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MA 408

Measure Theory

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§1. Measures

§§1.1. Introduction

Theorem 1.1.1 (Non existence of ideal measure)**.** There is no map $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$,
2. $\mu(E) = \mu(x + E)$ for all $x \in \mathbb{R}$ and $E \in \mathcal{P}(\mathbb{R})$, where $x + E := \{x + y \mid y \in E\}$,
3. for any disjoint countable collection $\{E_i\}_i^\infty$ of subsets of $\mathbb{R}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i),$$

4. $\mu([0, 1]) = 1$.

Note that the last is a "normalisation" property. Otherwise $\mu \equiv 0$ or $\mu(X) = \begin{cases} 0 & X = \varnothing, \\ \infty & \text{otherwise} \end{cases}$ would also satisfy and give us "useless" functions.

Replacing "countable union" with "finite union" also won't do the trick in general due to the Banach-Tarski "paradox" (theorem).

Both the above required a use of the Axiom of Choice.

§§1.2. σ-algebras

Definition 1.2.1 (Algebra)**.** Let X be a non-empty set.
An algebra ("field") on X is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying

1. (Closure under complements) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
2. (Closure under finite unions) $A_1, \dots, A_n \in \mathcal{F} \implies \bigcup_{i=1}^n A_i \in \mathcal{F}$.

Definition 1.2.2 (σ-algebra)**.** Let X be a non-empty set.
A σ-algebra ("σ-field") on X is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying

1. (Closure under complements) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
2. (Closure under countable unions) $A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{i=1}^\infty A_i \in \mathcal{F}$.

Example 1.2.3 (Countable-cocountable σ-algebra)**.** Let $X \neq \varnothing$. Then,

$$\mathcal{F} = \{E \subseteq X \mid E \text{ or } E^c \text{ is countable}\}$$

is a σ-algebra on X.

Definition 1.2.4 (σ-algebra generated by a set)**.** Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then,

$$\mathcal{M}(\mathcal{E}) := \bigcap_{\substack{\mathcal{E}\subseteq\mathcal{B} \\ \mathcal{B} \text{ is a } \sigma-\text{algebra}}} \mathcal{B}$$

is a σ-algebra. Moreover, it is the smallest σ-algebra containing $\mathcal{B}$.

This is called the σ-algebra generated by $\mathcal{B}$.

Definition 1.2.5 (Borel σ-algebra)**.** Let $(X, \mathcal{T})$ be a topological space. The σ-algebra generated by $\mathcal{T}$ is called the Borel σ-algebra on X, denoted $\mathcal{B}(X)$.

In other words, $\mathcal{B}(X)$ is the σ-algebra generated by the open sets of X.

Proposition 1.2.6. All of the following are contained in $\mathcal{B}(\mathbb{R})$:

1. All closed sets.
2. All open sets.
3. All F_σ and G_δ sets.

Recall that an F_σ set is a set which can be written as countable union of closed sets. Similarly, G_δ as countable intersection of open sets.

Proposition 1.2.7. $\mathcal{B}(\mathbb{R})$ is generated by any of the following collections.

1. $\{(a, b) \mid a < b\}$ or $\{[a, b] \mid a < b\}$,
2. $\{(a, b] \mid a < b\}$ or $\{[a, b) \mid a < b\}$,
3. $\{(a, \infty) \mid a \in \mathbb{R}\}$ or $\{(-\infty, b) \mid b \in \mathbb{R}\}$,
4. $\{[a, \infty) \mid a \in \mathbb{R}\}$ or $\{(-\infty, b] \mid b \in \mathbb{R}\}$.

Definition 1.2.8 (Product of σ-algebras)**.** Let $(X_\alpha)_{\alpha\in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the projection (coordinate) maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α (for all α), the product σ-algebra on X is the σ-algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}.$$

This σ-algebra is denoted by $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$.

If $A = \{1, \dots, n\}$, we also write this as $\bigotimes_{j=1}^n \mathcal{M}_j$ or $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$.

Proposition 1.2.9. If A is countable, then $\bigotimes_\alpha \mathcal{M}_\alpha$ is the σ-algebra generated by $\{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$.

Proposition 1.2.10. For each $\alpha \in A$, let $\mathcal{E}_\alpha$ be a generating set for $\mathcal{M}_\alpha$. Then, $\mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 := \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\}$.
Furthermore, if A is countable and $\underline{X_\alpha \in \mathcal{E}_\alpha}$ for all α, then $\bigotimes_\alpha \mathcal{M}_\alpha$ is also generated by $\mathcal{F}_2 := \{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$.

With the above, we get two (possibly different) σ-algebras on $\mathbb{R}^n$. One is the Borel σ-algebra on it, by virtue of it being a topological space, i.e., $\mathcal{B}(\mathbb{R}^n)$ and the other is the product of σ-algebra, i.e., $\prod_{i=1}^n \mathcal{B}(\mathbb{R})$. As it turns out, both are equal.

Theorem 1.2.11. Let $X_1, \dots, X_n$ be metric spaces, and let $X := \prod_j X_j$ be equipped with the product metric. Then, $\bigotimes_j \mathcal{B}(X_j) \subseteq \mathcal{B}(X)$. Furthermore, if each X_j is separable, then $\bigotimes_j \mathcal{B}(X_j) = \mathcal{B}(X)$.

In particular, $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R})$.

Definition 1.2.12. An elementary family is a collection $\mathcal{E}$ of subsets of X such that

1. $\varnothing \in \mathcal{E}$,
2. if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$,
3. if $E \in \mathcal{E}$, then E^c is a finite disjoint union of members of $\mathcal{E}$.

Proposition 1.2.13. If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint union of members of $\mathcal{E}$ is an algebra.

§§1.3. Measures

Definition 1.3.1 (Measure)**.** Suppose X is a non-empty set and $\mathcal{M}$ a σ-algebra on X. A measure on X is a map

$$\mu : X \longrightarrow [0, \infty]$$

satisfying

1. $\mu(\varnothing) = 0$,
2. (countable additivity) if $\{E_i\}_1^\infty \subseteq \mathcal{M}$ are pairwise disjoint, then

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

$(X, \mathcal{M}, \mu)$ is called a measure space.

Note that $\mu\left(\bigsqcup E_i\right)$ makes sense because $\mathcal{M}$ is a σ-algebra and hence $\bigsqcup E_i \in \mathcal{M}$.

Remark 1.3.2. Countable additivity implies finite additivity: If $E_1, \ldots, E_n$ are disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_j E_j\right) = \sum_j \mu(E_j)$.

If μ satisfies $\mu(\varnothing) = 0$ and finite additivity, then μ is called a finitely additive measure. (Note that this μ need not be a measure.)

Definition 1.3.3. If X is a set and $\mathcal{M} \subseteq \mathcal{P}(X)$ a σ-algebra, then $(X, \mathcal{M})$ is called a measurable space and sets in $\mathcal{M}$ are called measurable sets. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a measure space.

Definition 1.3.4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. If $\mu(X) < \infty$, then μ is called finite.
2. If $X = \bigcup_{j=1}^\infty E_j$, where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, then μ is called σ-finite.
3. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$, then μ is called semifinite.

Exercise 1.3.5. Every σ-finite measure is semifinite, but the converse is not true.

Proposition 1.3.6. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. All sets mentioned below are in

$\mathcal{M}$. Then,

1. (Monotonicity) $E \subseteq F \implies \mu(E) \leqslant \mu(F)$,
2. (Subadditivity) $\mu\left(\bigcup_1^\infty E_i\right) \leqslant \sum_1^\infty \mu(E_i)$,
3. (Continuity from below) If $E_i \uparrow$ (i.e., $E_1 \subseteq E_2 \subseteq \cdots$), then
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_i),$$
4. (Continuity from above) If $E_i \downarrow$ (i.e., $E_1 \supseteq E_2 \supseteq \cdots$), and $\mu(E_i) < \infty$ for some i, then
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_i).$$

Definition 1.3.7. If $(X, \mathcal{M}, \mu)$ is a measure set, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a null set. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true almost everywhere (a.e.), or for almost every x.

A measure whose domain includes all subsets of null sets is said to be complete.

Definition 1.3.8 (Completion)**.** Given a measure space $(X, \mathcal{M}, \mu)$, the completion of $\mathcal{M}$ with respect to μ, denoted $\overline{\mathcal{M}}$, is the collection of all subsets of the form $E \cup N$ where $E \in \mathcal{M}$ and N is a subset of a null set.

Note that the set N above itself need not be in $\mathcal{M}$.

Clearly, $\mathcal{M} \subseteq \overline{\mathcal{M}}$ since $\varnothing$ is a null set.

Proposition 1.3.9 (Extension to completion)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. $\overline{\mathcal{M}}$ is a σ-algebra.
2. There is a unique measure
$$\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$$
such that $\overline{\mu}|\mathcal{M} = \mu$.

$\overline{\mu}$ is called the completion of μ.

§§1.4. Outer Measures

Definition 1.4.1 (Outer measure)**.** An outer measure on a nonempty set X is a map

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

satisfying

1. $\mu^*(\varnothing) = 0$,
2. $A \subseteq B \implies \mu^*(A) \leqslant \mu^*(B)$,
3. $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} \mu(E_i)$.

Note that we don't demand equality even if disjoint.

Proposition 1.4.2 (A construction of an outer measure)**.** Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{F} \to [0, \infty]$ is a map such that

1. $\varnothing, X \in \mathcal{F}$,
2. $\rho(\varnothing) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{F},\ E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}.$$

Then, μ^* is an outer measure.

Note that the above had just the bare minimum requirement for both ρ and $\mathcal{F}$ and still gave us that μ^* is an outer measure.

Definition 1.4.3 (μ^*-measurable)**.** Given an outer measure μ^* on a set X, a set $A \subseteq X$ is said to be μ^*-measurable if for all $E \subseteq X$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Note that $\mu^*(E) \leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$ holds for all A and E, just by virtue of μ^* be an outer measure. Moreover, the reverse inequality also holds trivially if $E = \infty$. Thus, A is μ^*-measurable iff

$$\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subseteq X \text{ such that } \mu^*(E) < \infty.$$

Theorem 1.4.4 (Carathéodory)**.** Let μ^* be an outer measure on X. Let

$$\mathcal{M} := \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}.$$

Then,

1. $\mathcal{M}$ is a σ-algebra.
2. $\mu^*|\mathcal{M}$ is a complete measure.

Definition 1.4.5 (Premeasure)**.** Suppose $\mathcal{F}$ is an algebra on X. A map

$$\mu_0 : \mathcal{F} \to [0, \infty]$$

is called a premeasure if

1. $\mu_0(\varnothing) = 0$,
2. if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint such that $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{F}$, then

$$\mu_0\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

Note that by putting all but finitely many $A_i = \varnothing$, the above equality holds for finite unions as well. (The finite union *will* be in $\mathcal{F}$ since it's an algebra.)

Proposition 1.4.6. Suppose μ_0 is a premeasure on an algebra $\mathcal{F}$. Then, if μ^* is the outer measure as defined in Proposition 1.4.2 (with $\rho = \mu_0$), then

1. $\mu^*|\mathcal{F} = \mu_0$,
2. every set in $\mathcal{F}$ is μ^*-measurable.

Theorem 1.4.7. Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$ is an algebra and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{F}$.

Let μ_0 be a premeasure defined on $\mathcal{F}$ and let μ^* be the outer measure as before. Then

1. $\mu^*|\mathcal{M}$ is a measure on $(X, \mathcal{M})$. Put $\mu = \mu^*|\mathcal{M}$ for the next part.
2. If ν is any measure extending μ_0, then $\nu \leqslant \mu$, and

$$\nu(E) = \mu(E)$$

whenever $\mu(E) < \infty$.

3. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

§§1.5. Borel measures on the real line

Definition 1.5.1. A half-interval is a subset of $\mathbb{R}$ of one of the following forms:

1. $(a, b]$ for $-\infty \leqslant a < b < \infty$,
2. (a, ∞) for $-\infty \leqslant a < \infty$,
3. $\varnothing$.

Proposition 1.5.2. The collection $\mathcal{F}$ of all finite disjoint unions of half-intervals is an algebra on $\mathbb{R}$. The σ-algebra generated by $\mathcal{F}$ is $\mathcal{B}(\mathbb{R})$.

Proposition 1.5.3. Let $\mathcal{F}$ be the algebra consisting of finite unions of half-intervals. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing an right-continuous function. Define

$$\mu_0\left(\bigsqcup_{i=1}^{n}(a_j, b_j]\right) := \sum_{i=1}^{n}[F(b_j) - F(a_j)],$$

and let $\mu_0(\varnothing) = 0$.

Then, μ_0 is a well-defined premeasure on $\mathcal{F}$.

Definition 1.5.4 (Borel measure)**.** A measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called a Borel measure on $\mathbb{R}$.

Theorem 1.5.5. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right-continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) := \begin{cases} \mu\left((0, x]\right) & x > 0, \\ 0 & x = 0, \\ -\mu\left((-x, 0]\right) & x < 0, \end{cases}$$

then F is increasing and right-continuous, and $\mu = \mu_F$.

Let F be an increasing and right-continuous function on $\mathbb{R}$. The earlier theory gives us not only a Borel measure μ_F but also a complete measure $\overline{\mu_F}$, which is the completion of μ_F. We shall usually denote the completion also by μ_F; it is called the Lebesgue-Stieltjes measureassociated to F.

Proposition 1.5.6. Let μ be a complete Lebesgue-Stieltjes measure on $\mathbb{R}$, associated to an increasing right-continuous function F. Let $\mathcal{M}_\mu$ denote the domain of μ.

For any $E \in \mathcal{M}_\mu$, we have

$$\begin{aligned}\mu(E) &= \inf\left\{\sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_j (a_j, b_j]\right\} \\ &= \inf\left\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subseteq \bigcup_j (a_j, b_j)\right\} \\ &= \inf\{\mu(U) : U \supseteq E \text{ and } U \text{ is open}\} \\ &= \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}.\end{aligned}$$

If $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set $I \subseteq \mathbb{R}$ that is a finite union of open intervals such that $\mu(E \Delta I) < \epsilon$.

If $A \subseteq \mathbb{R}$, the following are equivalent.

1. $A \in M_\mu$.
2. $A = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$.
3. $A = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

Recall that a G_δ set is a countable intersection of open sets, and an F_σ set is a countable union of closed sets.

Now, consider F to be the identity function. The associated (complete) measure is denoted by m and called the Lebesgue measure. The domain of m is denoted by $\mathcal{L}$ and called the class of Lebesgue measurable sets.

Proposition 1.5.7 (Invariance of Lebesgue measure)**.** If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

§2. Integration

§§2.1. Measurable functions

Definition 2.1.1. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ between is called $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Proposition 2.1.2. Composition of measurable functions is measurable. If the measure on Y is generated by $\mathcal{E}$, then it suffices to check that $f^{-1}(E)$ is measurable for all $E \in \mathcal{E}$. Consequently, if X and Y are topological spaces with the Borel measure, then continuous functions are measurable.

If $(X, \mathcal{M})$ is a measurable space, a real-valued (resp. complex-valued) function on X will be called $\mathcal{M}$-measurable, or simply measurable, if f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}))$ (resp. $(\mathcal{M}, \mathcal{B}(\mathbb{C}))$) measurable. In particular, $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue (resp. Borel) measurable if f is $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$ (resp. $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$) measurable; likewise for $f : \mathbb{R} \to \mathbb{C}$.
Note that if $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable, it is not necessary that $f \circ g$ is Lebesgue measurable. This is because f and g are $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$-measurable and not (necessarily) $(\mathcal{L}, \mathcal{L})$.

Corollary 2.1.3. If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$, f being measurable is equivalent to any of the following:

1. $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,
2. $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,
3. $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,
4. $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$,

Proposition 2.1.4 (Universal property of products)**.** Let X and $(Y_\alpha)_\alpha$ be measurable spaces. Put $Y := \prod_\alpha Y$ and give Y the product σ-algebra. Let $\pi_\alpha : Y \to Y_\alpha$ denote the projection maps.

Then, $f : X \to Y$ is measurable iff $f \circ \pi_\alpha$ is measurable for all α. Moreover, each π_α is measurable.

Corollary 2.1.5. A function $X \to \mathbb{C}$ is measurable iff its real and imaginary parts are measurable functions $X \to \mathbb{R}$.

Recall the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$ is a metrisable topological space. We may talk about measurable functions $X \to \overline{\mathbb{R}}$ by giving $\overline{\mathbb{R}}$ the Borel measure. Explicitly, this is given by $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.

Proposition 2.1.6. If $f, g : X \to \mathbb{C}$ are measurable, so are $f + g$ and fg.

If $(f_n)_{n\geqslant 1}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions, then the functions

$$g_1(x) := \sup_n f_n(x), \quad g_2(x) := \inf_n f_n(x),$$
$$g_3(x) := \limsup_{n\to\infty} f_n(x), \quad g_4(x) := \liminf_{n\to\infty} f_n(x)$$

are all measurable.

If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x, then f is measurable. This is also true if f_n were complex measurable functions.

If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then so are $\min(f, g)$ and $\max(f, g)$.

Definition 2.1.7. If $f : X \to \mathbb{R}$, we define the positive and negative parts of f to be

$$f^+(x) := \max(f(x), 0) \quad \text{and} \quad f^-(x) := \max(-f(x), 0).$$

Note that f^+ and f^- are nonnegative valued measurable functions. Moreover, $f = f^+ - f^-$.

If f is complex-valued, we have the polar decomposition as

$$f = (\operatorname{sign}(f))|f|,$$

where $\operatorname{sign}(z) = z/|z|$ for $z \neq 0$ and $\operatorname{sign}(0) = 0$.

Let $(X, \mathcal{M})$ be a measurable space as usual. If $E \subseteq X$, the characteristic (or indicator) function χ_E is defined on X by

$$\chi_E(x) := \begin{cases} 1 & x \in E, \\ 0 & x \notin E. \end{cases}$$

Note that χ_E is measurable iff E is measurable. A simple function on X is a finite linear combination, with complex coefficients, of characteristic functions of sets in $\mathcal{M}$. Equivalently, $f : X \to \mathbb{C}$ is simple iff f is measurable and the image of f is a finite subset of $\mathbb{C}$. Explicitly, we have

$$f = \sum_{j=1}^{n} z_j \chi_{E_j},$$

where $\operatorname{im}(f) = \{z_1, \ldots, z_n\}$ and $E_j = f^{-1}(\{z_j\})$. This is called the standard representation of f. It exhibits f as a linear combination, with distinct coefficients, of characteristic functions of disjoint sets whose union is X. (It is possible that one z_j is 0 but we still consider it as a part of the representation.)

Exercise 2.1.8. If f and g are simple, then so are $f+g$ and fg.

Theorem 2.1.9. Let $(X, \mathcal{M})$ be a measurable space.

1. If $f : X \to [0, \infty]$ is measurable, then there is a sequence $(\phi_n)_{n\geqslant 1}$ of real-valued simple functions such that $0 \leqslant \phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant f$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.
2. If $f : X \to \mathbb{C}$ is measurable, then there is a sequence $(\phi_n)_{n\geqslant 1}$ of simple functions such that $0 \leqslant |\phi_1| \leqslant |\phi_2| \leqslant \cdots \leqslant |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

Proposition 2.1.10. Let $(X, \mathcal{M}, \mu)$ be a measure space. The following implications are valid iff μ is complete:

1. If f is measurable and $f = g$ μ a.e., then g is measurable.
2. If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ a.e., then f is measurable.

Proposition 2.1.11. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, then there is an $\mathcal{M}$-measurable function g such that $f = g$ $\bar{\mu}$ almost everywhere.

§§2.2. Integration of nonnegative functions

In this subsection, we fix a measure space $(X, \mathcal{M}, \mu)$. We define

$$\mathcal{L}^+ := \text{the space of all measurable functions from } X \text{ to } [0, \infty].$$

The above may be denoted by $L^+(X)$ or $L^+(\mu)$ or $L^+(X, \mu)$.

If $\phi \in L^+$ is a simple function with *standard representation* $\phi = \sum a_j \chi_{E_j}$, then we define the integral of ϕ with respect to μ by

$$\int_X \phi \, d\mu := \sum_{j=1}^{n} a_j \mu(E_j).$$

Note that there is no question of "well-defined-ness" since there is a unique standard representation. We make the convention $0 \cdot \infty = 0$. The usual conventions of alternate notations apply. Some are shown below.

$$\int_A \phi \, d\mu = \int_A \phi = \int_A \phi(x) \, d\mu = \int_X \phi\chi_A \, d\mu, \qquad \int = \int_X .$$

(In the above, A is any measurable subset of X. Note that $\phi\chi_A$ is a simple function on A.)

Proposition 2.2.1. Let ϕ and ψ be simple function in L^+.

1. If $c \geqslant 0$, then $\int c\phi = c \int \phi$.
2. $\int(\phi + \psi) = \int \phi + \int \psi$.
3. If $\phi \leqslant \psi$, then $\int \phi \leqslant \int \psi$.
4. $A \mapsto \int_A d\mu$ is a measure on $\mathcal{M}$.

Extend the definition of $\int$ to all $f \in L^+$ by

$$\int f \, d\mu := \sup \left\{ \int \phi \, d\mu : 0 \leqslant \phi \leqslant f, \ \phi \text{ simple} \right\}.$$

By the previous proposition, the above agrees with the earlier definition when f is simple. The definition quickly also implies

$$\int f \leqslant \int g \text{ whenever } f \leqslant g, \quad \text{and} \quad \int cf = c \int f \text{ for all } c \in [0, \infty].$$

Theorem 2.2.2 (Monotone Convergence Theorem)**.** If $(f_n)_n$ is a sequence in L^+ such that $f_j \leqslant f_{j+1}$ for all j, and $f = \lim_n f_n = \sup_n f_n$, then $\int f = \lim_n \int f_n$.

Corollary 2.2.3. If $(\phi_n)_n$ is a sequence of simple L^+ functions increasing to f, then $\int f = \lim_n \int \phi_n$.

If $(f_n)_n$ is a finite or infinite sequence in L^+ and $f = \sum_n f_n$, then $\int f = \sum_n \int f_n$.

Proposition 2.2.4. If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.

If $(f_n)_n$ is a sequence in L^+, $f \in L^+$, and $f_n(x)$ increases to $f(x)$ for a.e. x, then $\int f = \lim_n \int f_n$.

Theorem 2.2.5 (Fatou's Lemma)**.** If $(f_n)_n$ is any sequence in L^+, then

$$\int (\liminf f_n) \leqslant \liminf \int f_n.$$

Example 2.2.6. Consider $f_n = \chi_{(n,n+1)}$ or $f_n = n\chi_{(0,1/n)}$ to see that the inequality can be strict. Note that in either case $\int f_n = 1$ for all n but $f_n \to 0$ pointwise.

Corollary 2.2.7. If $(f_n)_n$ is a sequence in L^+, $f \in L^+$, and $f_n \to f$ a.e., then $\int f \leqslant \liminf \int f_n$.

Proposition 2.2.8. If $f \in L^+$ and $\int f < \infty$, then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite.

§§2.3. Integration of complex functions

We continue to work on a fixed measure space $(X, \mathcal{M}, \mu)$. Let $f : X \to \mathbb{R}$ be measurable. Note that then f^+, f^-, and $|f|$ are all in L^+. (In fact, $|f| = f^+ + f^-$.) Thus, it makes sense to talk about their integrals.
f is said to be integrable if either (and hence both) of the two equivalent conditions hold:

1. $\int f^+$ and $\int f^-$ are finite.
2. $\int |f|$ is finite.

In this case, we define

$$\int f := \int f^+ - \int f^-.$$

Proposition 2.3.1. The set of all integrable real-valued functions on X is a real vector space, and the integral is a linear functional on it.

Now, if f is a complex-valued measurable function, we say that f is integrable if $\int |f| < \infty$. More generally, if $E \in \mathcal{M}$, f is integrable on E if $\int_E |f| < \infty$. Check that f is integrable iff its real and imaginary parts are so. In this case, we define

$$\int f := \int \Re(f) + \iota \int \Im(f).$$

It follows that the space of complex-valued functions on X is a complex vector space, and the integral is a linear functional on it. This space is denoted by L^1 (or $L^1(\mu)$ or ...).

Proposition 2.3.2. If $f \in L^1$, then

1. $|\int f| \leqslant \int |f|$,
2. $\{x : f(x) \neq 0\}$ is σ-finite,
3. if $g \in L^1$, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f - g| = 0$ iff $f = g$ a.e.

The above tells us that it makes no difference if we alter functions on null sets. In this fashion, we can treat $\overline{\mathbb{R}}$-valued functions that are finite a.e. as real-valued functions for the purposes of integration.

We find it useful to treat $L^1(\mu)$ to be the set of *equivalence classes* of integrable functions modulo the relation $f \sim g$ if $f = g$ a.e. This new L^1 continues to be a complex vector space. Moreover, L^1 now becomes a metric space with metric $\rho(f, g) := \int |f - g|$.

Theorem 2.3.3 (Dominated convergence theorem)**.** Let $(f_n)_n$ be a sequence in L^1 such that

1. $f_n \to f$ a.e., and
2. there exists $g \in L^1$ such that $|f_n| \leqslant g$ a.e. for all n.

Then, $f \in L^1$ and $\int f = \lim_n \int f_n$.

Corollary 2.3.4. Suppose that $(f_n)_n$ is a sequence in L^1 such that $\sum \int |f_n| < \infty$. Then, $\sum f_n$ converges a.e. to a function $f \in L^1$ and $\int f = \sum \int f_n$.

Theorem 2.3.5. If $f \in L^1(\mu)$ and $\epsilon > 0$, then there is an integrable simple function ϕ such that $\int |f - \phi| < \epsilon$. (Simple functions are dense.)

If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, the sets in the definition of $\phi = \sum a_j \chi_{E_j}$ can be taken to be finite unions of open intervals; moreover, there is a continuous function g that vanishes outside a bounded interval such that $\int |f - g| < \epsilon$.

Theorem 2.3.6. Suppose that $f : X \times [a, b] \to \mathbb{C}$ (here $-\infty < a < b < \infty$) and that $f(-, t) : X \to \mathbb{C}$ is integrable for each $t \in [a, b]$. Let $F(t) := \int_X f(x, t)\, d\mu(x)$ for $t \in [a, b]$.

1. Suppose that there exists $g \in L^1(\mu)$ such that $|f(x, t)| \leqslant g(x)$ for all x, t.
   If $\lim_{t \to t_0} f(x, t) = f(x, t_0)$ for all $x \in X$, then $\lim_{t \to t_0} F(t) = F(t_0)$; in particular, if $f(x, -)$ is continuous for every x, then F is continuous.
2. Suppose that $\partial f/\partial t$ exists and there is a $g \in L^1(\mu)$ such that $|(\partial f/\partial t)(x, t)| \leqslant g(x)$ for all x, t. Then, F is differentiable and $F'(t) = \int (\partial f/\partial t)(x, t)\, d\mu(x)$.

In the special case that μ is the Lebesgue measure on $\mathbb{R}$, the integral developed is called the Lebesgue integral.

Theorem 2.3.7. Let f be a bounded real-function on $[a, b]$.

1. f is Riemann integrable iff $\{x \in [a, b] : f \text{ is discontinuous at } x\}$ has Lebesgue measure zero.

2. If f is Riemann integrable, then f is the Lebesgue measurable, and $\int_a^b f = \int_{[a,b]} f\,dm$.

§§2.4. Modes of convergence

Definition 2.4.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of complex-valued measurable functions on X, and $f : X \to \mathbb{C}$ be measurable.

1. $(f_n)_n$ is Cauchy in measure if for every $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f_m(x)| \geqslant \epsilon\}) \to 0 \quad \text{as } n, m \to \infty.$$

2. $f_n \to f$ in measure if for every $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f(x)| \geqslant \epsilon\}) \to 0 \quad \text{as } n \to \infty.$$

We already know what it means for f_n to converge pointwise, a.e., uniformly, and in L^1.

Example 2.4.2. Consider the following examples of sequences of measurable functions on $\mathbb{R}$:

(i) $f_n = \frac{1}{n}\chi_{(0,n)}$.

(ii) $f_n = \chi_{(n,n+1)}$.

(iii) $f_n = n\chi_{[0,1/n]}$.

(iv) $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, and in general, $f_n = \chi_{j/2^k,(j+1)/2^k}$, where $n = 2^k + j$ with $0 \leqslant j < 2^k$.

In (i), (ii), and (iii), $f_n \to 0$ uniformly, pointwise, and a.e., respectively, but $f_n \not\to 0$ in L^1 since $\int |f_n| = 1$ for all n.

In (i) and (iii), $f_n \to 0$ in measure (but not in L^1).

In (iv), $f_n \to 0$ in L^1, but $f_n(x)$ converges for no x.

Proposition 2.4.3. If $f_n \to f$ a.e. and $|f_n| \leqslant g \in L^1$, then $f_n \to f$ in L^1.

Suppose $(f_n)_n$ is Cauchy in measure. Then, there is a measurable function f such that $f_n \to f$ in measure, and there is a subsequence $(f_{n_j})_j$ such that $f_{n_j} \to f$ a.e. Moreover, if also $f_n \to g$ in measure, then $g = f$ a.e.

If $f_n \to f$ in L^1, then $f_n \to f$ in measure. Moreover, there is a subsequence (f_{n_j}) such that

$f_{n_j} \to f$ a.e.

Theorem 2.4.4 (Egoroff's theorem)**.** Suppose $\mu(X) < \infty$, and $f, f_1, f_2, \dots$ are complex-valued measurable functions on X such that $f_n \to f$ a.e.
Then, for every $\epsilon > 0$, there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

§§2.5. Product measures

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We define a measure on the measurable space $(X \times Y, \mathcal{M} \otimes \mathcal{N})$.

Define a rectangle to be a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Check that the class of rectangles is closed under finite intersections and complements. Thus, the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra (Proposition 1.2.13) and the σ-algebra it generates is $\mathcal{M} \otimes \mathcal{N}$ (by definition).

Exercise 2.5.1. If $A \times B = \bigsqcup_j A_j \times B_j$, then $\mu(A)\nu(B) = \sum_j \mu(A_j)\nu(B_j)$. ($j$ may run over a finite or countably infinite set.)

Thus, if $E \in \mathcal{A}$ is the disjoint union of rectangles $A_1 \times B_1, \dots, A_n \times B_n$, we may define

$$\pi(E) := \sum_{j=1}^{n} \mu(A_j)\nu(B_j).$$

Moreover, π is a premeasure on $\mathcal{A}$. By our earlier theory, π generates an outer measure on $X \times Y$ whose restriction to $\mathcal{M} \otimes \mathcal{N}$ is a measure that extends π. We call this the product measure. If μ and ν are σ-finite, then so is $\mu \times \nu$. In this case, $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ for all rectangles $A \times B$.

The above (and the below) constructions (and results) can be extended to more factors but we work with only two.

If $E \subseteq X \times Y$, for $x \in X$ and $y \in Y$, we define the x-section E_x and the y-section E^y of E by

$$E_x = \{y \in Y : (x, y) \in E\} \subseteq Y \quad \text{and} \quad E^y := \{x \in X : (x, y) \in E\} \subseteq X.$$

Similarly, if f is a function on $X \times Y$, we define the x-section f_x and the y-section f^y of f by

$$f_x(y) := f(x, y) =: f^y(x).$$

Note: $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

Proposition 2.5.2.

1. If $E \in \mathcal{M} \otimes N$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.
2. If f is $\mathcal{M} \otimes N$-measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.

Theorem 2.5.3. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on X and Y, respectively, and

$$(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Theorem 2.5.4 (The Fubini-Tonelli theorem)**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

1. (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) := \int f_x\, d\nu$ and $h(y) := \int f^y\, d\mu$ are in $L^+(X)$ and $L^+(Y)$ respectively, and

$$\begin{aligned} \int f\, d(\mu \times \nu) &= \int \left[\int f(x, y)\, d\nu(y) \right] d\mu(x) \\ &= \int \left[\int f(x, y)\, d\mu(x) \right] d\nu(y). \end{aligned} \tag{2.1}$$

2. (Fubini) If $f \in L^1(X \times Y)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$. The a.e.-defined functions $g(x) := \int f_x\, d\nu$ and $h(y) := \int f^y\, d\mu$ are in $L^1(X)$ and $L^1(Y)$ respectively, and (2.1) holds.

The product measure is almost never complete, even if μ and ν are so.

Theorem 2.5.5 (The Fubini-Tonelli Theorem for Complete Measures)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

If f is $\mathcal{L}$-measurable and either

(i) $f \geqslant 0$, or

(ii) $f \in L^1(\lambda)$,

then f_x is $\mathcal{N}$-measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. y, and in case (ii), f_x

and f^y are also integrable for a.e. x and y. Moreover, $x \mapsto \int f_x \, d\nu$ and $y \mapsto \int f^y \, d\mu$ are measurable, and in case (ii) also integrable, and

$$\int f \, d\lambda = \iint f(x,y) \, d\mu(x) \, d\nu(y) = \iint f(x,y) \, d\nu(y) \, d\mu(x).$$

§§2.6. Integration in Polar Coordinates

If $x \in \mathbb{R}^n \setminus \{0\}$, the polar coordinates of x are

$$r := \|x\| \in (0,\infty), \quad x' := \frac{x}{r} \in S^{n-1}.$$

The map $\Phi(x) := (r, x')$ is a homeomorphism from $\mathbb{R}^n \setminus \{0\}$ onto $(0,\infty) \times S^{n-1}$.

m_* is the Borel measure on $(0,\infty) \times S^{n-1}$ defined by

$$m_*(E) := m(\Phi^{-1}(E)).$$

We define the measure $\rho = \rho_n$ on $(0,\infty)$ by $\rho(E) := \int_E r^{n-1} \, dr$.

Theorem 2.6.1. There is a unique measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that $m_* = \rho \times \sigma$. If f is Borel measurable on $\mathbb{R}^n$ and $f \geqslant 0$ or $f \in L^1(m)$, then

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty \int_{S^{n-1}} f(rx') r^{n-1} \, d\sigma(x') \, dr.$$

σ above is defined as follows: given a Borel set $E \subseteq S^{n-1}$, define $E' = \Phi^{-1}((0,1] \times E) = \{rx' : 0 < r \leqslant 1, x' \in E\}$, and set $\sigma(E) := n \cdot m(E')$.

Corollary 2.6.2. If f is a measurable function on $\mathbb{R}^n$ such that $f \in L^+ \cup L^1$ and $f(x) = g(\|x\|)$ for some function g on $(0,\infty)$, then

$$\int_{\mathbb{R}^n} f(x) \, dx = \sigma(S^{n-1}) \int_0^\infty g(r) r^{n-1} \, dr.$$

Proposition 2.6.3. For $a > 0$, we have

$$\int_{\mathbb{R}^n} \exp(-a\|x\|^2) \, dx = \left(\frac{\pi}{a}\right)^{n/2}.$$

Moreover,

$$\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

If $B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$, then

$$m(B^n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n + 1\right)}.$$

§3. Signed Measures and Differentiation

§§3.1. Signed measures

Definition 3.1.1. Let $(X, \mathcal{M})$ be a measurable space. A signed measure on $(X, \mathcal{M})$ is a function $\nu : X \to [-\infty, \infty]$ such that

1. $\nu(\varnothing) = 0$;
2. ν assumes at most one of the values $\pm\infty$;
3. if $(E_j)_j$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_j E_j) = \sum_j \nu(E_j)$, where the sum converges absolutely if it is finite.

Measures as defined earlier are examples of signed measures. For emphasis, we may use the term positive measure for the usual measures.

Example 3.1.2. Here are two examples, which are essentially the only examples of signed measures.

1. If μ_1, μ_2 are positive measures on $\mathcal{M}$ and at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.
2. If μ is a positive measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ \, d\mu$ or $\int f^- \, d\mu$ is finite (in which case we call f an extended μ-integrable function), then the function ν defined on $\mathcal{M}$ by

   $$\nu(E) := \int_E f \, d\mu$$

   is a signed measure.

   We denote the above relationship by

   $$d\nu = f \, d\mu. \tag{3.1}$$

   By abuse, we may even refer to ν by $f \, d\mu$.

Remark 3.1.3. Note that monotonicity is not a property of a signed measure. (In fact, monotonicity is a property iff the measure is positive.)

Proposition 3.1.4. Let ν be a signed measure on $(X, \mathcal{M})$. If $(E_j)_j$ is an increasing sequence in $\mathcal{M}$, then $\nu(\bigcup_j E_j) = \lim_j \nu(E_j)$. If $(E_j)_j$ is a decreasing sequence with some $\nu(E_j)$ finite, then $\nu(\bigcap_j E_j) = \lim_j \nu(E_j)$.

Definition 3.1.5. If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called positive (resp. negative, null) for ν if $\nu(F) \geqslant 0$ (resp. $\nu(F) \leqslant 0$, $\nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subseteq E$.

Example 3.1.6. In the earlier example of $\nu(E) = \int_E f \, d\mu$, we have that E is positive, negative, or null precisely when $f \geqslant 0$, $f \leqslant 0$, $f = 0$ μ-a.e. on E.

Remark 3.1.7. Note that $\nu(E) = 0$ is not enough for E to be null. (Similar comments for positive and negative.)

Proposition 3.1.8. Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive.

The statement is true for "positive" replaced with "negative" and "null" as well.

Theorem 3.1.9 (The Hahn Decomposition Theorem)**.** If ν is a signed measure on $(X, \mathcal{M})$, there exists a positive set P and a negative set N for ν such that $X = P \sqcup N$ (and $P \cap N = \varnothing$). If P', N' is another such pair, then $P \Delta P'$ ($= N \Delta N'$) is null for ν.

The decomposition $X = P \sqcup N$ of X as a disjoint union of a positive set and a negative set is called a Hahn decomposition for ν.

Definition 3.1.10. Two signed measures μ and ν on $(X, \mathcal{M})$ are mutually singular, or that ν is singular with respect to μ, or vice-versa, if there exist disjoint sets $E, F \in \mathcal{M}$ such that

1. $X = E \sqcup F$,
2. E is null for μ,
3. F is null for ν.

This is denoted by $\mu \perp \nu$.

Theorem 3.1.11 (The Jordan Decomposition Theorem)**.** If ν is a signed measure on $(X, \mathcal{M})$, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Given a Hahn decomposition $X = P \sqcup N$, we have $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$ for all $E \in \mathcal{M}$.

The measures ν^+ and ν^- are called the positive and negative variations of ν, and $\nu =$

$\nu^+ - \nu^-$ is called the Jordan decomposition of ν. The total variation of ν is the positive measure $|\nu|$ defined by $|\nu| = \nu^+ + \nu^-$.

Exercise 3.1.12. $E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$.

$\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Observation 3.1.13. Note that in general, ν is not bounded by $\nu(X)$. However, ν *is* bounded by $\nu^+(X) = \nu(P)$. In particular, if ν omits the value ∞, then $\nu^+(X) < \infty$. Similarly for $-\infty$.
Consequently, if the range of ν is contained in $\mathbb{R}$, then ν is finite.

Observation 3.1.14. Let ν be a signed measure on $(X, \mathcal{M})$, and $X = P \sqcup N$ be a Hahn decomposition, and set $f := \chi_P - \chi_N$. If we set $\mu = |\nu|$, then μ is a positive measure and we have

$$\nu(E) = \int_E f \, d\mu.$$

Integration with respect to a signed measure ν is defined as follows:

$$\begin{aligned} &L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-), \\ &\int f \, d\nu := \int f \, d\nu^+ - \int f \, d\nu^- \quad (f \in L^1(\nu)). \end{aligned}$$

A signed measure ν is called finite (resp. σ-finite) if $|\nu|$ is so.

Proposition 3.1.15. Let ν be a signed measure on $(X, \mathcal{M})$, and $E \in \mathcal{M}$. Then,

$$\begin{aligned} &\nu^+(E) = \sup\{\nu(F) : F \subseteq E, \, F \in \mathcal{M}\}, \\ &\nu^-(E) = -\inf\{\nu(F) : F \subseteq E, \, F \in \mathcal{M}\}, \\ &|\nu|(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, \, E = \bigsqcup_{j=1}^n E_j, \, E_1, \dots, E_n \in \mathcal{M}\right\}. \end{aligned}$$

§§3.2. The Lebesgue-Radon-Nikodym Theorem

Definition 3.2.1. Let ν be a signed measure and μ a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ, denoted $\nu \ll \mu$, if

$$\mu(E) \Rightarrow \nu(E)$$

for all $E \in \mathcal{M}$.

Exercise 3.2.2. The following are equivalent:

1. $\nu \ll \mu$,
2. $|\nu| \ll \mu$,
3. $\nu^{+} \ll \mu$ and $\nu^{-} \ll \mu$.

Exercise 3.2.3. $\nu \perp \mu$ and $\nu \ll \mu$ implies $\nu = 0$.

Theorem 3.2.4. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. The following are equivalent:

1. $\nu \ll \mu$,
2. for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

Note that $\nu \ll \mu$ iff $|\nu| \ll \mu$ and hence, the "$|\nu(E)| < \epsilon$" in the second statement can also be replaced with "$|\nu|(E) < \epsilon$".

Remark 3.2.5. Given a positive measure μ and an extended μ-integrable function f, the signed measure ν defined by $\nu(E) = \int_E f \, d\mu$ is absolutely continuous with respect to μ. (That is, $d\nu = f \, d\mu$.)

Moreover, ν is finite iff $f \in L^1(\mu)$.

Exercise 3.2.6. ν being finite cannot be dropped. Check that in the following two examples that $\nu \ll \mu$ but the ϵ-δ condition is not satisfied. (Note that ν is σ-finite in both cases.)

1. $d\nu(x) = dx/x$ and $d\mu = dx$ on $(0, 1)$.
2. ν is the counting measure and $\mu(E) = \sum_{n \in E} 2^{-n}$ on $\mathbb{N}$.

Corollary 3.2.7. If $f \in L^1(\mu)$, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\left|\int_E f \, d\mu\right| < \epsilon$

whenever $\mu(E) < \delta$.

Proposition 3.2.8. Suppose that ν and μ are finite positive measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$, or there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and E is a positive set for $\nu - \epsilon\mu$.

Theorem 3.2.9 (The Lebesgue-Radon-Nikodym Theorem)**.** Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu, \tag{3.2}$$

$$\nu = \lambda + \rho. \tag{3.3}$$

Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f\,d\mu$, and any two such functions are equal a.e.

(Recall (3.1) for the last notation.) The decomposition $\nu = \lambda + \rho$ satisfying (3.2) is called the Lebesgue decomposition of ν with respect to μ.

Corollary 3.2.10 (Radon-Nikodym theorem)**.** (Continuing the same hypothesis.) In particular, if $\nu \ll \mu$, then $d\nu = f\,d\mu$ for some f.

f above is called the Radon-Nikodym derivative of ν with respect to μ and is denoted by $d\nu/d\mu$. (Technically, this is a class of functions equal to f a.e.)

Exercise 3.2.11. σ-finiteness is necessary. Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}([0, 1])$, $m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{M}$. Show that

1. $m \ll \mu$ but $dm \neq f\,d\mu$ for any f,
2. μ has no Lebesgue decomposition with respect to m.

Proposition 3.2.12. Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite signed measures on $(X, \mathcal{M})$ such that $\nu \ll \mu \ll \lambda$.

1. If $g \in L^1(\nu)$, then $g \cdot \frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\int g\,d\nu = \int g \frac{d\nu}{d\mu}\,d\mu.$$

2. We have $\nu \ll \lambda$, and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Corollary 3.2.13. If $\mu \ll \lambda$ and $\lambda \ll \mu$, then $(\frac{d\lambda}{d\mu})(\frac{d\mu}{d\lambda}) = 1$ a.e. (with respect to either μ or λ).

Observation 3.2.14. If $\mu_1, \dots, \mu_n$ are positive measures on $(X, \mathcal{M})$, then $\mu := \sum_j \mu_j$ is a positive measure such that $\mu_j \ll \mu$ for all j.

§§3.3. Complex measures

Definition 3.3.1. A complex measure on a measurable space $(X, \mathcal{M})$ is a map $\nu : \mathcal{M} \to \mathbb{C}$ such that

1. $\nu(\varnothing) = 0$,
2. if $(E_j)_j$ is a sequence if disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_j E_j) = \sum_j \nu(E_j)$, where the sum converges absolutely.

Note that ν cannot take infinite values. So, a usual positive measure is a complex measure only if it is finite.

Example 3.3.2. If μ is a positive measure, and $f \in L^1(\mu)$, then $f\,d\mu$ is a complex measure.

If ν is a complex measure, we write ν_r and ν_i for the real and imaginary parts of ν. ν_r and ν_i are signed measures which do not take the values $\pm\infty$ and hence, finite. Thus, ν is a bounded subset of $\mathbb{C}$.

Integration: $L^1(\nu) := L^1(\nu_r) \cap L^1(\nu_i)$, and for $f \in L^1(\nu)$, we define
$$\int f\,d\nu := \int f\,d\nu_r + \iota \int f\,d\nu_i.$$

If ν and μ are complex measures, we say $\nu \perp \mu$ if $\nu_a \perp \mu_b$ for all $\{a, b\} \subseteq \{i, r\}$. If λ is a positive measure, we say $\nu \ll \lambda$ if $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$.

Theorem 3.3.3 (The Lebesgue-Radon-Nikodym Theorem)**.** If ν is a complex measure and μ a σ-finite positive measure on $(X, \mathcal{M})$, there exist a complex measure λ and an $f \in L^1(\mu)$

such that $\lambda \perp \mu$ and $d\nu = d\lambda + f\,d\mu$.
If also $\lambda' \perp \mu$ and $d\nu = d\lambda' + f\,d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ-a.e.

As before, if $\nu \ll \mu$, we denote f above by $d\nu/d\mu$.

Given any complex measure ν, we can write ν as $d\nu = f\,d\mu$ for some positive measure μ (one candidate is $\mu = |\nu_r| + |\nu_i|$). The total variation of ν is the positive measure $|\nu|$ determined by

$$d|\nu| = |f|\,d\mu.$$

One can check that this ν is independent of f and μ. Moreover, this coincides with the earlier definition for a (finite) signed measure.

Proposition 3.3.4. Let ν be a complex measure on $(X, \mathcal{M})$.

1. $|\nu(E)| \leqslant |\nu|(E)$ for all $E \in \mathcal{M}$.
2. $\nu \ll |\nu|$, and $d\nu/d|\nu|$ has absolute value 1 $|\nu|$-a.e.
3. $L^1(\nu) = L^1(|\nu|)$, and if $f \in L^1(\nu)$, then $|\int f\,d\nu| \leqslant \int |f|\,d|\nu|$.

Proposition 3.3.5. $|\nu_1 + \nu_2| \leqslant |\nu_1| + |\nu_2|$ for complex measures ν_1, ν_2 on $(X, \mathcal{M})$.

§§3.4. Differentiation on Euclidean Space

In this section, we look at the special case of the Lebesgue measure m on $\mathbb{R}^n$. The terms "integrable" and "almost everywhere" will mean with respect to the Lebesgue measure.

Proposition 3.4.1. Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B \in \mathcal{C}} B$. If $c < m(U)$, there exist disjoint $B_1, \ldots, B_k \in \mathcal{C}$ such that $\sum_{j=1}^{k} m(B_j) > 3^{-n}c$.

Definition 3.4.2. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is called locally integrable if $\int_K |f| < \infty$ for every bounded measurable set $K \subseteq \mathbb{R}^n$. (Equivalently, for every compact set $K \subseteq \mathbb{R}^n$.)

The space of locally integrable functions is denoted by L^1_{loc}. If $f \in L^1_{\text{loc}}$, $x \in \mathbb{R}^n$, and $r > 0$, we define $A_r f(x)$ by

$$A_r f(x) := \frac{1}{m(B(r,x))} \int_{B(r,x)} f.$$

Proposition 3.4.3. If $f \in L^1_{loc}$, $A_r f(x)$ is jointly continuous in r and x ($r > 0$, $x \in \mathbb{R}^n$).

Definition 3.4.4. If $f \in L^1_{loc}$, we define its Hardy-Littlewood maximal function Hf by

$$Hf(x) := \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f|.$$

Hf is a measurable function.

Theorem 3.4.5. Fix n. There is a constant $C > 0$ such that for all $f \in L^1(\mathbb{R}^n)$ and all $\alpha > 0$,

$$m(\{x : Hf(x) > \alpha\}) \leqslant \frac{C}{\alpha} \int_{\mathbb{R}^n} |f|.$$

Theorem 3.4.6. If $f \in L^1_{loc}$, then $\lim_{r\to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

Definition 3.4.7. For $f \in L^1_{loc}$, define the Lebesgue set L_f of f to be

$$L_f := \left\{ x \in \mathbb{R}^n : \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(x) - f(y)| \, dy \right\}.$$

Theorem 3.4.8. If $f \in L^1_{loc}$, then $m((L_f)^c) = 0$.

Note that this is a strengthening of the previous theorem.

Definition 3.4.9. A family $(E_r)_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to shrink nicely to $x \in \mathbb{R}^n$ if

1. $E_r \subseteq B(r, x)$ for each r;
2. there is a constant $\alpha > 0$, independent of r, such that $m(E_r) > \alpha m(B(x, r))$ for all r.

Note that $x \in E_r$ is not necessary.

Theorem 3.4.10 (The Lebesgue Differentiation Theorem). Suppose $f \in L^1_{loc}$. For every $x \in L_f$ – in particular, for almost every x – we have

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\, dy = 0 \quad \text{and}$$
$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f = f(x)$$

for every family $(E_r)_{r>0}$ that shrinks nicely to x.

Definition 3.4.11. A Borel measure ν on $\mathbb{R}^n$ will be called regular if

1. $\nu(K) < \infty$ for every compact K;
2. $\nu(E) = \inf\{\nu(U) : U \supseteq E,\ U \text{ open}\}$ for every $E \in \mathcal{B}(\mathbb{R}^n)$.

A signed or complex Borel measure ν will be called regular if $|\nu|$ is regular.

The second condition is actually implied by the first. For $n = 1$, this follows from results in the first section.

Every regular measure is σ-finite.

Example 3.4.12. If $f \in L^+(\mathbb{R}^n)$, the measure $f\,dm$ is regular iff $f \in L^1_{loc}$.

Theorem 3.4.13. Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f\,dm$ be its Lebesgue-Radon-Nikodym representation. Then, for m-almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every family $(E_r)_{r>0}$ that shrinks nicely to x.

§§3.5. Functions of Bounded Variation

Notations: For a function $F : \mathbb{R} \to \mathbb{R}$, $F(x+)$ denotes the right limit $\lim_{y\to x^+} F(y)$. (This will exist, for example, when F is increasing.) $F(x-)$ is defined similarly.
If F is increasing and right-continuous, μ_F is the Borel measure on $\mathbb{R}$ determined by $\mu_F((a, b]) = F(b) - F(a)$.

Theorem 3.5.1. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing, and let $G(x) = F(x+)$.

1. The set of discontinuities of F is countable.
2. F and G are differentiable a.e., and $F' = G'$ a.e.

Definition 3.5.2. If $F : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$, we define

$$T_F(x) := \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \right\}.$$

T_F is called the total variation of F.

If $a < b$, we have

$$T_F(b) - T_F(a) = \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b \right\}. \tag{3.4}$$

T_F is an increasing function with values in $[0, \infty]$.

Definition 3.5.3. If $T_F(\infty) = \lim_{x\to\infty} T_F(x)$ is finite, we say that F is of bounded variation on $\mathbb{R}$, and we denote the space of all such F by BV.

BV forms a complex vector space.

The supremum on the right in (3.4) is called the total variation of F on $[a, b]$. The space of functions $F : [a, b] \to \mathbb{C}$ whose total variation on $[a, b]$ is finite is denoted $BV([a, b])$.

Remark 3.5.4. If $F \in BV$, then $F|[a, b]$ is in $BV([a, b])$ for all $a, b \in \mathbb{R}$ with $a < b$.

Conversely, if $F \in BV([a, b])$ and we set $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$, then $F \in BV$.

Example 3.5.5.

1. If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then $F \in BV$.
2. $\sin \in BV([a, b])$ for all reals $a < b$ but $\sin \notin BV$.
3. If F is differentiable and F' is bounded, then $F \in BV([a, b])$ for all reals $a < b$.
4.

Proposition 3.5.6. If $F \in BV$ is real-valued, then $T_F + F$ and $T_F - F$ are increasing.

Theorem 3.5.7. Let $F : \mathbb{R} \to \mathbb{C}$.

1. $F \in BV$ iff $\Re(F) \in BV$ and $\Im(F) \in BV$.
2. If F is real valued, then $F \in BV$ iff F is the difference of two bounded increasing functions; for $F \in BV$, these functions may be taken to be $\frac{1}{2}(T_F \pm F)$.
3. If $F \in BV$, then $F(x+)$ and $F(x-)$ exist for all $x \in \mathbb{R}$, as do $F(\pm\infty)$.
4. If $F \in BV$, the set of discontinuities of F is countable.
5. If $F \in BV$ and $G(x) := F(x+)$, then F' and G' exist a.e. and are equal a.e.

The representation

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$$

of a real-valued $F \in BV$ is called a Jordan decomposition of F, and $\frac{1}{2}(T_F + F)$ and $\frac{1}{2}(T_F - F)$ are called the positive and negative variations of F.

For $x \in \mathbb{R}$, define $x^+ := \max(x, 0) = \frac{1}{2}(|x| + x)$ and $x^- := \max(-x, 0) = \frac{1}{2}(|x| - x)$. We then have

$$\frac{1}{2}(T_F \pm F)(x) = \sup\left\{ \sum_{j=1}^{n} \left[F(x_j) - F(x_{j-1})\right]^{\pm} : n \in \mathbb{N},\, x_0 < \cdots < x_n = x \right\} \pm \frac{1}{2}F(-\infty).$$

We define the space NBV (N for "normalised"):

$$\mathrm{NBV} := \{F \in BV : F \text{ is right-continuous and } F(-\infty) = 0\} \subseteq BV\,.$$

If $F \in BV$, then the function defined by $G(x) := F(x+) - F(-\infty)$ is in NBV and $F' = G'$ a.e.

Proposition 3.5.8. If $F \in BV$, then $T_F(-\infty) = 0$. If F is also right-continuous, then so is T_F.

Theorem 3.5.9. If μ is a complex Borel measure on $\mathbb{R}$ and $F(x) := \mu((-\infty, x])$, then $F \in$ NBV.
Conversely, if $F \in$ NBV, then there is a unique complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$; moreover, $|\mu_F| = \mu_{T_F}$.

Proposition 3.5.10. If $F \in$ NBV, then $F' \in L^1(m)$.

1. $\mu_F \perp m$ iff $F' = 0$ a.e.
2. $\mu_F \ll m$ iff F is absolutely continuous iff $F(x) = \int_{-\infty}^{x} F'(t)\, dt$.

If $f \in L^1(m)$, then the function $F(x) := \int_{-\infty}^{x} f(t)\,dt$ is in NBV and is absolutely continuous, and $f = F'$ a.e.

Recall that $F : \mathbb{R} \to \mathbb{C}$ is absolutely continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any finite set of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$,

$$\sum_j (b_j - a_j) < \delta \Rightarrow \sum_j |F(b_j) - F(a_j)| < \epsilon.$$

More generally, F is said to be absolutely continuous on $[a, b]$ if this condition is satisfied whenever (a_j, b_j) all lie in $[a, b]$.

If F is differentiable on $\mathbb{R}$ and F' is bounded, then F is absolutely continuous.

For the following results, a and b are reals with $a < b$.

Proposition 3.5.11. If F is absolutely continuous on $[a, b]$, then $F \in BV([a, b])$.

Theorem 3.5.12 (The Fundamental Theorem of Calculus for Lebesgue Integrals)**.** For $F : [a, b] \to \mathbb{C}$, the following are equivalent:

1. F is absolutely continuous on $[a, b]$.
2. $F(x) - F(a) = \int_x^a f(t)\,dt$ for some $f \in L^1([a, b], m)$.
3. F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_x^a F'(t)\,dt$.

Definition 3.5.13. A complex measure μ on $\mathbb{R}^n$ is called discrete if there is a countable set $\{x_j\}_{j \geqslant 1} \subseteq \mathbb{R}^n$ and complex numbers $(c_j)_{j \geqslant 1}$ such that $\sum |c_j| < \infty$ and $\mu = \sum_j c_j \delta_{x_j}$, where δ_x is the point mass at x.

μ is called continuous if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^n$.

Any complex measure μ can be uniquely written as $\mu = \mu_d + \mu_c$ where μ_d is discrete and μ_c continuous.
μ is discrete $\Rightarrow \mu \perp m$.
$\mu \ll m \Rightarrow \mu$ is continuous.

Any (regular) complex Borel measure on $\mathbb{R}^n$ can be written uniquely as

$$\mu_d + \mu_{ac} + \mu_{sc},$$

where μ_d is discrete, μ_{ac} is absolutely continuous with respect to m, and μ_{sc} is a "singular continuous" measure, that is, μ_{sc} is continuous but $\mu_{sc} \perp m$.

If $F \in NBV$, we denote the integral of a function g with respect to $\mu_F$by $\int g\,dF$ or $\int g(x)\,dF(x)$; such integrals are Lebesgue-Stieltjes integrals.

Theorem 3.5.14. If F and G are in NBV and at least one of them is continuous, then for $-\infty < a < b < \infty$,

$$\int_{(a,b]} F\,dG + \int_{(a,b]} G\,dF = F(b)G(b) - F(a)G(a).$$

If F and G are absolutely continuous on $[a, b]$, then so is FG, and

$$\int_a^b (FG' + G'F) = F(b)G(b) - F(a)G(a).$$