - \int_{|y|>1} \frac{\varphi(0)}{y(1+y^2)}\, dy$$ both odd

$$= 0.$$

Thus, $\lim_{t\to 0} Q_t = \lim_{t\to 0} \Psi_t = \text{P.V. } i$, as desired. ▣

Cor. Hence, $Hf(x) = \lim_{t\to 0} \frac{1}{\pi} (f * Q_t)(x)$. $f \in S(\mathbb{R})$.

Obs. P_t is an approximate identity but Q_t is not.

Summary. $f \in S(\mathbb{R})$

(1) $Hf(x) := \frac{1}{\pi}(\text{P.V. } i * f)(x)$.

$$= \frac{1}{\pi} \lim_{t\to 0} (Q_t * f)(x).$$

(2) $(\widehat{Hf})(\xi) = (-i\,\text{sign}(\xi))\, \hat{f}(\xi)$.

· Properties of Hilbert Transform

(1) $\|Hf\|_2 = \|f\|_2 \quad \forall f \in S(\mathbb{R})$

$\left(\because \; \|Hf\|_2 = \|\widehat{(Hf)}\|_2 = \|(-i\,\mathrm{sgn}\,)\hat{f}\|_2 = \|\hat{f}\|_2 = \|f\|. \right)$

This shows that Hf can be defined for $f \in L^2$.

(2) $\underbrace{H(Hf)}_{\in L^2} = -f \qquad \forall f \in \mathcal{S}(\mathbb{R})$

$$\widehat{H(Hf)}(\xi) = (-i\,\mathrm{sgn}\,\xi)\,\widehat{Hf}(\xi) = (-i\,\mathrm{sgn}\,\xi)^2 \hat{f}(\xi) = -\hat{f}(\xi).$$

(3) $\langle Hf, g\rangle_{L^2} = -\langle f, Hg\rangle_{L^2}$; $\quad f, g \in \mathcal{S}(\mathbb{R})$.

$$\langle Hf, g\rangle = \langle \widehat{(Hf)}, \hat{g}\rangle$$
$$= \langle (-i\,\mathrm{sgn})\hat{f}, \hat{g}\rangle$$
$$= \langle \hat{f}, (i\,\mathrm{sgn})\hat{g}\rangle = \langle \hat{f}, -\widehat{Hg}\rangle$$
$$= -\langle f, Hg\rangle$$

Thus, $H^* = -H$.

Thm. The Hilbert transform is weak (1,1) and strong (1,1) for $1<p<\infty$.

Proof. To show: H is weak (1,1).

We will show: for every $\lambda > 0$,

$$|\{x : Hf(x) > \lambda\}| \leq c \frac{\|f\|_1}{\lambda}.$$

Assume $f \geq 0$. Then, by Calderon-Zygmund decomposition, $\exists$ a sequence of dyadic intervals ([,) type) $\{I_j\}_j$ s.t.

(i) $f(x) \leq \lambda$ for a.e. $x \notin \bigcup_j I_j =: \Omega$,

(ii) $|\Omega| = |\bigcup_j I_j| \leq \frac{\|f\|_1}{\lambda}$

(iii) $\lambda \leq \frac{1}{|I_j|}\int_{I_j} f \leq 2\lambda \quad \forall j.$

Using this decomposition, we will break f into two parts: g (good part) and b (bad part).

Define
$$g(x) = \begin{cases} f(x) & ; \ x \notin \Omega, \\ \frac{1}{|I_j|}\int_{I_j} f & ; \ x \in I_j, \end{cases}$$

and $b(x) = \sum_j b_j(x)$, where

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f \right) \chi_{I_j}(x).$$

Clearly, $f(x) = g(x) + b(x) \quad \forall x \in \mathbb{R}.$

Also, $g(x) \leq 2\lambda$ for a.e. $x \in \mathbb{R}$.
Also, $\text{supp}(b_j) \subseteq \overline{I_j}$ and $\int_{\mathbb{R}} b_j = 0.$

Thus, $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g.$

Note, $f = g + b \Rightarrow Hf = Hg + Hb.$
Thus,

$$|\{x : Hf(x) > \lambda\}| \leq |\{x : Hg(x) > \lambda/2\}| + |\{x : Hb(x) > \lambda/2\}|.$$

H is strong type $(2,2)$ $\quad \|Hf\|_2 = \|f\|_2.$
$\Downarrow$
weak type $(2,2)$

$$\left|\{x : g(x) > \lambda/2\}\right| \leq C \cdot \left(\frac{\|g\|_2}{\lambda/2}\right)^2$$

$$= 4c \frac{1}{\lambda^2} \int (g(x))^2 dx$$

$$\leq 8c \cdot \frac{\|f\|_1}{\lambda}$$

Now, will prove : $\left|\{x : Hb(x) > \lambda\}\right| \leq c \cdot \frac{\|f\|_1}{\lambda}$.

Let $2I_j$ be the interval with same center as I_j and twice the length. Let

$$\Omega^* = \bigcup_j (2I_j).$$

$$\left|\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}\right| \leq |\Omega^*| + \left|\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}\right|$$

$$\leq 2\frac{\|f\|_1}{\lambda} + \frac{2}{\lambda} \int_{\mathbb{R} \setminus \Omega^*} |Hb(x)|\, dx$$

Also, $|Hb(x)| \leq \sum_j |Hb_j(x)|$.

Thus, if we prove : $\sum_j \int |Hb_j| \leq c \cdot \|f\|_1$, we are done.

Lecture 21 (02-04-2021)

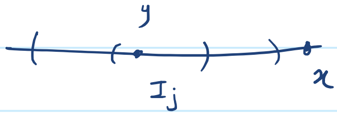
02 April 2021 09:36

Let $x \notin 2I_j$.

$$Hb_j(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{b_j(x-y)}{y}\, dy$$

$$= \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{b_j(y)}{x-y}\, dy$$

Note that b_j is supported on I_j.



Thus, only need to consider $y \in I_j$.

Thus, $|y-x|$ is greater than a fixed +ve quantity

$$Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x-y}\, dy.$$

Let α_j be the center of I_j.

$$\int_{\mathbb{R}-2I_j} |Hb_j(x)|\, dx = \int_{\mathbb{R}-2I_j} \left| \int_{I_j} \frac{b_j(y)}{x-y}\, dy \right| dx$$

$\because \int_{I_j} b_j = 0$

$$= \int_{\mathbb{R}-2I_j} \left| \int_{I_j} b_j(y) \left(\frac{1}{x-y} - \frac{1}{x-\alpha_j} \right) dy \right| dx$$

$$\leq \int_{I_j} |b_j(y)| \int_{\mathbb{R}-2I_j} \frac{|y-\alpha_j|}{|x-y| \cdot |x-\alpha_j|}\, dx\, dy$$

Note: $|y-\alpha_j| \geq |I_j|/2$ and $|x-y| \geq \frac{|x-\alpha_j|}{2}$

$$\leq \int_{I_j} |b_j(y)| \left(\int_{\mathbb{R}-2I_j} \frac{|I_j|}{|x-\alpha_j|^2}\, dx \right) dy$$

Use $2I_j = [\alpha-|I_j|, \alpha+|I_j|)$

$$= 2\int_{I_j} |b_j(y)|\, dy$$

$b_j = f - \frac{1}{|I_j|}\int_{I_j} f$

$$= 2\left(\int_{I_j} |f| + \int_{I_j} |f| \right) = 4\int_{I_j} |f|$$

$\therefore \sum_j \left| \int H b_j \right| \leq 4\|f\|_1$. Thus, H is weak $(1,1)$.

Moreover, H is strong $(2,2)$. By interpolation, H is strong (p,p) for $1 < p \leq 2$.

By duality, H is strong type (p,p) for $1 < p < \infty$.

Lecture 22 (07-04-2021)

07 April 2021 09:33

<u>Multipliers</u>

Let $m \in L^\infty(\mathbb{R}^n)$. We define a bounded op T_m on $L^2(\mathbb{R}^n)$ by

$$\widehat{T_m f}(\xi) = m(\xi)\,\hat{f}(\xi) \qquad \text{(Equality a.e.)}$$

<u>Note</u> T_m is a bounded operator on $L^2(\mathbb{R}^n)$

Pf.

$$\|T_m f\|_2 = \|\widehat{T_m f}\|_2 = \left(\int_{\mathbb{R}^n} |m(\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2}$$

$$\leq \|m\|_\infty \left(\int_{\mathbb{R}^n} |\hat{f}|^2\right)^{1/2} = \|m\|_\infty \|\hat{f}\|_2$$

$$= \|m\|_\infty \|f\|_2$$

$\therefore \|T_m f\|_2 \leq \|m\|_\infty \|f\|_2 \qquad \forall f \in L^2(\mathbb{R}^n)$

$\therefore \|T_m\|_{op} \leq \|m\|_\infty$

We now show that $\|T_m\| = \|m\|_\infty$

Fix $\epsilon > 0$. Let A be a measurable subset of $\{x \in \mathbb{R}^n \; |m(x)| > \|m\|_\infty - \epsilon\}$ whose measure is finite and positive.

By Plancherel, $\exists\, g \in L^2(\mathbb{R}^n)$ s.t. $\hat{g} = \chi_A$. Fix such a g.

$$\therefore \|T_m g\|_2^2 = \|\widehat{T_m g}\|_2^2$$

$$= \int_{\mathbb{R}^n} |\widehat{T_m g}(\xi)|^2 = \int_A |m(\xi)|^2 \, d\xi$$

$$= \int_{\mathbb{R}^n} |\widehat{T_m g}(\xi)|^2 = \int_A |m(\xi)|^2 \, d\xi$$

$$> (\|m\|_\infty - \varepsilon) \underbrace{\|g\|_2^2}_{\text{measure of } A}$$

$$\|T_m g\|_2 > (\|m\|_\infty - \varepsilon)^{1/2} \|g_2\| \qquad \boxtimes$$

Def^n The function $m \in L^\infty(\mathbb{R}^n)$ is called a L^2 multiplier for the operator T_m

- Let $m \in L^\infty(\mathbb{R}^n)$ be a function on $\mathbb{R}^n$ s.t. the operator T_m defined by

$$\widehat{T_m f}(\xi) = m(\xi) \hat{f}(\xi)$$

is a bounded operator on L^p. Then, the function m is called an L^p - multiplier for T_m

Examples ① Hilbert transform

$$\widehat{Hf}(\xi) = -i\,\mathrm{sign}(\xi)\,\hat{f}(\xi)$$

$m(\xi) = -i\,\mathrm{sign}(\xi)$ acts as an L^p - multiplier for Hilbert transform. $1 < p < \infty$

② Let $a, b \in \mathbb{R}$ with $a < b$. Let $m_{a,b}(\xi) := \chi_{(a,b)}(\xi)$

To find an operator $S_{a,b}$ s.t. $m_{a,b}$ is an L^p multiplier for $S_{a,b}$

That is, a bdd. op. $S_{a,b}$ $L^p \to L^p$ s.t.

$$\widehat{(S_{a,b} f)}(\xi) = \chi_{(a,b)}(\xi) \hat{f}(\xi).$$

Let $M_a f(x) = e^{2\pi i a x} f(x)$.

$$(\widehat{M_a f})(\xi) = \hat{f}(\xi - a)$$

$$\widehat{M_a H M_{-a} f}(\xi) = (\widehat{H M_{-a} f})(\xi - a)$$

$$= -i\,\mathrm{sign}(\xi - a)\, \widehat{M_{-a} f}(\xi - a)$$

$$= -i\,\mathrm{sign}(\xi - a)\,\hat{f}(\xi)$$

$\therefore$ $\widehat{(M_a H M_{-a} f)}(\xi) = -i\,\mathrm{sign}(\xi - a)\,\hat{f}(\xi)$

ily $\widehat{(M_b H M_{-b} f)}(\xi) = -i\,\mathrm{sign}(\xi - b)\,\hat{f}(\xi)$

$$\Rightarrow \widehat{(M_a H M_{-a} - M_b H M_{-b}) f}(\xi) = i\left(\mathrm{sign}(\xi - b) - \mathrm{sign}(\xi - a)\right)\hat{f}(\xi)$$

Now, note $\mathrm{sign}(\xi - b) - \mathrm{sign}(\xi - a) = \begin{cases} -2 & ; \ a < \xi < b \\ 0 & , \ \xi < a \text{ or } b < \xi \\ -1 & , \ \xi = b \text{ or } \xi = a \end{cases}$

$$\widehat{(M_a H M_{-a} - M_b H M_{-b}) f}(\xi) = -2i\,\chi_{(a,b)}(\xi)\,\hat{f}(\xi) \qquad \begin{pmatrix}\text{for} \\ a.e. \ \xi\end{pmatrix} \ [\xi \neq a, b]$$

Thus, if $S_{a,b} := \frac{i}{2}\left(M_a H M_{-a} - M_b H M_{-b}\right)$, then

$$\widehat{S_{a,b} f}(\xi) = \chi_{(a,b)}(\xi)\,\hat{f}(\xi)$$

Is $S_{a,b} : L^p \to L^p$ bounded? $(1 < p < \infty)$

We know $H : L^p \to L^p$ is bounded for $p \in (1, \infty)$

Thus, $\|M_a H M_{-a} f\|_p = \left(\int_{\mathbb{R}} |M_a H M_{-a} f(x)|^p dx\right)^{1/p}$

$$= \left(\int_{\mathbb{R}} \left| e^{2\pi i a x} H M_{-a} f(x)\right|^p\right)^{1/p} \qquad \left(|e^{2\pi i a x}| = 1\right)$$

$$= \|H M_{-a} f\|_p$$

$$\leq \|M_{-a} f\|_p = \|f\|_p$$

$\therefore$ $S_{a,b}$ is a bdd op

Hence, $\chi_{(a,b)}$ is an L^p- multiplier

- <u>L^p convergence of Fourier transform</u>

Let $f \in \mathcal{S}(\mathbb{R})$. Define

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi$$

Ques Does $S_R f \xrightarrow[R\to\infty]{} f$ in L^p?

$$S_R f(x) = \int_{-R}^{R} \left(\int_{\mathbb{R}} f(y) e^{-2\pi i \xi y} dy \right) e^{2\pi i \xi x} dx$$

$$= \int_{\mathbb{R}} f(y) \left(\int_{-R}^{R} e^{2\pi i \xi (x-y)} dx \right) dy$$

$$= (f * D_R)(x), \quad \text{where}$$

$$D_R(x) = \int_{-R}^{R} e^{2\pi i \xi x} d\xi$$

$$= \frac{1}{\pi x} \sin(2\pi R x).$$

$$\widehat{\chi_{(-R,R)}}(x) = D_R(x)$$

$$\therefore S_R f(x) = (f * \widehat{\chi}_{(-R,R)})(x)$$

$$\Rightarrow \widehat{S_R f}(\xi) = \hat{f}(\xi) \, \chi_{(-R,R)}(\xi) \qquad \left(\check{\chi}_{(-R,R)} = \chi_{(R,R)} \right)$$

$\therefore$ $\chi_{(-R,R)}$ is an L^p-operator for S_R.

Thus, $S_{-R,R} = S_R$ ($S_{R,R}$ from prev example)

$$S_{-R,R} = \frac{i}{2} \left(M_{-R} H M_R - M_R H M_{-R} \right)$$

Also, $\|S_R f\|_p \leq 2 \|f\|_p$ Thus implies $S_R f \to f$ in L^p (next propⁿ)

Prop^n If $f \in L^p(\mathbb{R})$ and $1 < p < \infty$, then $\lim_{R\to\infty} \|S_R f - f\|_p = 0$

Proof Let $X = \{ f \in \mathcal{S}(\mathbb{R}) .\ \hat{f}$ is compactly supported$\}$

Ex X is dense in $L^p(\mathbb{R})$.

If $g \in X$, $\exists R_0 > 0$ s.t. $S_R g = g \quad \forall R \geq R_0$

Let $f \in L^p(\mathbb{R})$. Then, $\exists g \in X$ s.t. $\|g - f\|_p < \varepsilon$

$$\|S_R f - f\|_p = \|S_R f - S_R g + S_R g - g + g - f\|_p$$

$$\leq 2\|f - g\|_p + \|f - g\|_p < 3\varepsilon$$

$S_R f \longrightarrow f$ in L^p. ∎

- $n > 1$ $\mathbb{R}^n$? The char f^n of a ball is not an L^p- multiplier. $(p > 1)$

— Feffermem

Moreover, the previous result is also not true for $\mathbb{R}^n$, $n > 1$.