Fourier Inversion for L^1 Functions

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Overview

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1. $(f * h_t)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi t \|\xi\|^2} e^{2\pi \iota x \cdot \xi} \, \mathrm{d}\xi$ for <u>all</u> $x \in \mathbb{R}^n$.

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2. Using DCT, we let $t \to 0$ in the above via $\{t_n\}$ to conclude that

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   $$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi \iota x \cdot \xi} \, \mathrm{d}\xi$$

   for those $x \in \mathbb{R}^n$ for which $(\star)$ holds.

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We will actually prove the result for a broader class of approximate identities.

Note that if $f \in L^1$, then f is finite a.e. Thus, we may assume that f is finite everywhere by changing it on a null set.

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for all $x \in \text{Leb}(f)$.

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It is now clear that proving the Main Theorem will show that ($\star$) holds for $x \in \mathrm{Leb}(f)$.

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$$\int_{B(x,r)} 1 = V_n r^n$$

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Recap on Polar Coordinates

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Note that for all $t > 0$, we have $\displaystyle\int_{\mathbb{R}^n} \varphi_t = \int_{\mathbb{R}^n} \varphi = \int_{\mathbb{R}^n} |\varphi| = 1$.

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Thus, for all $t > 0$, we have

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The A Game

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Using this, we first show that $I_2(t) \xrightarrow{t \to 0} 0$.

Taking I_2 down

$I_2(t)$

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$$I_2(t) = \left| \int_{\|u\| \geq \delta} [f(x-u) - f(x)] \varphi_t(u) \, \mathrm{d}u \right|$$

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With these notations, we do some more calculations.

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Integrate by parts

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I_1(t) &\leq \int_{\|u\|<\delta} |f(x-u) - f(x)| t^{-n} \varphi(u/t)\, \mathrm{d}u \\
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&= G(r) t^{-n} \psi_0(r/t)\Big|_0^\delta - \int_0^\delta G(r)\, \mathrm{d}(t^{-n} \psi_0(r/t))
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This completes the proof.

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Reference: *Introduction to Fourier Analysis on Euclidean Spaces* by *Stein and Weiss*