MA-412
Complex Analysis

Aryaman Maithani
https://aryamanmaithani.github.io/

Spring Semester 2019-20

Contents

Chapter 1

The Basics

This chapter lists the basic definitions and facts that will be assumed later on.
Notation: The following notation will be used through the notes:

1. $\mathbb{N}$ will denote the set of **positive** integers. That is, $\mathbb{N} = \{1, 2, \ldots\}$.
2. $\mathbb{Z}$ will denote the set of integers.
3. $\mathbb{Z}_{\geq 0}$ will denote the set of all **non-negative** integers. That is, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$.
4. $\mathbb{Q}$ will denote the set of rationals.
5. $\mathbb{R}$ will denote the set of real numbers.
6. $\mathbb{R}^+$ will denote the set of **positive** reals. That is, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.
7. $\mathbb{R}_{\geq 0}$ will denote the set of **non-negative** reals. That is, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\} = \mathbb{R}^+ \cup \{0\}$.
8. $\mathbb{R}^\times$ will denote the set of **non-zero** reals. That is, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.
9. $\mathbb{C}$ will denote the set of complex numbers.
10. $\mathbb{C}^\times$ will denote the set of **non-zero** complex numbers. That is, $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.
11. For $\Omega \subset \mathbb{R}^2$, $\mathcal{C}^n(\Omega)$ denotes the set of real-valued functions defined on Ω with all n-th partial derivatives continuous. (Partial derivatives refer to the usual derivatives of real functions.)
12. We shall write $a_n \to l$ as a shorthand for $\lim\limits_{n\to\infty} a_n = l$.

By abuse of notation, the above symbols will also be used to denote not only the set but the algebraic structure as well. That is, $\mathbb{R}$ will also denote the *field* of real numbers, et cetera.
More abuse of notation will be done when we will regard a set Ω as a subset of both $\mathbb{C}$ and $\mathbb{R}^2$.
The letter i will always be used to denote a root of $x^2 + 1 = 0$. We fix one such root of the rest of our discussion.

Definition 1.1 (Real and imaginary part)**.** Given any complex number $z \in \mathbb{C}$, it can be written as

$$z = x + iy$$

for unique real numbers x and y, which we call the *real* part and *imaginary* part of z, respectively.
We also denote them by $\Re z$ and $\Im z$, respectively.

Definition 1.2 (Conjugate)**.** Given any complex number $z \in \mathbb{C}$, its conjugate $\bar{z}$ is defined as

$$\bar{z} = z - 2i\Im z.$$

In other words, if $z = x + iy$ for $x, y \in \mathbb{R}$, then $\bar{z} = x - iy$.

Definition 1.3 (Absolute value)**.** The *absolute value* (or *modulus*) of a complex number z is denoted by $|z|$ and is defined as

$$|z| = \sqrt{(\Re z)^2 + (\Im z)^2}.$$

Where $\sqrt{\cdot}$ denotes the nonnegative square of a real number.

Definition 1.4 (Principal argument)**.** Assume $z \in \mathbb{C}^\times$, so that $|z| \neq 0$ and $\dfrac{\Re z}{|z|}$ and $\dfrac{\Im z}{|z|}$ are both real numbers in $[-1, 1]$. Moreover, the point $\left(\dfrac{\Re z}{|z|}, \dfrac{\Im z}{|z|}\right)$ lies on the unit circle centered at the origin.
Them, there exists a *unique* $\theta \in (-\pi, \pi]$ such that

$$\frac{\Re z}{|z|} = \cos\theta, \quad \frac{\Im z}{|z|} = \sin\theta.$$

This θ is called the *principal argument* of the complex number z and is denoted by $\operatorname{Arg} z$.

With the above, every nonzero complex number can be written as

$$z = \Re z + i\Im z = |z| \left(\cos\left(\operatorname{Arg} z\right) + \sin\left(\operatorname{Arg} z\right)\right).$$

Theorem 1.5 (n-th roots)**.** Every non-zero complex number has precisely n n-th roots.

Proof. Let $r = |z|$ and $\theta = \operatorname{Arg} z$.
Define $\zeta := \cos\left(\dfrac{2\pi}{n}\right) + i\sin\left(\dfrac{2\pi}{n}\right)$.
Define $\xi = r^{1/n}\left(\cos\left(\dfrac{\theta}{n}\right) + i\sin\left(\dfrac{\theta}{n}\right)\right)$.
(Where $r^{1/n}$ denotes the unique positive real number whose n-th power is r. Its existence is given by Real Analysis.)
One can verify that $\zeta, \ldots, \zeta^n$ are n distinct n-th roots of 1 and $\xi^n = z$. Thus, we get that

$$\xi\zeta, \xi\zeta^2, \ldots, \xi\zeta^n$$

are n distinct n-th roots of z. $\square$

We shall assume that the reader is familiar with basic topological terms such as open sets, closed sets, connected sets, closure (and interior) of a set, limit points, et cetera.

Definition 1.6 (Domain)**.** A *domain* is a non-empty, open-connected subset of $\mathbb{C}$.

In these notes, Ω will always denote an open subset of $\mathbb{C}$. (Which need not necessarily be a domain.)

Definition 1.7 (Convex domain)**.** A domain is said to be *convex* if the line segment joining any of its points lies entirely within it.

Definition 1.8 (Star-shaped domain)**.** A domain Ω is said to be *star-shaped* if there exists $z_0 \in \Omega$ such that given any $z \in \Omega$, the line segment joining z and z_0 lies within Ω.

Definition 1.9 (Balls)**.** For $\delta > 0$ and $z \in \mathbb{C}$, the open ball $B_\delta(z)$ is defined as

$$B_\delta(z) = \{z' \in \mathbb{C} \mid |z - z'| < \delta\}.$$

We now state a lemma for $\mathbb{C}$ that will be useful later.

Lemma 1.10. Let $\Omega \subset \mathbb{C}$ be open.
Suppose $C \subset \Omega$ is compact. Then, there exists a compact set $D \subset \Omega$ such that $C \subset \operatorname{int} D$.

[↓]

Chapter 2

Differentiation

§2.1. Definition

Definition 2.1 (Differentiable at a point)**.** Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a function. We say that f is differentiable at $z_0 \in \Omega$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, in which case, we denote it by $f'(z_0)$ and call it the derivative of f at z_0.

Definition 2.2 (Holomorphy)**.** We say that $f : \Omega \to \mathbb{C}$ is holomorphic if f is differentiable at each point in Ω.

Note very carefully that we have only talked about differentiability of functions defined on *open* sets.

Definition 2.3 ($A(\Omega)$)**.** For $\Omega \subset \mathbb{C}$ open, we define $A(\Omega)$ to be the ring of holomorphic functions defined on Ω.

(The ring operations are the natural point-wise ones. That this is a ring is an easy check.)
The usual rules of differentiation from real analysis still hold and can be derived similarly. To name a few, we have:

1. If f is differentiable at a point, then f is continuous at that point.
2. $(f + g)'(z_0) = f'(z_0) + g'(z_0)$.
3. $(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$.
4. Constant functions are differentiable with derivative 0.

5. For any $n \in \mathbb{N}$, the function $f : \mathbb{C} \to \mathbb{C}$ defined as $z \mapsto z^n$ is differentiable with $f'(z_0) = nz_0^{n-1}$.

6. Polynomials are differentiable.

7. If f and g are differentiable at z_0 with $g(z_0) \neq 0$, then f/g is differentiable at z_0 with
$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - g'(z_0)f(z_0)}{(f(z_0))^2}.$$

Proposition 2.4 (Increment lemma)**.** Let $f : \Omega \to \mathbb{C}$ be differentiable at z_0. TFAE:

(i) f is complex differentiable at z_0.

(ii) There exists function $\psi : \Omega \to \mathbb{C}$ such that
$$f(z) = f(z_0) + (z - z_0)\psi(z)$$
and ψ is continuous at z_0.
In this case $\psi(z_0) = f'(z_0)$.

Proposition 2.5 (Chain rule)**.** Suppose $f : \Omega_1 \to \mathbb{C}$ is differentiable at z_0 and $g : \Omega_2 \to \mathbb{C}$ is differentiable at $f(z_0)$. (Of course, $f(z_0) \subset \Omega_2$.)
Then, $g \circ f$ is differentiable at z_0 with
$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

§2.2. Cauchy Riemann Equations

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose it into its real and imaginary parts as follows:
Define the functions $u, v : \Omega \to \mathbb{C}$ as
$$u(z) := \Re f(z), \quad v(z) := \Im f(z).$$
For the remainder of these notes, whenever we write $f = u + iv$, it is to be understood that u and v have the above meaning.
We will also regard u and v as real valued functions of the real part and imaginary part of z.
To be more precise, we also have the functions $\tilde{u}, \tilde{v}$ defined on $\tilde{\Omega} = \{(x, y) \in \mathbb{R}^2 \mid x + iy \in \Omega\}$ as
$$\tilde{u}(x, y) := u(x + iy), \quad \tilde{v}(x, y) := v(x + iy).$$
By abuse of notation, we will drop the ˜ and just write u and v. (Note that $\tilde{\Omega}$ is just Ω regarded as a subset of $\mathbb{R}^2$.)

Theorem 2.6 (The Cauchy-Riemann Equations)**.** Suppose $f : \Omega \to \mathbb{C}$ is differentiable at $z_0 = x_0 + iy_0$. Then,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \text{ and} \tag{2.1}$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.2}$$

Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Note that the last equation only has partial derivatives with respect to x.
(2.1) to (2.2) are called the Cauchy-Riemann equations or CR equations, for short.

Corollary 2.7. Suppose Ω is an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ holomorphic.
Regard f as a map $\Omega \to \mathbb{R}^2$ and Ω as an open set in $\mathbb{R}^2$. Then,

$$(\text{Jacobian } f)(x, y) = |f'(x + iy)|^2.$$

Theorem 2.8 (A converse)**.** Suppose $u, v \in \mathcal{C}^1(\Omega)$ and u, v satisfy (2.1) to (2.2).
Then, $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$.

Corollary 2.9. If u, v satisfy (2.1) to (2.2) throughout Ω, then $f = u + iv \in A(\Omega)$.

Definition 2.10 (Harmonic conjugate)**.** Suppose Ω is an open set in $\mathbb{C}$ and (u, v) is a pair of real values $\mathcal{C}^1$ functions satisfying the CR equations. Then, we say that v is a harmonic conjugate of u.

Note that if v is a harmonic conjugate of u, then a harmonic conjugate of v is $-u$.

Definition 2.11 (Harmonic functions)**.** The Δ operator is defined as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Any solution of the equation $\Delta f = 0$ is called a *harmonic function*.

Proposition 2.12. Suppose $u, v \in \mathcal{C}^2(\Omega)$ and the pair (u, v) satisfies the CR equations. Then, $\Delta u = \Delta v = 0$.
In other words, u and v are harmonic functions.

Definition 2.13 (Entire functions)**.** An entire function is a function which is holomorphic on $\mathbb{C}$.

Chapter 3

Power Series

The reader familiar with basic definition like those of e, absolute convergence, root test, et cetera can skip to section 3.4.

§3.1. Preliminaries

Theorem 3.1 (AM-GM-HM)**.** Let $a_1, \ldots, a_n \in \mathbb{R}^+$. Then,

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

[↓]

Corollary 3.2. (i) if $a > 0$, then $\sqrt[n]{a} \to 1$,

(ii) $\sqrt[n]{n} \to 1$,

(iii) For any $x \in \mathbb{R}$,
$a_n := \left(1 + \dfrac{x}{n}\right)^n$ is bounded above and eventually monotonically increasing,
$b_n := \left(1 - \dfrac{x}{n}\right)^{-n}$ is bounded below and eventually monotonically decreasing.
Moreover, both (a_n) and (b_n) have a common limit, denoted by e^x.

[↓]

Note that the above is the definition of e^x for real x. The constant e is, by definition, e^1.

Theorem 3.3 (Cauchy's first limit theorem)**.** Suppose (a_n) is a sequence of complex or real numbers and $a_n \to l$.
Then, $\frac{1}{n}(a_1 + \cdots + a_n) \to l$.

[↓]

Corollary 3.4. Let (a_n) be a sequence of positive reals converging to l.
Then, $(a_1 \cdots a_n)^{1/n} \to l$.

[↓]

Theorem 3.5 (Cauchy's second limit theorem)**.** If (a_n) is a sequence of positive reals such that

$$\frac{a_{n+1}}{a_n} \to l,$$

then

$$\sqrt[n]{a_n} \to l.$$

[↓]

Corollary 3.6.

$$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}.$$

[↓]

§3.2. Infinite series

Definition 3.7 (Infinite series)**.** Given a sequence (z_n) of complex numbers, the formal expression $\displaystyle\sum_{n=1}^{\infty} z_n$ is called an infinite series.

Define the sequence of *partial sums* as $S_n := \displaystyle\sum_{k=1}^{n} z_k$.

We say the series $\sum z_n$ converges if S_n converges and we write $\sum z_n = \lim_{n\to\infty} S_n$.
Otherwise, we say the series $\sum z_n$ diverges.

Theorem 3.8 (Cauchy criterion)**.** A series $\sum z_n$ of complex numbers converges if and only if the following condition (known as Cauchy criterion) holds:

Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left|\displaystyle\sum_{k=N}^{N+m} z_n\right| < \epsilon$ for all $m \in \mathbb{Z}_{\geq 0}$.

The above is simply a consequence of the fact that $\mathbb{C}$ is a complete metric space since the criterion above is just the usual Cauchy criterion for sequences applied to the sequence of partial sums.

Proposition 3.9 (Some results). 1. If $\sum a_n$ converges, then $a_n \to 0$.

2. The series $\sum ar^n$ converges for any $a \in \mathbb{C}$ if $|r| < 1$. If $a = 0$, it converges for all $r \in \mathbb{C}$. If $a \neq 0$, then it diverges for $|r| \geq 1$. ($r \in \mathbb{C}$.)

3. A series of (real) nonnegative terms converges if and only if its partial sums form a bounded sequence.

Theorem 3.10 (Comparison test)**.** Let (a_n) be a sequence of complex numbers and $(c_n), (d_n)$ be real sequences.

(i) If $|a_n| \leq c_n$ for n sufficiently large, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(ii) If $a_n \geq d_n \geq 0$ for n sufficiently large, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Part (ii) assumes that a_n is eventually real and nonnegative.

[↓]

Theorem 3.11 (Cauchy's condensation test)**.** Suppose (a_n) is a monotone decreasing sequence of nonnegative reals. Then,

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converges.}$$

[↓]

Corollary 3.12. For $p \in \mathbb{R}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Theorem 3.13 (Alternating series test)**.** Suppose that (a_n) is a monotone decreasing sequence of real numbers.

The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges if and only if $a_n \to 0$.

[↓]

Definition 3.14 ($\limsup$)**.** Recall the definition of $\limsup$ of a sequence (a_n) of real numbers.
For each $n \in \mathbb{N}$, define the sequence s_n as

$$s_n := \sup\{a_m \mid m \geq n.\}$$

Then (s_n) is a decreasing sequence and thus, $\alpha = \lim\limits_{n\to\infty} s_n$ exists. We denote α as $\limsup_{n\to\infty} a_n$.

Note in the above that s_n can be eventually ∞ or that α can be ∞.

Theorem 3.15 (Root test)**.** Given $\sum a_n$, put $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$.
Then,

(i) if $\alpha < 1$, $\sum a_n$ converges,

(ii) if $\alpha > 1$, $\sum a_n$ converges,

(iii) if $\alpha = 1$, the test gives no information.

[↓]

Note that the above test is particularly useful when $\left(\sqrt[n]{|a_n|}\right)$ is a decreasing sequence for then $\limsup$ can be replaced with $\lim$.

Theorem 3.16. For $x \in \mathbb{R}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ equals e^x.

[↓]

§3.3. Absolute convergence

Definition 3.17 (Absolutely converging series)**.** The series $\sum a_n$ (of complex numbers) is said to be absolutely convergent if $\sum |a_n|$ converges.

Proposition 3.18. An absolutely convergent series is convergent.

The above follows directly from the Comparison test.

Definition 3.19 (Conditionally convergent series)**.** A series which is convergent but not absolutely convergent is called conditionally convergent.

Definition 3.20 (Rearrangements)**.** Suppose that $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection.
We say that $\displaystyle\sum_{n=1}^{\infty} a_{\sigma(n)}$ is a rearrangement of the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

In general, the rearrangement of a series can behave differently from the original one. It may be possible that one diverges while the other converges or that both converge but to different limits. The following theorem sheds more light on this.

Theorem 3.21 (Riemann)**.** Given a conditionally convergent series of *real numbers,* and $c \in \mathbb{R}$, we can find a rearrangement of the series such that it converges to c.

However, absolutely convergent series behave much better as seen by the following theorem.

Theorem 3.22 (Dirichlet)**.** Every rearrangement of an absolutely convergent series (of complex numbers) is absolutely convergent and converges to the same limit.

[↓]

Definition 3.23 (Cauchy product)**.** Suppose $\displaystyle\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ are two infinite series.
Their Cauchy product is the series $\displaystyle\sum_{n=0}^{\infty} c_n$ where $\displaystyle c_n := \sum_{j=0}^{n} a_j b_{n-j}$.

Theorem 3.24 (Cauchy product convergence)**.** If $\displaystyle\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge absolutely, then their Cauchy product converges absolutely to $\displaystyle\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

[↓]

§3.4. Power series

Definition 3.25 (Power series)**.** Let (a_n) be a sequence of complex numbers and $z_0 \in \mathbb{C}$. The series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{3.1}$$

is said to be a power series with center at z_0.

In the above, it is to be understood that $(z - z_0)^0 = 1$ for all z.

Proposition 3.26. If a power series (3.1) converges at a point z_1 such that $z_1 \neq z_0$, then it converges absolutely throughout the open disc

$$D = \{z \in \mathbb{C} \mid |z - z_0| < |z - z_1|\}.$$

[↓]

The above proposition then lets us characterise precisely the region of convergence of a power series.

Theorem 3.27 (Region of convergence)**.** Given a power series (3.1), precisely one of the following holds:

(i) The series converges for $z = z_0$ and no other $z \in \mathbb{C}$,

(ii) The series converges for all $z \in \mathbb{C}$, or

(iii) There exists a real number $R > 0$ such that the power series converges absolutely for all $z \in \{z \mid |z - z_0| < R\}$ and diverges for all $z \in \{z \mid |z - z_0| > R\}$. Moreover, the above R is unique.

[↓]

Note that in the third case, we make no comment about the convergence on the *boundary* itself.

Definition 3.28 (Radius of convergence)**.** Given a power series $\displaystyle\sum_{n=0}^{\infty} a_n(z - z_0)^n$, as per the three possibilities above, the radius of convergence is defined to be

(i) 0,

(ii) ∞, or

(iii) R.

In other words, it equals $\sup \left\{ R \in \mathbb{R} \;\middle|\; \displaystyle\sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ converges for all } z \text{ satisfying } |z - z_0| < R \right\}$.

Examples.

1. $\displaystyle\sum_{n=1}^{\infty} n^n z^n$ has radius of convergence 0.

2. $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence ∞.

3. $\displaystyle\sum_{n=0}^{\infty} z^n$ has radius of convergence 1.

Theorem 3.29 (Calculating the radius of convergence)**.** Given the power series $\sum a_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \quad R = \alpha^{-1}.$$

(If $\alpha = 0$, then $R = \infty$ and if $\alpha = \infty$, then $R = 0$.)
Then, R is the radius of convergence.

[↓]

Definition 3.30 (Disc of convergence)**.** Given a power series with nonzero radius of convergence, the union of all the open discs on which it converges it called its disc of convergence.

If the radius of convergence is ∞, then the disc is $\mathbb{C}$, else it is the disc $D = \{z \mid |z - z_0| < R\}$ where z_0 is the center of the power series and R is its radius of convergence.
Note that D is always open. Moreover, we have that the series converges *absolutely* within D.

Lemma 3.31. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has D as its disc of convergence with positive radius, then $\displaystyle\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ also converges absolutely on D.

[↓]

Likewise, we have the convergence of $\displaystyle\sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}$.

Theorem 3.32 (Differentiation theorem)**.** Suppose $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a power series with positive radius of convergence and D as its disc of convergence. The sum f is holomorphic on D.
Further,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

[↓]

Theorem 3.33 (Abel's limit theorem)**.** Suppose $\sum_{n=0}^{\infty} a_n$ converges and $f(z) := \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Then,

$$\lim_{\substack{z \to 1^- \\ z \in \mathbb{R}}} f(z) = \sum_{n=0}^{\infty} a_n.$$

[↓]

Note that the above limit reads

$$\lim_{z \to 1^-} \lim_{n \to \infty} \sum_{j=0}^{n} a_j z^j = \lim_{n \to \infty} \lim_{z \to 1^-} \sum_{j=0}^{n} a_j z^j.$$

That is, there is an interchange of limits at play.
We note the following corollary of the theorem which states another result about the Cauchy product.

Corollary 3.34. Suppose $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ are two series converging to A and B, respectively.
Assume that their Cauchy product also converges. Let C be this sum.
Then, $C = AB$.

[↓]

Definition 3.35 (Some familiar functions)**.** For $z \in \mathbb{R}$, we define the following functions as power series:

1. $\exp z := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
2. $\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.
3. $\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

Note that the above are entire functions.
By Theorem 3.16, $\exp x$ agrees with $e^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ for real x. (Well, the earlier proof had been incomplete and only proven the result for $x \geq 0$. The case $x < 0$ required the following theorem. Note, however, that there has not been any circular reasoning.)

Theorem 3.36 (Exponential addition theorem)**.** For $z, w \in \mathbb{C}$, we have

$$\exp(z+w) = \exp(z)\exp(w).$$

[↓]

We shall also use the notation e^z instead of $\exp z$.

Chapter 4

Cauchy Integral Theorem and elementary properties of holomorphic functions

§4.1. Preliminaries

In what follows, Ω is an open subset of $\mathbb{R}^2$.

Definition 4.1 (Paths)**.** A *path* in Ω is a continuous, piecewise smooth function $\gamma : [a, b] \to \Omega$.
Namely, there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that for all $j \in \{1, \ldots, n\}$,:

(i) γ is differentiable on (t_{j-1}, t_j),

(ii) γ' is continuous on (t_{j-1}, t_j),

(iii) $\lim\limits_{t \to t_{j-1}^+} \gamma'(t)$ an d $\lim\limits_{t \to t_j^-} \gamma'(t)$ exist.

Definition 4.2 (Perimeter)**.** The *perimeter* of a path $\gamma : [a, b] \to \Omega$ is defined to be

$$\int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

The above is an ordinary Riemann integral which will exist since γ' is continuous on $[a, b]$ except possibly on finitely many points. Note that γ' above is the one from real multivariable calculus. To be explicit, if $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, then $|\gamma'(t)| = \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))}$.

Definition 4.3 (Reverse path)**.** Let $\gamma : [a, b] \to \Omega$ be a path. The *reverse path* of γ is denoted by $\bar{\gamma}$ where $\bar{\gamma} : [a, b] \to \Omega$ is defined by

$$\bar{\gamma}(t) = \gamma(a + b - t).$$

Definition 4.4 (Juxtaposition of two paths)**.** Suppose $\gamma : [a, b] \to \Omega$ and $\sigma : [b, c] \to \Omega$ are two paths such that $\gamma(b) = \sigma(b)$. Their *juxtaposition* is the path $\gamma * \sigma : [a, c] \to \Omega$ given by

$$(\gamma * \sigma)(t) := \begin{cases} \gamma(t) & a \leq t \leq b \\ \sigma(t) & b < t \leq c. \end{cases}$$

Note that it should be verified that the juxtaposition is indeed a path.

Definition 4.5 (A proper reparameterisation)**.** Suppose $\gamma : [a, b] \to \Omega$ is a path. A *proper reparameterisation* is a path $\sigma \circ \gamma : [c, d] \to \Omega$ where $\sigma : [c, d] \to [a, b]$ is a strictly increasing bijection such that σ' exists and is positive except possibly at finitely many points such that the one sided limits exist at these exceptional points.

Definition 4.6 (Line integrals of vector fields)**.** Suppose $\Phi : \Omega \to \mathbb{R}^2$ is a continuous function and γ is a path in Ω.
Suppose $\Phi(x, y) = (P(x, y), Q(x, y))$ and $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Then,

$$\int_\gamma \Phi := \int_a^b [P(\gamma(t))\gamma_1'(t) + Q(\gamma(t))\gamma_2'(t)] \, \mathrm{d}t$$

The integral on the right exists as an ordinary Riemann integral by our assumption on Φ and γ.
We may also denote (by abuse) the above integral as

$$\int_\gamma P\mathrm{d}x + Q\mathrm{d}y \text{ or } \int_\gamma \Phi \cdot \mathrm{d}r.$$

Definition 4.7 (Integrals of complex functions)**.** Let $f : \Omega \to \mathbb{C}$ be a complex continuous function. As usual, let $f = u + iv$. We define

$$\int_\gamma f := \int_\gamma u\mathrm{d}x - v\mathrm{d}y + i \int_\gamma u\mathrm{d}y + v\mathrm{d}x.$$

Where the integrals on the right were defined earlier. We may also denote the above integral as

$$\int_\gamma f(z)\mathrm{d}z.$$

One can check that the above integral is the same as

$$\int_a^b f(\gamma(t))(\gamma_1'(t) + i\gamma_2'(t))\mathrm{d}t.$$

Note that in the above, we have regarded Ω as a subset of $\mathbb{R}^2$. We shall now resume to use Ω as a subset of $\mathbb{C}$ and thus, it would make sense to talk about holomorphic functions. It is clear how one can define γ to be either of the form $(\gamma_1(t), \gamma_2(t))$ or of the form $\gamma_1(t) + i\gamma_2(t)$.

Examples

1. Let $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$.
   If $k \in \mathbb{Z} \setminus \{-1\}$, then
   $$\int_\gamma z^k \mathrm{d}z = 0.$$
   (The above being true for negative k as well.)
   On the other hand,
   $$\int_\gamma \frac{1}{z}\mathrm{d}z = 2\pi i.$$

Theorem 4.8 (FTC)**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and f' is continuous. The, for any path γ is Ω, we have

$$\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)).$$

In particular, if γ is closed, the integral is zero.

[↓]

Note that f' above is of course, the complex derivative.

Corollary 4.9. Suppose $f : \Omega \to \mathbb{C}$ admits a primitive F. (That is, a function $F : \Omega \to \mathbb{C}$ such that $F' = f$.)
Then, $\displaystyle\int_\gamma f = 0$ for any closed path γ in Ω.

Lemma 4.10 (M-L inequality)**.** Let $f : \Omega \to \mathbb{C}$ be a continuous function and $\gamma : [a, b] \to \Omega$ be a path. Then,

$$\left|\int_\gamma f\right| \le \left(\sup_{t\in[a,b]} |f(\gamma(t))|\right) \cdot (\text{perimeter}(\gamma))\,.$$

[↓]

§4.2. Some results

Definition 4.11 (Some useful notations).

1. For complex numbers z_1, z_2, we shall use $l[z_1, z_2]$ to denote the line segment joining z_1 and z_2. That is,
$$l[z_1, z_2] = \{(1-t)z_1 + tz_2 \mid t \in [0, 1]\}.$$
2. For complex numbers z_1 and z_2, we shall write $\displaystyle\int_{z_1}^{z_2} f(z)\mathrm{d}z$ to mean the integral $\displaystyle\int_\gamma f$ where $\gamma : [0, 1] \to \mathbb{C}$ is defined as $\gamma(t) = (1-t)z_1 + tz_2$, the line segment starting at z_1 and ending at z_2.

Theorem 4.12 (Goursat's lemma). Suppose Ω is open and convex and $f : \Omega \to \mathbb{C}$ is differentiable. Then,
$$\int_T f = 0$$
for any closed triangle $T \subset \Omega$.

[↓]

Note that we cannot appeal to Theorem 4.8 (or Corollary 4.9) since we do not know, a priori, that f admits a primitive.
Also, note from the proof that we can actually weaken the hypothesis to conclude the following:

Proposition 4.13 (Stronger Goursat's lemma). Suppose Ω is open (not necessarily convex) and $f : \Omega \to \mathbb{C}$ is holomorphic.
Let $T \subset \Omega$ be a triangle satisfying the following:
There exists a set $C \subset \Omega$ such that $\widehat{T} \subset \operatorname{int} C$, where $\widehat{T}$ denotes the convex hull of T and $\operatorname{int} C$, the interior of C.
Then,
$$\int_T f = 0.$$

Corollary 4.14 (Cauchy's theorem for convex domains). Let $\Omega \subset \mathbb{C}$ be open and convex and $f : \Omega \to \mathbb{C}$ be holomorphic. Then, f admits a primitive. That is, there exists $F : \Omega \to \mathbb{C}$ such that F is holomorphic and $F' = f$.
In particular,
$$\int_\gamma f(z)\mathrm{d}z$$
for all closed paths γ.

[↓]

Corollary 4.15 (Cauchy's theorem for star-shaped domains)**.** If $\Omega \subset \mathbb{C}$ is open and star-shaped and $f : \Omega \to \mathbb{C}$ is holomorphic, then f admits a primitive.
In particular,

$$\int_\gamma f(z)\mathrm{d}z$$

for all closed paths γ.

[↓]

§4.3. Consequences of holomorphy

Theorem 4.16 (Cauchy integral formula)**.** Let Ω be an open convex set in $\mathbb{C}$.
Suppose $f : \Omega \to \mathbb{C}$ is holomorphic.
Let $p \in \Omega$ and $r > 0$ be such that $\overline{B_r(p)} \subset \Omega$.
Then, for any point $z \in B_r(p)$, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi,$$

where γ is the circle $\partial\overline{B_r(p)}$ traced counterclockwise.
(That is, $\gamma(t) = p + re^{it}$ for $t \in [0, 2\pi]$.)

[↓]

Corollary 4.17 (Holomorphic functions are analytic)**.** Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Then, for each $p \in \Omega$, there exists $r > 0$ such that $B_r(p) \subset \Omega$ and on $B_r(p)$, f admits a power series representation. That is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - p)^n \tag{4.1}$$

for all $z \in B_r(p)$.
In fact, the above is true for any r such that $B_r(p) \subset \Omega$.
In particular, if f is differentiable once, then f is infinitely differentiable. (Since power series are clearly infinitely differentiable in view of Differentiation theorem.)

[↓]

As the proof of the above theorem shows, we have an explicit formula for a_n.

Corollary 4.18. With everything as in the above theorem, we have

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - p)^{n+1}}\mathrm{d}\xi, \tag{4.2}$$

where $\gamma(t) = p + re^{it}$ for $t \in [0, 2\pi]$.

Theorem 4.19 (Cauchy's estimate)**.** With notation as earlier, if $\overline{B_R(p)} \subset \Omega$, then

$$|f^{(k)}(p)| \le \frac{k!}{R^k}\sup_{B_R(p)}|f|.$$

[↓]

Theorem 4.20 (Bounded entire functions)**.** A bounded entire function is constant.

[↓]

Corollary 4.21 (Fundamental Theorem of Algebra)**.** Every non-constant polynomial with complex coefficients has a complex root.

Recall Green's theorem from multivariable calculus.

Theorem 4.22 (Green's Theorem)**.** Suppose that P and Q are continuously differentiable on an open set $\Omega \subset \mathbb{R}^2$ and γ is a simple closed curve lying in Ω such that $\mathrm{int}(\gamma) \subset \Omega$. Then,

$$\oint_\gamma P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_{\mathrm{int}\,\gamma}\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)\mathrm{d}(x,y),$$

where the curve γ is traced counterclockwise.

(Note that in the above, $\mathrm{int}\,\gamma$ is not the usual interior.)
Note that now we know that if f is differentiable once, then its derivative is continuous. Using Green's theorem, we can now strengthen the result of Cauchy's theorem for convex domains as follows.

Theorem 4.23 (Cauchy's Integral Theorem (basic))**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Let γ be a simple closed curve in Ω such that $\mathrm{int}\,\gamma \subset \Omega$. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

[↓]

Note that similarly, Cauchy integral formula can be improved to the following. The proof is left to the reader.

Theorem 4.24 (Cauchy's Integral Formula)**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Let γ be a simple closed curve in Ω (oriented positively) such that $\operatorname{int}\gamma \subset \Omega$. If $z_0 \in \operatorname{int}\gamma$, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z.$$

Now, we see an important lemma that helps us prove other powerful results about holomorphic functions.

Lemma 4.25. Let Ω be a **connected** open set in $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $p \in \Omega$. Suppose f is constant in a neighbourhood of p. Then, f is constant on Ω.

[↓]

Theorem 4.26. Let Ω be a **connected** open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be holomorphic. TFAE:

(i) $f \equiv 0$ on Ω,

(ii) There exists a sequence (p_n) of distinct points of Ω such that $p_n \to p \in \Omega$ and $f(p_n) = 0$ for all $n \in \mathbb{N}$, (note that the limit is in Ω)

(iii) There exists a point $p \in \Omega$ such that $f^{(k)}(p) = 0$ for all $k \geq 0$.

[↓]

Corollary 4.27 (Identity Theorem)**.** Suppose Ω is a **connected** open set in $\mathbb{C}$ and $f, g : \Omega \to \mathbb{C}$ are holomorphic. If the set

$$\{z \in \Omega \mid g(z) = f(z)\}$$

has a limit point *in* Ω, then $f \equiv g$.

Note very carefully that we require the limit point to be in Ω.

Corollary 4.28. The zeroes of a non-constant holomorphic function defined on an open connected set must be isolated.

Now, we state (and prove) a converse of sorts to Cauchy's theorem.

Theorem 4.29 (Morera's Theorem)**.** Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be continuous. Assume that

$$\int_T f = 0$$

for all triangles $T \subset \Omega$. Then, f is holomorphic on Ω.

[↓]

Corollary 4.30 (Montel's theorem)**.** Let Ω be an open set in $\mathbb{C}$ and (f_n) be a sequence of functions in $A(\Omega)$ converging uniformly to f on compact subsets of Ω.
Then, the limit function f is also holomorphic. Further, $f_n' \to f'$ uniformly on compact subsets of Ω.

[↓]

In particular, if $\sum f_n$ is an infinite series of holomorphic functions converging uniformly on compact subset of Ω to f, then $f' = \sum f_n'$.

Lemma 4.31. Let Ω be a connected open set and $f : \Omega \to \mathbb{C}$ be holomorphic. If $|f|$ is constant, then so is f.

[↓]

Theorem 4.32 (Maximum Modulus Theorem)**.** Suppose Ω is a connected open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is holomorphic such that $|f|$ attains a local maximum at a point $p \in \Omega$. Then, f is constant.

[↓]

Corollary 4.33. Let D be a (bounded) closed disc and $f : D \to \mathbb{C}$ be a non-constant continuous function which is holomorphic on the interior of D.
Then,

$$\sup_{z \in D} |f(z)| = \sup_{z \in \partial D} |f(z)| .$$

Furthermore, for any $p \in D^\circ$, we have

$$f(p) < \sup_{z \in \partial D} |f(z)| .$$

The proof is simple. Since f is continuous and D is compact, we know that f achieves its supremum. By Maximum Modulus Theorem, we know that this cannot be achieved at any point in the interior.

Theorem 4.34 (Open Mapping Theorem)**.** If Ω is a **connected** open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function, then f is an open mapping.
That is, for every open $U \subset \Omega$, the set $f(U)$ is open in $\mathbb{C}$. In particular, $f(\Omega)$ is open.

[↓]

Chapter 5

Proofs

Lemma 1.10. Let $\Omega \subset \mathbb{C}$ be open.
Suppose $C \subset \Omega$ is compact. Then, there exists a compact set $D \subset \Omega$ such that $C \subset \operatorname{int} D$.

[↑]

Proof. For $x \in C$ let $\delta_x > 0$ be such that $B_{\delta_x}(x) \subset \Omega$. (Which exists because Ω is open.)
Then, $C \subset \bigcup\limits_{x \in C} B_{\delta_x/2}(x)$.
Since C is compact, only finitely many of $B_{\delta_x/2}(x)$ cover C. Let $I = \{x_1, \ldots, x_n\}$ be such that $C \subset \bigcup\limits_{x \in I} B_{\delta_x/2}(x)$.
Define $D = \bigcup\limits_{x \in I} \overline{B_{\delta_x/2}(x)}$.
Clearly, $C \subset \bigcup\limits_{x \in I} B_{\delta_x/2}(x) \subset \operatorname{int} D$.
Moreover, D is union of finitely many closed balls and hence, is closed and bounded and thus, compact.
Lastly, $D = \bigcup\limits_{x \in I} \overline{B_{\delta_x/2}(x)} \subset \bigcup\limits_{x \in I} B_{\delta_x}(x) \subset \Omega$, completing the proof. $\square$

Theorem 3.1 (AM-GM-HM)**.** Let $a_1, \ldots, a_n \in \mathbb{R}^+$. Then,

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \le \sqrt[n]{a_1 \cdots a_n} \le \frac{a_1 + \cdots + a_n}{n}.$$

[↑]

Proof. Note that it is enough to prove AM $\geq$ GM since the other inequality will follow by considering the reciprocals.
We prove AM $\geq$ GM via induction. The case $n = 2$ (and $n = 1$) is trivial and follows from manipulating $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$.
Assume that it is true for n. We prove it for $n + 1$.
Let $A_k := \dfrac{1}{k}(a_1 + \cdots + a_k)$ and $G_k := (a_1 \cdots a_k)^{1/k}$.
Then, we have

$$\begin{aligned}
A_{n+1} &= \frac{1}{n+1}(a_1 + \cdots + a_{n+1}) \\
&= \frac{1}{n+1}(nA_n + a_{n+1}) \\
&\geq \frac{1}{n+1}(nG_n + a_{n+1}) \\
&= \frac{1}{n+1}\left(nG_{n+1}\left(\frac{G_{n+1}}{a_{n+1}}\right)^{1/n} + a_{n+1}\right) \qquad (G_{n+1}^{n+1} = G_n^n a_{n+1}) \\
&= G_{n+1} \cdot \frac{1}{n+1}\left(n\left(\frac{G_{n+1}}{a_{n+1}}\right)^{1/n} + \frac{a_{n+1}}{G_{n+1}}\right) \\
\implies \frac{A_{n+1}}{G_{n+1}} &\geq \frac{1}{n+1}\underbrace{\left(n\left(\frac{G_{n+1}}{a_{n+1}}\right)^{1/n} + \frac{a_{n+1}}{G_{n+1}}\right)}_{(*)}
\end{aligned}$$

If we prove that $(*) \geq n + 1$, then we are done.
Call $\dfrac{G_{n+1}}{a_{n+1}} =: \theta^{-n}$.
Thus, we want to show that $n\theta^{-1} + \theta^n \geq n + 1$.
Note the following equivalences

$$\begin{aligned}
n\theta^{-1} + \theta^n \geq n + 1 &\iff \theta^{n+1} - \theta n - \theta + n \geq 0 \\
&\iff \theta(\theta^n - 1) - n(\theta - 1) \geq 0 \\
&\iff \theta(\theta - 1)(\theta^{n-1} + \cdots + 1) - n(\theta - 1) \geq 0 \\
&\iff (\theta - 1)[\theta^n + \cdots + \theta - n] \geq 0 \\
&\iff (\theta - 1)[(\theta^n - 1) + \cdots + (\theta - 1)] \geq 0.
\end{aligned}$$

The last inequality holds for $\theta = 1$, $\theta > 1$, and $0 < \theta < 1$. Thus, we are done! $\square$

Corollary 3.2. (i) if $a > 0$, then $\sqrt[n]{a} \to 1$,

(ii) $\sqrt[n]{n} \to 1$,

(iii) For any $x \in \mathbb{R}$,
$a_n := \left(1 + \frac{x}{n}\right)^n$ is bounded above and eventually monotonically increasing,
$b_n := \left(1 - \frac{x}{n}\right)^{-n}$ is bounded below and eventually monotonically decreasing.
Moreover, both (a_n) and (b_n) have a common limit, denoted by e^x.

[↑]

Proof. (i) Clearly true for $a = 1$. We assume $a > 1$. Proving this case is sufficient. (Why?)
For $n \geq 2$, apply AM-GM to $\underbrace{1, \ldots, 1,}_{n-1} a$ to get

$$1 \leq \sqrt[n]{a} \leq \frac{1}{n}(n - 1 + a) \to 1.$$

Sandwich theorem yields the answer.

(ii) For $n \geq 3$, apply AM-GM to $\underbrace{1, \ldots, 1,}_{n-2} \sqrt{n}, \sqrt{n}$ to get

$$1 \leq \sqrt[n]{n} \leq \frac{1}{n}(n - 2 + 2\sqrt{n}) \to 1.$$

Sandwich theorem yields the answer.

(iii) First assume $x > 0$.
We apply AM-GM to

$$\underbrace{1 + \frac{x}{n}, \ldots, 1 + \frac{x}{n}}_{n}, 1$$

to get

$$\begin{aligned}
\left(\left(1 + \frac{x}{n}\right)^n\right)^{1/(n+1)} &\leq \frac{1}{n+1}\left[n\left(1 + \frac{x}{n}\right) + 1\right] \\
&= 1 + \frac{x}{n+1} \\
\implies \left(1 + \frac{x}{n}\right)^n &\leq \left(1 + \frac{x}{n+1}\right)^{n+1} \\
\implies a_n &\leq a_{n+1}.
\end{aligned}$$

Similarly, we get $b_n \geq b_{n+1}$ for n sufficiently large. (By taking n large enough that $1 - x/n > 0$.)

Let $N \in \mathbb{N}$ be such that $1 - x/N > 0$.
Then, for all $n \ge N$, we have $a_N \le a_n \le b_n \le b_N$.
Thus, both (a_n) and (b_n) have a positive limit. It suffices to show that $\dfrac{a_n}{b_n} \to 1$.
Note that

$$\begin{aligned} \frac{b_n}{a_n} &= \left\{\left(1 - \frac{x^2}{n^2}\right)^{-n^2}\right\}^{1/n} \\ &=: R_n^{1/n}. \end{aligned}$$

Note that R_n is eventually bounded between positive constants, say α and β.
Thus, we have

$$1 \leftarrow \alpha^{1/n} \le R_n^{1/n} \le \beta^{1/n} \to 1.$$

The result follows from Sandwich Theorem.
The case $x < 0$ is handled by considering the reciprocals. (The case $x = 1$ is trivial.)

□

Theorem 3.3 (Cauchy's first limit theorem)**.** Suppose (a_n) is a sequence of complex or real numbers and $a_n \to l$.
Then, $\dfrac{1}{n}(a_1 + \cdots + a_n) \to l$.

[↑]

Proof. Let $\epsilon > 0$. There exists $N' \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon/2 \quad \forall n \ge N'.$$

Since a_n converges, $|a_n|$ is bounded. Let M be one such bound.
For $n > N'$, note that

$$\begin{aligned} \left|\frac{a_1 + \cdots + a_n}{n} - l\right| &= \frac{1}{n}\left|(a_1 - l) + \cdots + (a_n - l)\right| \\ &\le \frac{1}{n}\left\{|a_1 - l| + \cdots + |a_{N'} - l| + (n - N')\frac{\epsilon}{2}\right\} \\ &\le \frac{2N'M}{n} + \frac{\epsilon}{2}. \end{aligned}$$

Now, choose $N > N'$ such that $\dfrac{2N'M}{n} < \dfrac{\epsilon}{2}$ for all $n \ge N$.
Thus, we get that

$$\left|\frac{a_1 + \cdots + a_n}{n} - l\right| < \epsilon \quad \forall n > N,$$

as desired. □

Corollary 3.4. Let (a_n) be a sequence of positive reals converging to l. Then, $(a_1 \cdots a_n)^{1/n} \to l$.

[↑]

Proof.

Case 1. $l = 0$.
Then, $0 \leq \text{GM} \leq \text{AM} \to l$.

Case 2. $l > 0$.
Note that $\dfrac{1}{a_n} \to \dfrac{1}{l}$.
And thus, $\dfrac{1}{n}\left(\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n}\right) \to \dfrac{1}{l}$.
The result then follows from AM-GM-HM and Sandwich theorem.

□

Theorem 3.5 (Cauchy's second limit theorem)**.** If (a_n) is a sequence of positive reals such that

$$\frac{a_{n+1}}{a_n} \to l,$$

then

$$\sqrt[n]{a_n} \to l.$$

[↑]

Proof. Define (b_n) as $b_1 := a_1$, $b_n := \dfrac{a_n}{a_{n-1}}$ for $n \geq 2$.
By hypothesis, we have $\lim\limits_{n\to\infty} b_n = l$. By the previous corollary, we see that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (b_1 \cdots b_n)^{1/n} = \lim_{n\to\infty} b_n = l,$$

as desired. □

Corollary 3.6.

$$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}.$$

[↑]

Proof. Let $a_n := \dfrac{n!}{n^n}$. We wish to show that $\sqrt[n]{a_n} \to e^{-1}$.
By the previous corollary, it suffices to show that $\dfrac{a_{n+1}}{a_n} \to e$.
Note that

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}\\ &= \frac{(n+1)}{(n+1)^{n+1}}\frac{n^n}{1}\\ &= \frac{n^n}{(n+1)^n}\\ &= \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e},\end{aligned}$$

a desired. □

Theorem 3.10 (Comparison test)**.** Let (a_n) be a sequence of complex numbers and $(c_n), (d_n)$ be real sequences.

(i) If $|a_n| \leq c_n$ for n sufficiently large, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(ii) If $a_n \geq d_n \geq 0$ for n sufficiently large, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Part (ii) assumes that a_n is eventually real and nonnegative.

[↑]

Proof. We prove part (i) since (ii) follows from it.
Let $N_0 \in \mathbb{N}$ be such that $|a_n| \leq c_n$ for all $n > N_0$.
By the Cauchy criterion, there exists $N \geq N_0$ such that

$$\left|\sum_{k=N}^{N+m} c_k\right| < \epsilon$$

for all $m \in \mathbb{Z}_{\geq 0}$.
Hence, we also get that

$$\left|\sum_{k=N}^{N+m} a_k\right| \leq \sum_{k=N}^{N+m} |a_k| \leq \left|\sum_{k=N}^{N+m} c_k\right| < \epsilon$$

for all $m \in \mathbb{Z}_{\geq 0}$ concluding that $\sum a_n$ converges, again, by the Cauchy criterion. □

Theorem 3.11 (Cauchy's condensation test)**.** Suppose (a_n) is a monotone decreasing sequence of nonnegative reals. Then,

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converges.}$$

[↑]

Proof. Note that since $a_n \geq 0$, proving convergence of either of $\sum a_n$ or $\sum 2^n a_{2^n}$ just requires us to show that the corresponding sequence of partial sums is bounded. (Since the sequence of partial sums will be increasing.)

Assume that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Let $S_N := \sum_{k=1}^{N} a_k$.

Since (S_n) is increasing and $n \leq 2^n - 1$ for all $n \in \mathbb{N}$, we have $S_n \leq S_{2^n-1}$. This gives us

$$S_n \leq S_{2^n-1} \leq a_1 + \underbrace{(a_2 + a_3)}_{\leq 2a_2} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{\leq 4a_4} + \cdots + \underbrace{(a_{2^{n-1}} + \cdots + a_{2^n-1})}_{\leq 2^{n-1}a_{2^{n-1}}},$$

where the underbraced inequalities follow due to the monotonicity of (a_n). Thus, we get

$$S_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1}a_{2^{n-1}} \leq \sum_{n=0}^{\infty} 2^n a_{2^n},$$

showing that S_n is bounded and proving the convergence of $\sum a_n$ as desired.

Conversely, suppose that $\sum a_n$ converges and let (S_n) be as before. Note that

$$\begin{aligned} S_{2^n} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n}) \\ &\geq a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1}a_{2^n} \\ &= \frac{1}{2}\left(\sum_{k=1}^{n} 2^k a_{2^k}\right). \end{aligned}$$

This gives us that

$$\sum_{k=0}^{n} 2^k a_{2^k} \leq 2S_{2^n} + a_1 \leq a_1 + 2\sum_{n=1}^{\infty} a_n.$$

Thus, the partial sums of $\sum 2^n a_{2^n}$ are also bounded, as desired. $\square$

Theorem 3.13 (Alternating series test)**.** Suppose that (a_n) is a monotone decreasing sequence of real numbers.

The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if and only if $a_n \to 0$.

[↑]

Proof. Note that ($\implies$) is clear.
Conversely, suppose that $a_n \to 0$. In particular, we must have that $a_n \geq 0$ since a_n is decreasing.
The sequence of even partial sums (S_{2n}) is monotone increasing since

$$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2} \geq S_{2n}.$$

Similarly, the sequence of odd partial sums is monotonically decreasing. Furthermore, both of these sequences are bounded for

$$S_1 \geq S_{2n+1} \geq S_{2n} + a_{2n+1} \geq S_{2n} \geq S_2.$$

holds for all $n \in \mathbb{N}$.
In particular, (S_{2n}) is convergent. Let l be the limit. We show that $S_n \to l$. It suffices to show that $S_{2n+1} \to l$.
Note that

$$|S_{2n+1} - l| = |S_{2n} - l - a_{2n}| \leq |S_{2n} - l| + |a_{2n}| \to 0.$$

□

Theorem 3.15 (Root test)**.** Given $\sum a_n$, put $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$.
Then,

(i) if $\alpha < 1$, $\sum a_n$ converges,

(ii) if $\alpha > 1$, $\sum a_n$ converges,

(iii) if $\alpha = 1$, the test gives no information.

[↑]

Proof. (i) Assume $\alpha < 1$. Let β be such that $\alpha < \beta < 1$.
Thus, there exists $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$. (Since the supremum of the tail is eventually $\leq \alpha$.)
Since $\beta < 1$, the series $\sum \beta^n$ converges and by the Comparison test, $\sum a_n$ converges.

(ii) Assume $\alpha > 1$. Let β be such that $\alpha > \beta > 1$.
Then, there are infinitely $n \in \mathbb{N}$ for which $\sqrt[n]{|a_n|} > \beta$. (Otherwise, we'd have that $\sqrt[n]{|a_n|}$ is eventually $\le \beta$ and thus, so would the $\limsup$.)
Thus, $\sqrt[n]{|a_n|} \not\to 0$ and the result follows from item 1 of Proposition 3.9.

(iii) The series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ show this.

□

Theorem 3.16. For $x \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ equals e^x.

[↑]

Proof. Note that our definition of e^x is $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$. We show that this equals the series written above.
For $x = 0$, it is clear.
Let $x > 0$ and denote the sum of the series by $E(x)$.
Fix $n \in \mathbb{N}$. Using binomial theorem, we see

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= 1 + n\frac{x}{n} + \frac{n(n-1)}{2}\frac{x^2}{n^2} + \cdots + \frac{n(n-1)\cdots(n-(n-1))}{n!}\frac{x^n}{n!} \\
&= 1 + x + \frac{x^2}{2!}\left(1 - \frac{1}{n}\right) + \frac{x^3}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \\
&\quad \frac{x^n}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\
&\le 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} < E(x).
\end{aligned}$$

Thus, we get that $e^x \le E(x)$. To get the other inequality, we fix $n \in \mathbb{N}$ and let $N > n$. Recall that (iii) of Corollary 3.2 showed that (a_n) was (eventually) increasing and thus, we may write $e^x \ge \left(1 + \frac{x}{N}\right)^N$. (For $x > 0$, this is actually valid for all $N \in \mathbb{N}$.)
Expanding the latter using binomial theorem and only retaining the first n terms gives us

$$\begin{aligned}
e^x &\ge \left(1 + \frac{x}{N}\right)^N \\
&\ge 1 + x + \frac{x^2}{2!}\left(1 - \frac{1}{N}\right) + \frac{x^3}{3!}\left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right) + \cdots + \\
&\quad \frac{x^n}{n!}\left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\cdots\left(1 - \frac{n-1}{N}\right).
\end{aligned}$$

The above is valid for all $N > n$ and thus, letting $N \to \infty$ gives us

$$e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Note that the above is for any arbitrary n. Thus, we may let $n \to \infty$ to obtain the reverse inequality $e^x \geq E(x)$.
For $x < 0$, we make use of the result that $E(-x)E(x) = E(0) = 1$, which we will prove later in more generality. (Theorem 3.36)
The fact that $e^{-x} = 1/e^x$ follows from part (iii) of Corollary 3.2. □

Theorem 3.22 (Dirichlet)**.** Every rearrangement of an absolutely convergent series (of complex numbers) is absolutely convergent and converges to the same limit.

[↑]

Proof. Let $\sum a_n$ be absolutely convergent with sum S and $\sigma : \mathbb{N} \to \mathbb{N}$ be an arbitrary bijection.
Showing that the rearrangement is absolutely convergent is easy for we just need to bound the partial sums. To this end, let $N \in \mathbb{N}$ be fixed and choose $M := \max\{\sigma(1), \ldots, \sigma(N)\}$. Then, we have

$$\sum_{k=1}^{N} |a_{\sigma(k)}| \leq \sum_{k=1}^{M} |a_k| \leq S.$$

Let $T = \sum a_{\sigma(n)}$. The main result of this theorem is to show that $T = S$. We do this by showing that $|S - T|$ can be made arbitrarily small.
Let $N \in \mathbb{N}$ be fixed. Choose M sufficiently large such that $\{1, \ldots, N\} \subset \{\sigma(1), \ldots, \sigma(M)\}$. (It is clear that any $M' > M$ will also have this property.)
We now estimate $|S - T|$ as

$$|S - T| \leq \left| S - \sum_{n=1}^{N} a_n \right| + \left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{M} a_{\sigma(n)} \right| + \left| \sum_{n=1}^{M} a_{\sigma(n)} - T \right|. \tag{5.1}$$

Since $M \geq N$, the first and last terms go to 0 as $N \to \infty$.
Our choice of M shows that the middle sum in (5.1) results in a finite sum of terms a_j with $j \geq N + 1$ and hence, the middle term can be estimated as

$$\left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{M} a_{\sigma(n)} \right| \leq \sum_{j=N+1}^{\infty} |a_n|,$$

the latter of which goes to 0 as $N \to \infty$. (Cauchy criterion.)
Thus, letting $N \to \infty$ in (5.1) finishes the proof. □

Theorem 3.24 (Cauchy product convergence)**.** If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge absolutely, then their Cauchy product converges absolutely to $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

[↑]

Proof. Let $c_n := \sum_{j=0}^{n} a_j b_{n-j}$.
First we show the convergence of $\sum |c_n|$. This is simple for

$$\begin{aligned}
\sum_{n=0}^{N} |c_n| &= \sum_{n=0}^{N} \left| \sum_{j=0}^{n} a_j b_{n-j} \right| \\
&\leq \sum_{n=0}^{N} \sum_{j=0}^{n} |a_j||b_{n-j}| \\
&\leq \sum_{n=0}^{N} \sum_{j=0}^{N} |a_j||b_j| \\
&= \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right) \\
&\leq \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).
\end{aligned}$$

Thus, $\sum c_n$ converges absolutely. Let $C := \sum c_n$. We have to show that show $C = AB$ where $A := \sum_{n=0}^{\infty} a_n, B := \sum_{n=0}^{\infty} b_n$. Note that

$$|C - AB| \leq \left| C - \sum_{n=0}^{2N} c_n \right| + \left| \sum_{n=0}^{2N} c_n - \left(\sum_{i=0}^{N} a_i\right)\left(\sum_{j=0}^{N} b_j\right) \right| + \left| \left(\sum_{i=0}^{N} a_i\right)\left(\sum_{j=0}^{N} b_j\right) - AB \right|.$$

The first and last terms can clearly be made arbitrarily small by choosing N large enough. We show that this is true for the middle term as well.

Note that

$$\begin{aligned}
\left|\sum_{n=0}^{2N} c_n - \left(\sum_{i=0}^{N} a_i\right)\left(\sum_{j=0}^{N} b_j\right)\right| &= \left|\sum_{\substack{i+j\leq 2N\\ i>N \text{ or } j>N}} a_i b_j\right| \\
&\leq \sum_{\substack{i+j\leq 2N\\ i>N \text{ or } j>N}} |a_i||b_j| \\
&\leq \sum_{i>N} |a_i||b_j| + \sum_{j>N} |a_i||b_j| \\
&\leq \left(\sum_{j=0}^{\infty} |b_j|\right)\left(\sum_{i=N+1}^{\infty} |a_i|\right) + \left(\sum_{i=0}^{\infty} |a_i|\right)\left(\sum_{j=N+1}^{\infty} |b_j|\right) \\
&= B\left(\sum_{i=N+1}^{\infty} |a_i|\right) + A\left(\sum_{i=0}^{\infty} |a_i|\right).
\end{aligned}$$

Note that both the sums above can be made arbitrarily small by choosing N sufficiently large. □

Proposition 3.26. If a power series (3.1) converges at a point z_1 such that $z_1 \neq z_0$, then it converges absolutely throughout the open disc

$$D = \{z \in \mathbb{C} \mid |z - z_0| < |z - z_1|\}.$$

[↑]

Proof. The convergence of (3.1) tells us that $a_n(z_1 - z_0)^n \to 0$. In particular, the sequence $(a_n(z_1 - z_0)^n)$ is bounded.
Let $M > 0$ be such that $|a_n(z_1 - z_0)^n| < M$ for all $n \in \mathbb{Z}_{\geq 0}$.
Thus, for any $z \in D$, we have

$$\begin{aligned}
|a_n(z - z_0)^n| &= |a_n(z_1 - z_0)^n| \cdot \left|\frac{z - z_0}{z_1 - z_0}\right|^n \\
&\leq M\rho^n,
\end{aligned}$$

where $\rho := \left|\dfrac{z - z_0}{z_1 - z_0}\right|$.
Since $z \in D$, we have $0 \leq \rho < 1$ and hence, the series $\sum M\rho^n$ converges. By Comparison test, the result follows. □

Theorem 3.27 (Region of convergence)**.** Given a power series (3.1), precisely one of the following holds:

(i) The series converges for $z = z_0$ and no other $z \in \mathbb{C}$,

(ii) The series converges for all $z \in \mathbb{C}$, or

(iii) There exists a real number $R > 0$ such that the power series converges absolutely for all $z \in \{z \mid |z - z_0| < R\}$ and diverges for all $z \in \{z \mid |z - z_0| > R\}$. Moreover, the above R is unique.

[↑]

Proof. It is clear that all three conditions are mutually exclusive. Let us assume that (i) and (ii) don't hold. We prove (iii).
Note that if some $R > 0$ satisfies the condition, then it must clearly be unique. Thus, we show only the existence.
Let Rad $= \{R > 0 \mid$ the power series converges for all $z \in B_R(z_0)\}$.
Note that Rad $\neq \varnothing$ since (i) does not hold. Moreover, Rad is bounded above since (iii) does not hold.
Let $R := \sup \text{Rad}$. Clearly, $R > 0$. The claim is that R satisfies the condition in (iii). The two following claims prove that.

Claim 1. Let $z \in \mathbb{C}$ be such that $|z - z_0| < R$. Then, (3.1) converges absolutely.

Proof. Choose R_0 such that $|z - z_0| < R_0 < R$. Then, $R_0 \in$ Rad.
Choose $z_1 \in \mathbb{C}$ such that $|z - z_0| < |z_0 - z_1| < R_0$. By definition of Rad, (3.1) converges for z_1 and by Proposition 3.26, it converges absolutely for z. □

Claim 2. Let $z \in \mathbb{C}$ be such that $|z - z_0| > R$. Then, (3.1) diverges.

Proof. Choose R_0 such that $|z - z_0| > R_0 > R$. Then, $R_0 \notin$ Rad.
If (3.1) converged for z, then it would converge (absolutely) for any $z \in B_{R_0}(z_0)$, contradicting the definition of Rad. □

□

Theorem 3.29 (Calculating the radius of convergence)**.** Given the power series $\sum a_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \quad R = \alpha^{-1}.$$

(If $\alpha = 0$, then $R = \infty$ and if $\alpha = \infty$, then $R = 0$.)
Then, R is the radius of convergence.

[↑]

Proof. This will be an application of the Root test.
For $z \in \mathbb{C}^\times$, put $b_n = a_n z^n$. We then get,

$$\limsup_{n\to\infty} \sqrt[n]{b_n} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z|\alpha.$$

If $\alpha = 0$ or ∞, the result follows.
If $\alpha \in \mathbb{R}^+$, then note that $\sum b_n$ converges if $|z|\alpha < 1$ and diverges if $|z|\alpha > 1$. The these cases correspond to $|z| < R$ and $|z| > R$, respectively which is how the radius of convergence was defined in this case. $\square$

Lemma 3.31. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has D as its disc of convergence with positive radius, then $\displaystyle\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ also converges absolutely on D.

[↑]

Proof. Let $z \in D$ be arbitrary. As D is open, we may find $z_1 \in D$ such that $|z-z_0| < |z_1 - z_0|$. Set $\rho := \dfrac{|z-z_0|}{|z_1-z_0|}$. We have $0 \le \rho < 1$.
Also, note that $\sum a_n(z_1-z_0)^n$ converges and hence, there exists $M > 0$ such that $|a_n(z_1-z_0)^n| < M$ for all $n \in \mathbb{Z}_{\ge 0}$. Fix any such M. We then have,

$$\begin{aligned} |na_n(z-z_0)^n| &= |na_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \\ &\le nM\rho^n. \end{aligned}$$

Since $\rho < 1$, $\sum nM\rho^n$ converges and we are done, by the comparison test. $\square$

Theorem 3.32 (Differentiation theorem)**.** Suppose $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series with positive radius of convergence and D as its disc of convergence. The sum f is holomorphic on D.
Further,

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}.$$

[↑]

Proof. WLOG, we let $z_0 = 0$. Fix $z \in D$. We show that f is differentiable at z. Choose $r > 0$ such that $\overline{B_r(z)} \subset D$.
In what follows, $h \neq 0$ is small enough such that $z+h \in B_r(z)$. In particular, $f(z+h)$ converges.
Let $\rho < R$ be such that $|z|, |z+h| \le \rho$ for all $h \in B_r(0)$.
We now note the following:

$$\begin{aligned}
f(z+h) - f(z) &= \sum_{n=1}^{\infty} a_n \left((z+h)^n - z^n\right) \\
\implies \frac{f(z+h)-f(z)}{h} &= \sum_{n=1}^{\infty} a_n \left[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1}\right] \\
\implies \frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^n &= \sum_{n=2}^{\infty} a_n \left[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1} - nz^{n-1}\right] \\
&= \sum_{n=2}^{\infty} a_n \begin{bmatrix} (z+h)^{n-1} - z^{n-1} \\ +(z+h)^{n-1}z - z^{n-1} \\ \vdots \\ +(z+h)z^{n-2} - z^{n-1} \end{bmatrix} \\
&= \sum_{n=2}^{\infty} a_n \left\{ \sum_{j=1}^{n-1} \left((z+h)^j - z^j\right) z^{n-1-j} \right\}
\end{aligned}$$

$$\begin{aligned}
\implies \left| \frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^n \right| &= |h| \left| \sum_{n=1}^{\infty} a_n \left\{ \sum_{j=1}^{n-1} \left(\frac{(z+h)^j - z^j}{h} \right) z^{n-1-j} \right\} \right| \\
&\le |h| \sum_{n=2}^{\infty} |a_n| \sum_{j=1}^{n-1} \left| \frac{(z+h)^j - z^j}{h} \right| \rho^{n-1-j} \\
&= |h| \sum_{n=2}^{\infty} |a_n| \sum_{j=1}^{n-1} \rho^{n-1-j} \left(\sum_{k=1}^{j-1} |z+h|^k |z|^{j-k-1} \right) \\
&\le |h| \sum_{n=2}^{\infty} |a_n| \sum_{j=1}^{n-1} \rho^{n-1-j} \left(\sum_{k=1}^{j-1} |\rho|^{j-1} \right) \\
&\le |h| \sum_{n=2}^{\infty} |a_n| \sum_{j=1}^{n-1} (j-1) \rho^{n-1-j} \rho^{j-1} \\
&= \frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1) |a_n| \rho^{n-2}
\end{aligned}$$

Note that the last sum converges and thus, the differentiability follows as we let $h \to 0$. ☐

Theorem 3.33 (Abel's limit theorem)**.** Suppose $\sum_{n=0}^{\infty} a_n$ converges and $f(z) := \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Then,

$$\lim_{\substack{z \to 1^- \\ z \in \mathbb{R}}} f(z) = \sum_{n=0}^{\infty} a_n.$$

[↑]

Proof. Let $s := \sum a_n$. (Which exists, by hypothesis.)
Let $s_n := a_0 + \cdots + a_n$ for $n \geq 0$ and let $s_{-1} := 0$.
Then, we have

$$\begin{aligned}
\sum_{j=0}^{n} a_j z^j &= \sum_{j=0}^{n} (s_j - s_{j-1}) z^j \\
&= \sum_{j=0}^{n} s_j z^j - z \sum_{j=1}^{n} s_{j-1} z^{j-1} \\
&= \sum_{j=0}^{n} s_j z^j - z \sum_{j=0}^{n-1} s_j z^j \\
&= (1 - z) \sum_{j=0}^{n} s_j z^j. \qquad (*)
\end{aligned}$$

Note that $s_n \to s$ and thus, $|s_n|$ is bounded, by say, M. Since $\sum M z^n$ converges on $|z| < 1$, so does $\sum s_n z^n$.
Letting $n \to \infty$ in $(*)$ gives us

$$f(z) = (1 - z) \sum_{j=0}^{\infty} s_j z^j. \quad (\star)$$

Also, note that $1 = (1 - z) \sum_{j=0}^{\infty} z^j$ for $|z| < 1$ and thus,

$$s = s(1 - z) \sum_{j=0}^{\infty} z^j \quad (\star\star)$$

for $|z| < 1$. Subtracting $(\star\star)$ from $(\star)$ gives

$$f(z) - s = (1 - z) \sum_{j=0}^{\infty} (s_j - s) z^j.$$

Now, let $\epsilon > 0$ be arbitrary and let $z \in (0,1)$.
Let $N \in \mathbb{N}$ be such that $|s_n - s| < \epsilon/2$ for $n \geq N$. Thus,

$$\begin{aligned}
|f(z) - s| &\leq |1-z| \sum_{j=0}^{N} |s_j - s||z|^j + \frac{\epsilon}{2}|1-z| \sum_{j=N}^{\infty} |z|^j \\
&\leq |1-z| \sum_{j=0}^{N} |s_j - s||z|^j + \frac{\epsilon}{2}|1-z| \sum_{j=0}^{\infty} |z|^j \\
&= |1-z| \sum_{j=0}^{N} |s_j - s||z|^j + \frac{\epsilon}{2} \underbrace{\frac{|1-z|}{1-|z|}}_{\substack{=1 \\ \because z \in (0,1)}} \\
&= |1-z| \sum_{j=0}^{N} |s_j - s||z|^j + \frac{\epsilon}{2} \\
&< |1-z| \sum_{j=0}^{N} |s_j - s| + \frac{\epsilon}{2}.
\end{aligned}$$

Note that $\displaystyle\sum_{j=0}^{N} |s_j - s|$ is a fixed quantity. Thus, letting $\delta > 0$ to be such that $\displaystyle\left(\sum_{j=0}^{N} |s_j - s|\right) |1-z| < \epsilon/2$ whenever $|1-z| < \delta$, we see that

$$|f(z) - s| < \epsilon \quad \text{for all } z < 1 \text{ such that } |1-z| < \delta.$$

□

Corollary 3.34. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ are two series converging to A and B, respectively.
Assume that their Cauchy product also converges. Let C be this sum.
Then, $C = AB$.

[↑]

Proof. Let $f(z) := \displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \displaystyle\sum_{n=0}^{\infty} b_n z^n$ be defined for $|z| < 1$. (Note that the series converge absolutely here.)

$f(z)g(z)$ can be computed for $|z| < 1$ using the Cauchy product. (Recall Theorem 3.24.)
Namely, the product is given as

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) z^n.$$

Since $\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right)$ is known to converge (to C), we may appeal to Abel's limit theorem and conclude

$$\lim_{z \to 1^-} f(z)g(z) = C.$$

On the other hand, we have

$$\begin{aligned} \lim_{z \to 1^-} f(z)g(z) &= \left(\lim_{z \to 1^-} f(z) \right) \left(\lim_{z \to 1^-} g(z) \right) \\ &= AB, \end{aligned}$$

once again, by Abel's limit theorem. □

Theorem 3.36 (Exponential addition theorem)**.** For $z, w \in \mathbb{C}$, we have

$$\exp(z + w) = \exp(z)\exp(w).$$

[↑]

Proof. The series $\sum z^n/n!$ and $\sum w^n/n!$ converge absolutely and thus, we may use the Theorem 3.24 to conclude that

$$\begin{aligned} \exp(z)\exp(w) &= \sum_{N=0}^{\infty} \sum_{j=0}^{N} \frac{z^j}{j!} \frac{w^{N-j}}{(N-j)!} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{j=0}^{N} \binom{N}{j} z^j w^{N-j} \\ &= \sum_{N=0}^{\infty} \frac{(z+w)^N}{N!} \\ &= \exp(z + w). \end{aligned}$$

□

Theorem 4.8 (FTC)**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and f' is continuous. The, for any path γ is Ω, we have

$$\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)).$$

In particular, if γ is closed, the integral is zero.

[↑]

Proof. Recall from Theorem 2.6 that $f' = u_x + iv_x$. This gives us that

$$\int_\gamma f' = \int_\gamma [u_x \mathrm{d}x - v_x \mathrm{d}y] + i \int_\gamma [u_x \mathrm{d}y + v_x \mathrm{d}x]$$

Using CR equations, the above can be written as

$$\int_\gamma f' = \int_\gamma [u_x \mathrm{d}x + u_y \mathrm{d}y] + i \int_\gamma [v_y \mathrm{d}y + v_x \mathrm{d}x]. \quad (*)$$

We evaluate the real part of the RHS as follows:

$$\begin{aligned} \int_\gamma [u_x \mathrm{d}x + u_y \mathrm{d}y] &= \int_a^b \left[u_x(\gamma(t))\gamma_1'(t) + u_y(\gamma(t))\gamma_2'(t)\right] \mathrm{d}t \\ &= \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t))\right] \mathrm{d}t && \text{(chain rule)} \\ &= u(\gamma(b)) - u(\gamma(a)) && \text{((real) FTC)} \end{aligned}$$

Similarly, we see that the imaginary part of $(*)$ is $v(\gamma(b)) - v(\gamma(a))$. Adding the two gives us the desired result. ☐

Lemma 4.10 (M-L inequality)**.** Let $f : \Omega \to \mathbb{C}$ be a continuous function and $\gamma : [a, b] \to \Omega$ be a path. Then,

$$\left| \int_\gamma f \right| \leq \left(\sup_{t \in [a,b]} |f(\gamma(t))| \right) \cdot (\text{perimeter}(\gamma)).$$

[↑]

Proof. Let $M := \sup_{t\in[a,b]} |f(\gamma(t))|$.

Let $w := \left|\int_\gamma f\right|$. If $w = 0$, we are done. Assume $w \neq 0$ and put $c = \frac{|w|}{w}$. Note that $|c| = 1$ and that $cw \in \mathbb{R}$ and hence,

$$c\int_\gamma f(z)\mathrm{d}z = \int_\gamma cf(z)\mathrm{d}z \in \mathbb{R}.$$

Note that $\int_\gamma cf(z)\mathrm{d}z = \int_a^b cf(z)(\gamma_1'(t) + i\gamma_2'(t))\mathrm{d}t$. Since the latter is purely real, it equals its imaginary part. Since the real part of an integral equal the integral of the real part of the integrand (why), we see that

$$\begin{aligned}
c\int_\gamma f(z)\mathrm{d}z &= \int_\gamma cf(z)\mathrm{d}z \\
&= \int_a^b cf(z)(\gamma_1'(t) + i\gamma_2'(t))\mathrm{d}t \\
&= \int_a^b \Re\Big(cf(z)(\gamma_1'(t) + i\gamma_2'(t))\Big)\mathrm{d}t \\
&\leq \int_a^b |cf(z)(\gamma_1'(t) + i\gamma_2'(t))|\,\mathrm{d}t \\
&\leq |c|M\int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\mathrm{d}t \\
&= M\cdot \mathrm{perimeter}(\gamma),
\end{aligned}$$

and the result follows. □

Theorem 4.12 (Goursat's lemma)**.** Suppose Ω is open and convex and $f : \Omega \to \mathbb{C}$ is differentiable. Then,

$$\int_T f = 0$$

for any closed triangle $T \subset \Omega$.

[↑]

Proof. Suppose not. Then, there is a triangle T such that $\int_T f \neq 0$.

Let $\alpha := \left|\int_T f\right| > 0$.

Using the midpoints of the sides of T, split T into four smaller triangles and call them T_1, T_2, T_3, T_4. Traverse them in such a way that

$$\int_T f = \int_{T_1} f + \int_{T_2} f + \int_{T_3} f + \int_{T_4} f.$$

Now, there must exist $i \in \{1, \ldots, 4\}$ such that $\left|\int_{T_i} f\right| \geq \alpha/4$. (Why?)
WLOG, let $i = 1$ be one such.
Now, repeating this process of a bisection, we get a sequence of triangles $T = T_0, T_1, T_2, \ldots$ such that $\left|\int_{T_j} f\right| \geq \alpha/4^j$ for all $j \geq 0$.
Let $\widehat{T_j}$ denote the convex hull of T_j, that is, the boundary of T_i union'ed with its "interior".
Then, $\widehat{T_0} \supset \widehat{T_1} \supset \widehat{T_2} \supset \cdots$ is a sequence of nested nonempty compact sets.
Thus, $\bigcap\limits_{j\geq 0} \widehat{T_j} \neq \varnothing$. Choose any $p \in \bigcap\limits_{j\geq 0} \widehat{T_j}$. (In fact, the intersection must be a singleton.)
By convexity of Ω, we have that $p \in \Omega$. (Since each $\widehat{T_j}$ must be contained in Ω.)

Now, choose any positive $\epsilon < \dfrac{\alpha}{4(\operatorname{per} T_0)^2}$. (Where per denotes $\operatorname{perimeter}$.)

Using the fact that f is differentiable at p, we choose $\delta > 0$ such that

$$0 < |h| < \delta \implies |f(p+h) - (f(p) + hf'(p))| < \epsilon|h|.$$

Choose $N \in \mathbb{N}$ such that $\widehat{T_N} \subset B_\delta(p)$. (Why does such an N exist?)
Thus, if $z \in T_N$, then $|z - p| < \delta$ and in turn, we would have

$$|f(z) - (f(p) + (z-p)f'(p))| < \epsilon|z-p| < \epsilon\delta. \quad (\star)$$

Now, note that $f(p) + (z-p)f'(p)$ is a polynomial (in z) and thus, admits a primitive and hence, by Corollary 4.9, we get that

$$\int_{T_N} \Big[f(p) + (z-p)f'(z)\Big] \mathrm{d}z = 0.$$

Thus, we have

$$\int_{T_N} f(z)\mathrm{d}z = \int_{T_N} \Big[f(z) - f(p) - (z-p)f'(z)\Big] \mathrm{d}z$$

and

$$\left|\int_{T_N} f(z)\mathrm{d}z\right| \geq \frac{\alpha}{4^N}.$$

Thus, we get

$$\left|\int_{T_N}\Big[f(z)-f(p)-(z-p)f'(z)\Big]\mathrm{d}z\right| \geq \frac{\alpha}{4^j}. \quad (*)$$

On the other hand, using M-L inequality (Lemma 4.10), we have

$$\begin{aligned}\left|\int_{T_N}\Big[f(z)-f(p)-(z-p)f'(z)\Big]\mathrm{d}z\right| &\leq \sup_{z\in T_N}\Big[\cdots\Big]\operatorname{per}T_N\\ &\leq \epsilon\delta\cdot\operatorname{per}T_N. \qquad (\because \star)\end{aligned}$$

Also, note that $\delta \leq 2\cdot|\text{longest side of } T_N| \leq \operatorname{per}T_N$ and thus, the above inequality becomes

$$\left|\int_{T_N}\Big[f(z)-f(p)-(z-p)f'(z)\Big]\mathrm{d}z\right| \leq \epsilon\left(\operatorname{per}T_N\right)^2. \quad (**)$$

From $(*)$ and $(**)$, we conclude that

$$\begin{aligned}\frac{\alpha}{4^N} &\leq \epsilon\left(\operatorname{per}T_N\right)^2\\ &= \epsilon\frac{(\operatorname{per}T_0)^2}{4^N}\\ &< \frac{\alpha}{4(\operatorname{per}T_0)^2}\cdot\frac{(\operatorname{per}T_0)^2}{4^N}\\ &= \frac{1}{4}\frac{\alpha}{4^N}\\ \implies \alpha &< \frac{\alpha}{4},\end{aligned}$$

a contradiction. □

Corollary 4.14 (Cauchy's theorem for convex domains)**.** Let $\Omega \subset \mathbb{C}$ be open and convex and $f:\Omega\to\mathbb{C}$ be holomorphic. Then, f admits a primitive. That is, there exists $F:\Omega\to\mathbb{C}$ such that F is holomorphic and $F' = f$.
In particular,

$$\int_\gamma f(z)\mathrm{d}z$$

for all closed paths γ.

[↑]

Proof. Fix $z_0 \in \Omega$. Since Ω is convex, $l[z_0, z] \subset \Omega$ for any $z \in \Omega$.
Define $F : \Omega \to \mathbb{C}$ as

$$F(z) := \int_{z_0}^{z} f(\xi)\mathrm{d}\xi.$$

We claim that $F' = f$. Let $z \in \mathbb{C}$ be arbitrary and $h \neq 0$ be small enough that $z + h \in \Omega$. Then,

$$\begin{aligned} F(z+h) - F(z) &= \int_{z_0}^{z+h} f(\xi)\mathrm{d}\xi - \int_{z_0}^{z} f(\xi)\mathrm{d}\xi \\ &= \int_{z}^{z+h} f(\xi)\mathrm{d}\xi, \end{aligned}$$

where the last equality follows from Goursat's lemma applied to the triangle with vertices $z_0, z, z+h$. Note that this triangle did completely lie within Ω.
Thus, we see

$$\begin{aligned} F(z+h) - F(z) - hf(z) &= \int_{z}^{z+h} f(\xi)\mathrm{d}\xi - hf(z) \\ &= \int_{z}^{z+h} f(\xi)\mathrm{d}\xi - f(z)\int_{z}^{z+h} 1\mathrm{d}\xi \\ &= \int_{z}^{z+h} f(\xi)\mathrm{d}\xi - \int_{z}^{z+h} f(z)\mathrm{d}\xi \\ \implies F(z+h) - F(z) - hf(z) &= \int_{z}^{z+h} [f(\xi) - f(z)]\mathrm{d}\xi \\ \implies |F(z+h) - F(z) - hf(z)| &\leq \sup_{\xi \in l[z,z+h]} |f(\xi) - f(z)||h| \\ \implies \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &\leq \sup_{\xi \in l[z,z+h]} |f(\xi) - f(z)|. \end{aligned}$$

Note that since f is continuous, we have that $\sup_{\xi \in l[z,z+h]} |f(\xi) - f(z)| \to 0$ as $h \to 0$ and we are done. □

Corollary 4.15 (Cauchy's theorem for star-shaped domains)**.** If $\Omega \subset \mathbb{C}$ is open and star-shaped and $f : \Omega \to \mathbb{C}$ is holomorphic, then f admits a primitive.
In particular,

$$\int_{\gamma} f(z)\mathrm{d}z$$

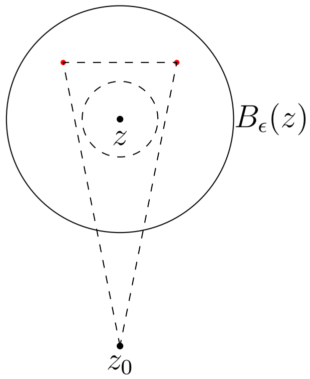
for all closed paths γ.

[↑]

Proof. Let $z_0 \in \Omega$ be as in Definition 1.8. Then, given any $z \in \Omega$, we have $l[z_0, z] \subset \Omega$. Like before, define $F : \Omega \to \mathbb{C}$ as

$$F(z) := \int_{z_0}^{z} f(\xi)\mathrm{d}\xi.$$

We claim that $F' = f$. The argument will be almost identical to the one last time.



Let $z \in \Omega$ be arbitrary. We show that $F'(z) = f(z)$.
Since Ω is open, we can find $\epsilon > 0$ such that $B_\epsilon(z) \subset \Omega$. We can choose points z_1, z_2 (in red) that belong to $B_\epsilon(x)$ such that z in the interior of the triangle T formed by z_0, z_1, z_2 (the dashed triangle).
Note that $\widehat{T} \subset \Omega$.
By Lemma 1.10, there exists a (compact) $C \subset \Omega$ such that $\widehat{T} \subset \operatorname{int} C$.
Let $\delta > 0$ be such that $B_\delta(z) \subset \operatorname{int} T$. (The dashed circle is $B_\delta(z)$.)
Now for all $h \neq 0$ such that $z + h \in B_\delta(z)$, we can use Stronger Goursat's lemma to conclude that

$$\begin{aligned} F(z+h) - F(z) &= \int_{z_0}^{z+h} f(\xi)\mathrm{d}\xi - \int_{z_0}^{z} f(\xi)\mathrm{d}\xi \\ &= \int_{z}^{z+h} f(\xi)\mathrm{d}\xi. \end{aligned}$$

The remainder of the proof then proceeds identically as earlier. □

Theorem 4.16 (Cauchy integral formula)**.** Let Ω be an open convex set in $\mathbb{C}$.
Suppose $f : \Omega \to \mathbb{C}$ is holomorphic.
Let $p \in \Omega$ and $r > 0$ be such that $\overline{B_r(p)} \subset \Omega$.
Then, for any point $z \in B_r(p)$, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi,$$

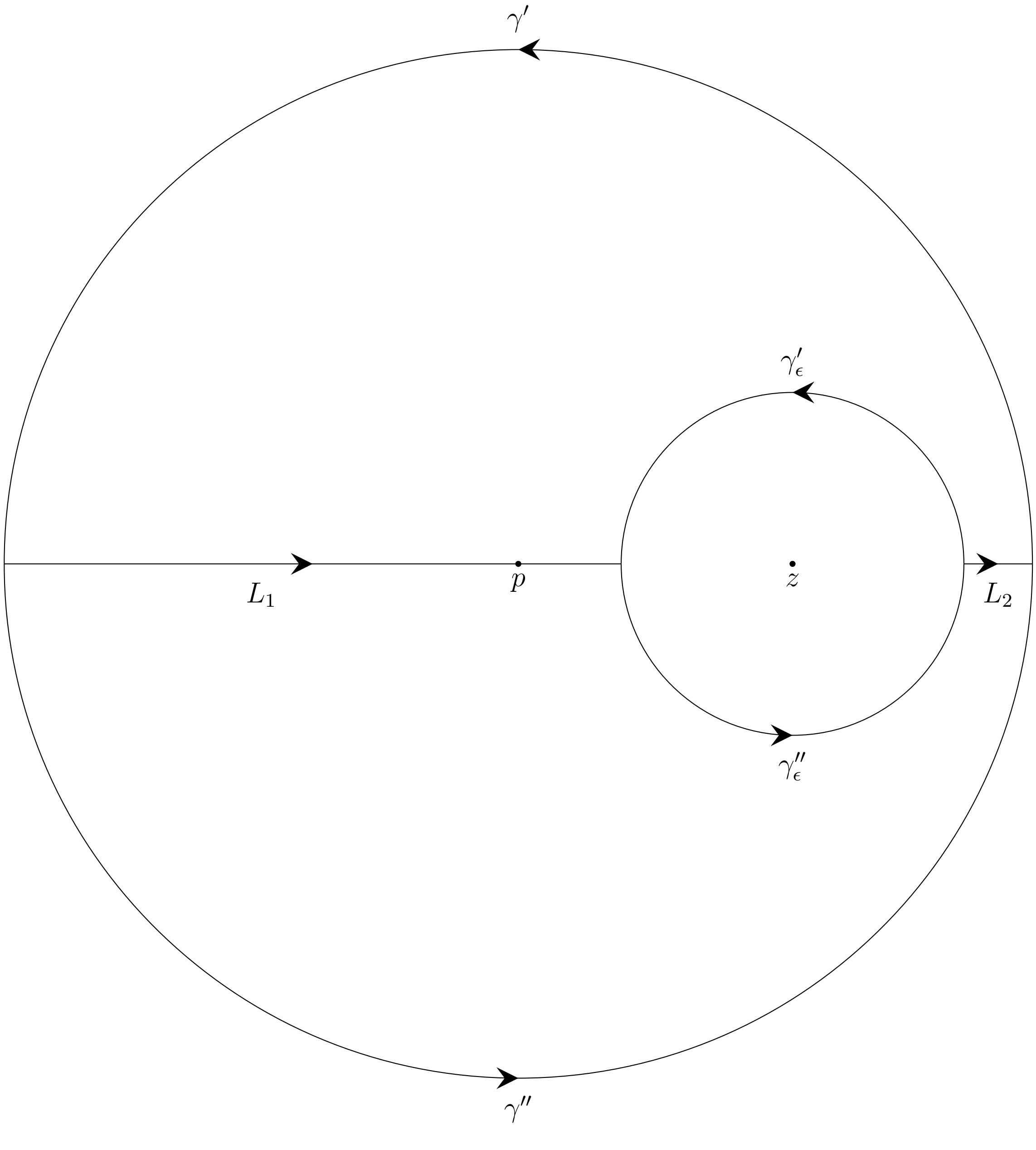
where γ is the circle $\partial\overline{B_r(p)}$ traced counterclockwise.
(That is, $\gamma(t) = p + re^{it}$ for $t \in [0, 2\pi]$.)

[↑]

Proof. First, we show that

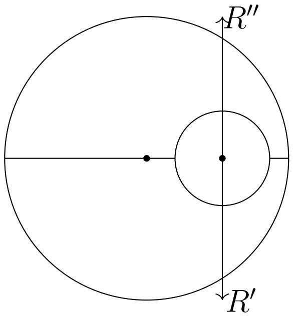
$$\int_\gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi = \int_{\gamma_\epsilon} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi,$$

where $\gamma_\epsilon(t) = z + \epsilon e^{it},\ t \in [0, 2\pi]$ for any $\epsilon > 0$ small enough that $\overline{B_\epsilon(z)} \subset B_r(p)$.



As shown in the figure, we get two paths as follows:

$$L' = L_1 * (\overline{\gamma'_\epsilon}) * L_2 * \gamma',$$
$$L'' = \gamma'' * (\overline{L_2}) * (\overline{\gamma''_\epsilon}) * (\overline{L_1}).$$



Note that L' is contained in the star-shaped region $(\mathbb{C} \setminus R') \cap \Omega$ and L'' in $(\mathbb{C} \setminus R'') \cap \Omega$. In these star-shaped regions, the function $\xi \mapsto \dfrac{f(\xi)}{\xi - z}$ is holomorphic.
Thus, by Cauchy's theorem for star-shaped domains, we see that

$$\int_{L'} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi = \int_{L''} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi = 0.$$

Adding the two integrals and using the facts $\gamma = \gamma' * \gamma''$, $\gamma_\epsilon = \gamma'_\epsilon * \gamma''_\epsilon$, we see that

$$\int_{\gamma} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi = \int_{\gamma_\epsilon} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi. \tag{5.2}$$

Now, we look at the integral on the right.

$$\begin{aligned} \int_{\gamma_\epsilon} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi &= \int_{\gamma_\epsilon} \frac{f(\xi) - f(z)}{\xi - z} \mathrm{d}\xi + \int_{\gamma_\epsilon} \frac{f(z)}{\xi - z} \mathrm{d}\xi \\ &= \int_{\gamma_\epsilon} \frac{f(\xi) - f(z)}{\xi - z} \mathrm{d}\xi + f(z) \int_{\gamma_\epsilon} \frac{1}{\xi - z} \mathrm{d}\xi \\ &= \int_{\gamma_\epsilon} \frac{f(\xi) - f(z)}{\xi - z} \mathrm{d}\xi + 2\pi i f(z). \qquad (*) \end{aligned}$$

We show that the integral in $(*)$ is zero.

$$\begin{aligned} \left| \int_{\gamma_\epsilon} \frac{f(\xi) - f(z)}{\xi - z} \mathrm{d}\xi \right| &\leq \sup_{\xi \in B_\epsilon(z)} \frac{|f(\xi) - f(z)|}{|\xi - z|} (2\pi\epsilon) \\ &= \sup_{\xi \in B_\epsilon(z)} \frac{|f(\xi) - f(z)|}{\epsilon} (2\pi\epsilon) \\ &= 2\pi \sup_{\xi \in B_\epsilon(z)} |f(\xi) - f(z)|. \end{aligned}$$

As we let $\epsilon \to 0$, we see that the integral in $(*)$ goes to 0. Thus, letting $\epsilon \to 0$ in (5.2) yields the result. $\square$

Corollary 4.17 (Holomorphic functions are analytic)**.** Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Then, for each $p \in \Omega$, there exists $r > 0$ such that $B_r(p) \subset \Omega$ and on $B_r(p)$, f admits a power series representation. That is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n \tag{4.1}$$

for all $z \in B_r(p)$.
In fact, the above is true for any r such that $B_r(p) \subset \Omega$.
In particular, if f is differentiable once, then f is infinitely differentiable. (Since power series are clearly infinitely differentiable in view of Differentiation theorem.)

[↑]

Proof. Fix $p \in \Omega$. Since Ω is open, there exists some $r > 0$ such that $\overline{B_r(p)} \subset \Omega$. Choose any such r.
Let $z \in B_r(p)$. By Cauchy integral formula, we see that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi,$$

where $\gamma(t) = p + re^{it}$ for $t \in [0, 2\pi]$.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - p - (z-p)} \mathrm{d}\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - p)\left(1 - \frac{z-p}{\xi - p}\right)} \mathrm{d}\xi. \end{aligned}$$

Note that z is fixed and ξ varies over $\partial B_r(p)$ and thus, $|z-p| \lneq |\xi - p| = r$ and hence,

$$f(z) = \frac{1}{2\pi i} \int_\gamma f(\xi) \sum_{k=0}^{\infty} \frac{(z-p)^k}{(\xi - p)^{k+1}} \mathrm{d}\xi.$$

Let $M > 0$ be such that $|f(\xi)| < M$ for all $\xi \in \partial B_r(p)$. (Which exists since f is holomorphic and in particular, continuous.)
Let $\rho := \dfrac{|z-p|}{r}$. Then, the sum $\sum f(\xi) \dfrac{(z-p)^k}{(\xi - p)^{k+1}}$ is dominated by $\sum (M/r)\rho^k$ and

thus, by Weierstrass' M-test, the series converges uniformly. This lets us switch the sum and integral as

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_\gamma f(\xi)\frac{(z-p)^k}{(\xi-p)^{k+1}}\mathrm{d}\xi \\ &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}(z-p)^k\left(\int_\gamma \frac{f(\xi)}{(\xi-p)^{k+1}}\mathrm{d}\xi\right) \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-p)^{k+1}}\mathrm{d}\xi\right)(z-p)^k. \end{aligned}$$

Note that $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\xi)}{(\xi-p)^{k+1}}\mathrm{d}\xi$ is independent of z and thus, the above shows that f admits a power series representation on $B_r(p)$. □

Theorem 4.19 (Cauchy's estimate)**.** With notation as earlier, if $\overline{B_R(p)} \subset \Omega$, then

$$|f^{(k)}(p)| \leq \frac{k!}{R^k}\sup_{B_R(p)}|f|.$$

[↑]

Proof. Differentiate (4.1) and put $z = p$ to get $a_k = \dfrac{1}{k!}f^{(k)}(p)$.
Comparing with (4.2), we get

$$\begin{aligned} \left|f^{(k)}(p)\right| &= \frac{k!}{2\pi}\left|\int_{\gamma_R}\frac{f(\xi)}{(\xi-p)^{k+1}}\mathrm{d}\xi\right| \\ &\leq \frac{k!}{2\pi}\sup_{\xi\in B_R(p)}|f(\xi)|\frac{1}{R^{k+1}}(2\pi R) \\ &= \frac{k!}{R^k}\sup_{B_R(p)}|f|. \end{aligned}$$

□

Theorem 4.20 (Bounded entire functions)**.** A bounded entire function is constant.

[↑]

Proof. Let f be entire and bounded. Let $M = \sup_{\mathbb{C}} |f|$.
Fix $p = 0$. The power series (4.1) converges for all $z \in \mathbb{C}$.
Fix $k \geq 1$. By Cauchy's estimate, we see

$$|f^{(k)}(0)| \leq \frac{k!M}{R^k}$$

for every $R > 0$. Letting $R \to \infty$, we see that $f^{(k)}(0) = 0$.
As k was arbitrary, we see that the power series simply reduces to $f(z) = a_0$, as desired. □

Theorem 4.23 (Cauchy's Integral Theorem (basic))**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Let γ be a simple closed curve in Ω such that $\operatorname{int}\gamma \subset \Omega$. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

[↑]

Proof.

$$\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &= \int_\gamma u\mathrm{d}x - v\mathrm{d}y + i\int_\gamma u\mathrm{d}y + v\mathrm{d}x \\
&= \iint_{\operatorname{int}\gamma} (v_x + u_y)\mathrm{d}(x, y) + i\iint_{\operatorname{int}\gamma} (u_x - v_y)\mathrm{d}(x, y) \\
&= 0.
\end{aligned}$$

We have used the fact that the vector field (u, v) is continuously differentiable and appealed to Green's Theorem. The last 0 follows by virtue of the CR equations. □

Lemma 4.25. Let Ω be a **connected** open set in $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $p \in \Omega$. Suppose f is constant in a neighbourhood of p. Then, f is constant on Ω.

[↑]

Proof. Note that an open and connected set in $\mathbb{C}$ is path connected.
Let $c = f(p)$.
We know that $f \equiv c$ in a neighbourhood $B_\delta(p)$. Choose any $q \in \Omega$ distinct from p.
We show that $f(z) = c$ for all z in a neighbourhood of q and thus, complete the proof.
Choose any continuous path $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$.
Define $S \subset [0, 1]$ as

$$S := \{t \in [0, 1] \mid f \equiv c \text{ in a neighbourhood of } \gamma(t)\}.$$

We know that S is nonempty since $0 \in S$. Moreover, $[0, \zeta] \subset S$ for some $\zeta > 0$. (Consider the intersection of $\gamma([0,1])$ with $B_\delta(p)$.)
Set

$$m := \sup\{t_0 \in [0,1] \mid [0, t_0] \subset S\} > 0.$$

We shall show that $m = 1$ and that will complete the proof.
Let $d := \operatorname{dist}\left(\gamma\left([0,1]\right), \partial\Omega\right)$.
Note that $d > 0$ since $\gamma\left([0,1]\right)$ is compact and does not intersect $\partial\Omega$.
Select $\eta > 0$ such that

$$|s - t| < \eta \implies |\gamma(s) - \gamma(t)| < \frac{d}{3}.$$

(Note that γ is continuous and thus, uniformly continuous.)
The important thing to remember is that for each $z_0 \in \Omega$, the power series representation of f is valid throughout $B_r(z_0)$ where $r = \operatorname{dist}(z_0, \partial\Omega)$.
Choose $t_0 \in S$ such that $m - \frac{\eta}{2} < t_0 \le m$ such $[0, t_0] \subset S$. (Exists by definition of m.)
Thus, $f \equiv c$ in a neighbourhood of $\gamma(t_0)$.
Thus, the power series of f at $\gamma(t_0)$ reduces to the constant and thus, $f \equiv c$ on $B_d(\gamma(t_0))$. (Recall d defined earlier.)
Also, note that $|t_0 - m| < \eta/2 < \eta$ and thus, $|\gamma(t_0) - \gamma(m)| \le \dfrac{d}{3}$, by choice of η and thus, $\gamma(m) \in B_d(\gamma(t_0))$.
Thus, $f \equiv c$ in a neighbourhood of m.
In fact, given any $t \in \left(m - \dfrac{\eta}{2}, m + \dfrac{\eta}{2}\right) \cap [0,1]$, $f \equiv c$ in a neighbourhood of t. (Since any such t would be at distance at most η from t_0.)
Thus, if $m < 1$, then we would get a contradiction as we would get an $m' > m$ which is in the set $\{t_0 \in [0,1] \mid [0, t_0] \subset S\}$.
This shows that $m = 1$. $\square$

Theorem 4.26. Let Ω be a **connected** open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be holomorphic. TFAE:

(i) $f \equiv 0$ on Ω,

(ii) There exists a sequence (p_n) of distinct points of Ω such that $p_n \to p \in \Omega$ and $f(p_n) = 0$ for all $n \in \mathbb{N}$, (note that the limit is in Ω)

(iii) There exists a point $p \in \Omega$ such that $f^{(k)}(p) = 0$ for all $k \ge 0$.

[↑]

Proof. (i) $\implies$ (ii) is trivial.
(ii) $\implies$ (iii).
Let (p_n) and p be as above. By continuity of f, we get that $f(p) = 0$. That is, $f^{(0)}(p) = 0$.
Assume that (iii) is not true. We arrive at a contradiction.
By our assumption, there exists $k \in \mathbb{N}$ such that $f^{(k)}(p) \neq 0$. Choose the smallest such k. Then, the power series around p reads

$$f(z) = (z-p)^k(a_k + a_{k+1}(z-p) + \cdots).$$

Note that $a_k \neq 0$.
Let $g(z) := a_k + a_{k+1}(z-p) + \cdots$. That is, $f(z) = (z-p)^k g(z)$ with g continuous at p.
Since $g(p) = a_k \neq 0$, there exists a neighbourhood U of p such that $g(z) \neq 0$ for any $z \in U$.
Choose N sufficiently large such that $p_N \in U$ and $p_N \neq p$. Then, note that $g(p_N) \neq 0$ and $(p_N - p)^k \neq 0$. However, $g(p_N)(p_N - p)^k = f(p_N) = 0$. A contradiction.
(iii) $\implies$ (i).
If $f^{(k)}(p) = 0$ for all $k \geq 0$, then $f \equiv 0$ on a neighbourhood of p. The result then follows by the previous lemma. □

Theorem 4.29 (Morera's Theorem)**.** Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be continuous. Assume that

$$\int_T f = 0$$

for all triangles $T \subset \Omega$. Then, f is holomorphic on Ω.

[↑]

Proof. Since holomorphy is a local property, it is enough to prove that f is holomorphic on each disc $D \subset \Omega$. WLOG, we may assume that Ω is a disc. (In particular, Ω will be convex.)
We first construct a primitive for f. Fix $z_0 \in \Omega$. Define $F : \Omega \to \mathbb{C}$ as

$$F(z) := \int_{z_0}^{z} f(\xi)\mathrm{d}\xi.$$

Let $h \neq 0$ be small enough that $z + h \in \Omega$. Then,

$$\begin{aligned}
F(z+h) - F(z) &= \int_{z_0}^{z+h} f(\xi)\mathrm{d}\xi - \int_{z_0}^{z} f(\xi)\mathrm{d}\xi \\
&= \int_{z_0}^{z+h} f(\xi)\mathrm{d}\xi + \int_{z}^{z_0} f(\xi)\mathrm{d}\xi \\
&= \int_T f + \int_z^{z+h} f(\xi)\mathrm{d}\xi \\
&= \int_z^{z+h} f(\xi)\mathrm{d}\xi,
\end{aligned}$$

where T is the triangle with vertices $z_0, z, z+h$.
The remainder of the proof now goes identically as that of Corollary 4.14. (See proof here.)
Thus, we get that $F' = f$. Since F is holomorphic, it is infinitely differentiable and so is f. □

Corollary 4.30 (Montel's theorem)**.** Let Ω be an open set in $\mathbb{C}$ and (f_n) be a sequence of functions in $A(\Omega)$ converging uniformly to f on compact subsets of Ω.
Then, the limit function f is also holomorphic. Further, $f_n' \to f'$ uniformly on compact subsets of Ω.

[↑]

Proof. To show that f is holomorphic, it is sufficient to prove it in the case that Ω is a disc.
Note that f is continuous since the convergence is uniform. We show that $\displaystyle\int_T f = 0$ for any triangle $T \subset \Omega$. The holomorphy of f will then follow from Morera's Theorem.
Let $T \subset \Omega$ be an arbitrary triangle. Note that T is compact and thus, $f_n \to f$ on T. This gives us that

$$\int_T f = \int_T \left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} \int_T f_n = 0,$$

as desired.

Now, we show that $f_n' \to f'$ uniformly on compact sets.
Let $K \subset \Omega$ be compact.
Set $d := \operatorname{dist}(K, \partial\Omega)$. (If $d = \infty$, then set $d := 43$.)

Let $z \in K$. Then, $\overline{B_{d/3}(z)} \subset \overline{B_{2d/3}(z)} \subset \Omega$. Applying Cauchy's estimate ($k = 1$) gives us:

$$|f_n'(z) - f'(z)| \le \left(\frac{d}{3}\right)^{-1} \sup_{\xi \in B_{d/3}(z)} |f_n(\xi) - f(\xi)|.$$

Let $\tilde{K} = \bigcup_{z \in K} B_{2d/3}(z)$. Then, $\overline{\tilde{K}} \subset \Omega$ is compact. Moreover, we have

$$\bigcup_{z \in K} B_{d/3}(z) \subset \overline{\tilde{K}}.$$

Thus, the above inequality gives us:

$$|f_n'(z) - f'(z)| \le \left(\frac{d}{3}\right)^{-1} \sup_{\xi \in \overline{\tilde{K}}} |f_n(\xi) - f(\xi)|.$$

Note that the RHS $\to 0$ as $n \to 0$ since (f_n) uniformly converges to f. Moreover, the inequality is true for all $z \in K$ and thus, we see that $f_n \to f'$ uniformly on K.

More elaboration on the last part:
Let $\epsilon > 0$ be given, then there exists $N \in \mathbb{N}$ such that $|f_n(\xi) - f(\xi)| < \epsilon$ for all $n \ge N$ and all $\xi \in \overline{\tilde{K}}$. (This is because of uniform convergence of f_n on $\overline{\tilde{K}}$.)
Choose any such N. Then, we have

$$\sup_{\xi \in \overline{\tilde{K}}} |f_n(\xi) - f(\xi)| \le \epsilon$$

for all $n \ge N$. In turn, we have, for all $z \in K$,

$$|f_n'(z) - f'(z)| \le 3\epsilon/d$$

for all $n \ge N$. This proves the desired uniform convergence. □

Lemma 4.31. Let Ω be a connected open set and $f : \Omega \to \mathbb{C}$ be holomorphic. If $|f|$ is constant, then so is f.

[↑]

Proof. Writing $f = u + iv$ as usual, we see that

$$u^2 + v^2 \equiv c.$$

If $c = 0$, then we are done. Assume $c \neq 0$.
Differentiating the above w.r.t. x gives us

$$uu_x + vv_x = 0. \quad (*)$$

Similarly, differentiating w.r.t. y gives us

$$uu_y + vv_y = 0.$$

Using CR equations, the last equation can be re-written as

$$-uv_x + vu_x = 0. \quad (**)$$

$(*)$ and $(**)$ together give us

$$\begin{bmatrix} u & v \\ v & -u \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that $\det \begin{bmatrix} u & v \\ v & -u \end{bmatrix} = c \neq 0$ and thus, $u_x = v_x \equiv 0$ on Ω.
This gives us that $f' \equiv 0$ on Ω and thus, f is constant, since Ω is connected. □

Theorem 4.32 (Maximum Modulus Theorem)**.** Suppose Ω is a connected open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is holomorphic such that $|f|$ attains a local maximum at a point $p \in \Omega$. Then, f is constant.

[↑]

Proof. Let p be as in the theorem. Let D be an open disc containing p such that $|f(p)| \geq |f(z)|$ for all $z \in D$.
Let $r > 0$ be arbitrary such that $\overline{B_r(p)} \subset D$. Let $\gamma(t) := p + re^{it}$ for $t \in [0, 2\pi]$. Then, Cauchy's Integral Formula gives us

$$\begin{aligned} f(p) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-p} \mathrm{d}z \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{it}) \mathrm{d}t. \end{aligned}$$

Hence,

$$\begin{aligned} |f(p)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(p + re^{it}) \right| \mathrm{d}t \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(p)| \, \mathrm{d}t \\ &= |f(p)| \, . \end{aligned}$$

Thus,

$$|f(p)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(p+re^{it})\right| \mathrm{d}t$$

or

$$\int_0^{2\pi} \left[|f(p)| - \left|f(p+re^{it})\right|\right] \mathrm{d}t = 0.$$

Note that the integrand is nonnegative and continuous. Thus, the integrand must be identically zero. This gives us that

$$|f(p)| = \left|f(p+re^{it})\right|$$

for all $t \in [0, 2\pi]$ and all r sufficiently small.
Thus, $|f|$ is constant in a neighbourhood of p. By Lemma 4.31, we see that f is constant in a neighbourhood of p. Since Ω is connected, we appeal to Lemma 4.25 and conclude that f is constant on Ω. □

Theorem 4.34 (Open Mapping Theorem)**.** If Ω is a **connected** open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function, then f is an open mapping.
That is, for every open $U \subset \Omega$, the set $f(U)$ is open in $\mathbb{C}$. In particular, $f(\Omega)$ is open.

[↑]

Proof. Let U be an arbitrary open subset of Ω.
Let $w_0 \in f(U)$ be arbitrary. We show that $f(U)$ contains a neighbourhood of w_0. This will prove the theorem.

Let $z_0 \in U$ be such that $f(z_0) = w_0$. Let $\epsilon > 0$ be such that $V := B_\epsilon(z_0) \subset U$ and $f(z) \neq w_0$ for $|z - z_0| = \epsilon$. (Such a choice of ϵ is possible because the zeroes of $z \mapsto f(z) - w_0$ are isolated.)

It follows that $0 < \delta := \inf\{|f(z)| \mid |z - z_0| = \epsilon\}$. (Since the $\inf$ is actually attained on a compact set.)

Claim. $B_{\delta/2}(w) \subset f(V)$.

Proof. Let $w \in \mathbb{C}$ be such that $|w - w_0| < \delta/2$. For the sake of contradiction, assume that $w \notin f(V)$.
By assumption, the function $g : V \to \mathbb{C}$ given by $z \mapsto f(z) - w$ does not vanish.
Since g is non-vanishing and holomorphic, so is $1/g$.
By Corollary 4.33, it follows that

$$\frac{1}{|g(z_0)|} < \sup\left\{\frac{1}{|f(z) - w|} \;\middle|\; z \in \partial V\right\}. \quad (*)$$

We estimate the quantity on the right side as follows:
For $z \in \partial V$, we know that $|f(z) - w_0| \geq \delta$. We then see that

$$\begin{aligned} &|f(z) - w_0| \geq \delta \\ \implies &|f(z) - w_0| - |w_0 - w| > \frac{\delta}{2} > 0 \\ \implies &|f(z) - w_0| - |w_0 - w| = \left||f(z) - w_0| - |w_0 - w|\right|. \end{aligned}$$

Using reverse triangle inequality, we see that

$$\begin{aligned} |f(z) - w| &\geq |f(z) - w_0| - |w_0 - w| \\ &\geq \delta - |w_0 - w| \\ \implies \frac{1}{|f(z) - w|} &\leq \frac{1}{\delta - |w_0 - w|} \\ \implies \sup_{z \in \partial V} \frac{1}{|f(z) - w|} &\leq \frac{1}{\delta - |w_0 - w|}. \end{aligned}$$

We now return to $(*)$ with the above inequality and the fact that $|w - w_0| = |g(z_0)|$ to obtain

$$\frac{1}{|w - w_0|} \leq \frac{1}{\delta - |w - w_0|}.$$

The above gives us $|w - w_0| \geq \dfrac{\delta}{2}$, a contradiction. □

The above claim proves the desired result. □