∫(͡° ͜ʖ ͡°)dx

MA 408

Measure Theory

Notes By: Aryaman Maithani

Spring 2020-21

Lecture 1

06 January 2021 22:06

Idea behind measure

Simplified case: Subsets of $\mathbb{R}$

Given $E \subseteq \mathbb{R}$, want to assign "length" or "content" to E.

Ideally, want a map

$$\mu : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$$

s.t.

(1) $\mu(\emptyset) = 0$

(2) For any $E \subset \mathbb{R}$ and $x \in E$,

$$\mu(E) = \mu(x + E).$$

$(x + E := \{ x + y : y \in E \})$ ← translation by x

(3) Given a countable collection $\{E_i\}_{i=1}^{\infty}$ of disjoint subsets of $\mathbb{R}$, we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

(So far, $\mu \equiv 0$ will satisfy above properties!)

(4) $\mu([0,1]) = 1$. ("Normalisation")

Any such μ would be a 'candidate' for our content.

However, no such μ exists!

Consider the following sets:

(1) Define $\sim$ on $\mathbb{R}$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
Clearly, $\sim$ is an equiv. relation.

Let $E \subseteq [0,1]$ be a set containing exactly one element from each equivalence class in $\mathbb{R}/\sim$.
(Existence is given by Axiom of Choice. Note that distinct equiv. classes are disjoint.)
and a small argument that lets you conclude $E \subset [0,1]$.

Q. What could $\mu(E)$ be?

Note that $\{E+r\}_{r \in \mathbb{Q} \cap [0,1]}$ is a collection of pairwise disjoint sets.

{Sketch. If $x \in (E+r_1) \cap (E+r_2)$, $(r_2 \geq r_1)$ then $x = r_1 + e_1 = r_2 + e_2$ for some $e_1, e_2 \in E$

$\Rightarrow e_1 - e_2 = r_2 - r_1 \in \mathbb{Q}$

$\Rightarrow e_1 \sim e_2 \Rightarrow e_1 = e_2$ }

Moreover, $[0,1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [0,1]} (E+r) \subseteq [0,2] = [0,1] \cup [1,2]$

(An easy consequence of (1) – (3) is that $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$.
Proof. $\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$. $\square$)

$$\Rightarrow \mu([0,1]) \leq \mu\left(\bigcup_{i=1}^{\infty}(E+r_i)\right) \leq \mu([0,1]) + \mu([1,2])$$

enumerate $\mathbb{Q} \cap [0,1]$ as $\{r_1, \ldots\}$

$$\Rightarrow 1 \leq \sum_{i=1}^{\infty} \mu(E+r_i) \leq 2$$

$[1,2] = [0,1]+1$

$$\Rightarrow \quad 1 \le \sum_{i=1}^{\infty} \mu(E + r_i) \le 2$$

$[1,2] = [0,1]+1$

$$\Rightarrow \quad 1 \le \sum_{i=1}^{\infty} \mu(E) \le 2$$

If $\mu(E)=0$: $1 \le 0$ $\rightarrow\leftarrow$

If $\mu(E) = r > 0$: $\sum_{i=1}^{\infty} \mu(E) = \infty \le 2$ $\rightarrow\leftarrow$

Possible way to salvage: Replace (3) to have "finite union" instead of "countable".

Turns out that that's still not enough.

(2) BANACH-TARSKI THEOREM ~~PARADOX~~ (1924): (Using AC)

For any open sets $U, V \subseteq \mathbb{R}^n$ where $n \geq 3$, there exists $k \in \mathbb{N}$ and set $U_1, \ldots, U_k$, $V_1, \ldots, V_k$ s.t.

(1) $U_i \cap U_j = \emptyset$, $V_i \cap V_j = \emptyset$, $1 \le i \ne j \le k$.

(2) $U = \bigsqcup_{i=1}^{k} U_i$, $V = \bigsqcup_{i=1}^{k} V_i$.

(3) $U_i \simeq V_i$, i.e.,
U_i is obtained from V_i by a sequence of rotations, reflections, and translations.
In other words, by isometries.

Thus, the analogue of (2) implies $\mu(U_i) = \mu(V_i)$ $\forall i$.

$\Rightarrow \mu(U) = \mu(V)$. $\rightsquigarrow$ Absurd conclusions.

As it turns out, the problem is NOT in the infinite union but rather the demand that μ is defined on all of $\mathcal{P}(\mathbb{R})$!

Thus, we restrict our attention to a smaller collection of subsets of $\mathbb{R}$. (Not too small!)

σ - ALGEBRAS

Let X be an arbitary set.

Def^n. (1) An algebra ("field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

(algebra, field)

① $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$

② $A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$ for any $n \in \mathbb{N}$.

(2) A σ-algebra ("σ-field") is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying:

(\sigma-algebra, \sigma-field)

① $A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}$

② $A_1, \dots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Note that complements and unions give us intersections. Also, $\phi, X \in \mathcal{F}$.

EXAMPLES

① $\mathcal{F} = \mathcal{P}(X)$ ← both

② (Countable - cocountable σ-algebra) (Countable-cocountable \sigma-algebra)

$\mathcal{F} = \{ E \subseteq X : E \text{ or } E^c \text{ is countable} \}$

Proof. Clearly closed under complement.

Let $A_1, \dots \in \mathcal{F}$.

If all A_i are countable, then $\bigcup A_i$ is.

Suppose A_1 not countable. Then, A_1^c is.

But

$$A_1 \subset \bigcup A_i \Rightarrow \left(\bigcup A_i\right)^c \subset A_1^c$$

$$\Rightarrow \left(\bigcup A_i\right)^c \text{ is countable. } \square$$

③ Given any $\mathcal{F} \subseteq \mathcal{P}(X)$, we can talk about σ-algebra generated by $\mathcal{F}$ denoted $\mathcal{M}(\mathcal{F})$ defined by

(\sigma-algebra generated)

$$\mathcal{M}(\mathcal{F}) = \bigcap_{\substack{\mathcal{F} \subseteq \mathcal{B} \\ \mathcal{B} \text{ is a } \sigma\text{-alg}}} \mathcal{B}$$

(Note that the intersection is non-empty because of $\mathcal{P}(X)$. Easy to see that intersection of σ-algebrae is again a σ-alg.)

By construction, $\mathcal{M}(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$.

Borel σ-Algebra.

Def^n: Let $(X, \mathcal{T})$ be a topological space. The σ-algebra generated by $\mathcal{T}$ is called the Borel σ-algebra on X, denoted $\mathcal{B}(X)$.

(Borel \sigma-algebra)

(Abuse of notation that we don't mention $\mathcal{T}$.)

In other words, it is generated by the open sets of X.

Borel σ-algebra on $\mathbb{R}$: Smallest σ-alg on $\mathbb{R}$ containing all the open sets.

Consequences:

① All open sets are in $\mathcal{B}(\mathbb{R})$.
② All closed sets are in $\mathcal{B}(\mathbb{R})$.
③ All F_σ, G_δ sets are in $\mathcal{B}(\mathbb{R})$.

$$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i \quad (F_i \text{ closed}) \; ; \qquad G_\delta \equiv \bigcap_{i=1}^{\infty} G_i \quad (G_i \text{ open})$$

Prop^n. Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$.
Then, $\mathcal{B}$ is also generated by any of the following:

(i) $\{(a, b) : a < b\}$ or $\{[a, b] : a < b\}$
(ii) $\{[a, b) : a < b\}$ or $\{(a, b] : a < b\}$
(iii) $\{(a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a) : a \in \mathbb{R}\}$
(iv) $\{[a, \infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a] : a \in \mathbb{R}\}$

Proof Easy. $\square$

Borel σ-algebra on $\mathbb{R}^n$:

Suppose $\{X_i\}_{i=1}^{n}$ are metric spaces.

Let $X = \prod_{i=1}^{n} X_i$ with the product metric.

def^n

If ρ_i is the metric on X_i, then ρ on $\prod X_i$ is defined as

$$\rho(x, y) = \max_{1 \le i \le n} \rho_i(x_i, y_i) \qquad \left(\begin{array}{l} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{array} \right)$$

Def^n. Suppose $(X_i, \mathcal{M}_i)$ are σ-algebrae. One can define a σ-algebra on $X := \prod X_i$ as follows:

(Product of \sigma-algebrae)

Consider the projection maps $\pi_i : X \to X_i$.
Let

$$\mathcal{F} = \{ \pi_i^{-1}(E) : E \in \mathcal{M}_i, \; i = 1, \dots, n \}$$

$$= \{E \times X_2 \times \cdots \times X_n : E \in \mathcal{M}_1\}$$
$$\cup \{X_1 \times E \times \cdots \times X_n : E \in \mathcal{M}_2\}$$
$$\cup \cdots \cup \{X_1 \times \cdots \times X_{n-1} \times E : E \in \mathcal{M}_n\}.$$

$\mathcal{M} := \mathcal{M}(\mathcal{F}) \subseteq \mathcal{P}(X)$ is the product σ-algebra induced by $\{\mathcal{M}_i\}_{i=1}^n$.

We often write the above as $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i$.

Caution. The above $\prod_{i=1}^{n}$ is NOT the set-theoretic cartesian product.

Now, we get two (possibly different) σ-algebrae on $\mathbb{R}^n$.

① Borel σ-alg. on $(\mathbb{R}^n, \mathcal{T})$

② Product of Borel σ-alg of $\mathcal{B}(\mathbb{R})$.

Prop^n: $\mathcal{B}(\mathbb{R}^n) = \prod_{i=1}^{n} \mathcal{B}(\mathbb{R})$. That is, both the σ-alg above are same.

Proof. We will prove this by a sequence of observations.

① Suppose $\{(X_i, \mathcal{M}_i)\}_{i=1}^n$ are σ-algebrae and $\mathcal{F}_i \subseteq \mathcal{M}_i$ are such that $\mathcal{M}_i = \mathcal{M}(\mathcal{F}_i)$. $(i=1,\dots,n)$

Then, if $X = \prod_{i=1}^{n} X_i$ and $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i$, then

$\mathcal{M}$ is generated by $\{\pi_i^{-1}(E) : E \in \mathcal{F}_i, i=1,\dots,n\}$.

② $\mathcal{M}$ is generated by $\{E_1 \times \cdots \times E_n : E_i \in \mathcal{F}_i\}$ if we further assume $X_i \in \mathcal{F}_i$.

Assuming ① and ② for now, we now note the following.

Clearly, one has $\prod_{i=1}^{n} \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^n)$.

[Proof: using ②, $\prod_{i=1}^{n} \mathcal{B}(\mathbb{R})$ is gen. by sets of the form $U_1 \times \cdots \times U_n$, each $U_i \subseteq \mathbb{R}$ open.
Each such set is open in the metric space $\mathbb{R}^n$.
Thus, it is in $\mathcal{B}(\mathbb{R}^n)$.]

We show $\mathcal{B}(\mathbb{R}^n) \subseteq \prod \mathcal{B}(\mathbb{R})$.

(*) It suffices to show that every set of the form

$U_1 \times \cdots \times U_n$ where $U_i \subset \mathbb{R}$ are open

are in the product $\prod \mathcal{B}(\mathbb{R})$.

[Why? Every open set in $\mathbb{R}^n$ is a countable union of sets of aforementioned form. In turn, the open sets generate $\mathcal{B}(\mathbb{R}^n)$.]

Proving (*) is easy because

$$U_1 \times \cdots \times U_n = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \cdots \cap \pi_n^{-1}(U_n).$$

these are in $\prod \mathcal{B}(\mathbb{R})$, by def^n

Proof of ①

Want to show that $\tilde{\mathcal{F}} = \{\pi_i^{-1}(E) : E \in \mathcal{F}_i,\ 1 \le i \le n\}$ gen. $\prod \mathcal{M}_i$.
$\mathcal{M}$

Clearly $\mathcal{M}(\tilde{\mathcal{F}}) \subseteq \mathcal{M}$. ($\tilde{\mathcal{F}} \subset \mathcal{M}$ and $\mathcal{M}$ is a σ-alg)

It now suffices to show that every (standard) generator of $\mathcal{M}$ is in $\mathcal{M}(\tilde{\mathcal{F}})$.

Note $\mathcal{M} = \langle \pi_i^{-1}(E) : E \in \mathcal{M}_i,\ 1 \le i \le n \rangle$

$\tilde{\mathcal{M}} := \langle \pi_i^{-1}(E) : E \in \mathcal{F}_i,\ 1 \le i \le n \rangle = \mathcal{M}(\tilde{\mathcal{F}})$

Let $\tilde{\mathcal{M}}_i := \{E \in \mathcal{M}_i : \pi_i^{-1}(E) \in \tilde{\mathcal{M}}\} \subseteq \mathcal{P}(X_i)$.

We shall show that $\tilde{\mathcal{M}}_i = \mathcal{M}_i$.

We know, by def^n that $\mathcal{F}_i \subseteq \tilde{\mathcal{M}}_i$. $\left(E \in \mathcal{F}_i \overset{\subseteq \mathcal{M}_i}{\Rightarrow} \pi_i^{-1}(E) \in \tilde{\mathcal{M}} \cap \mathcal{M}_i \Downarrow \in \tilde{\mathcal{M}}_i.\right)$

Moreover, $\mathcal{M}(\mathcal{F}_i) = \mathcal{M}_i$. Thus, it suffices to show that $\tilde{\mathcal{M}}_i$ is a σ-alg.

Also, $\tilde{\mathcal{M}}_i \subseteq \mathcal{M}_i$.

To that end, let $A \in \tilde{\mathcal{M}}_i$. Then, $\pi_i^{-1}(A) \in \tilde{\mathcal{M}}$.

Then, $\pi_i^{-1}(A)^c \in \tilde{\mathcal{M}}$. But $\pi_i^{-1}(A^c) = \pi_i^{-1}(A)^c \in \mathcal{M}$.

$\Rightarrow \pi_i^{-1}(A^c) \in \mathcal{M}_i$

Similarly, noting that $\pi_i^{-1}\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigcup_{j=1}^{\infty} \pi_i^{-1}(A_j)$ yields the result.

Proof of ②.

Let $\mathcal{F}' := \{E_1 \times \cdots \times E_n : E_i \in \mathcal{F}_i\}$.

Note that $E_1 \times \cdots \times E_n = \pi_1^{-1}(E_1) \cap \cdots \cap \pi_n^{-1}(E_n) \in \mathcal{M}$.

Thus, $\mathcal{M}(\mathcal{F}') \subseteq \mathcal{M}$.

OTOH, $\pi_1^{-1}(E_1) = E_1 \times X_2 \times \cdots \times X_n \in \mathcal{F}'$

or in general,

$$\pi_i^{-1}(E_i) = \prod_{j=1}^{n} A_j \in \mathcal{F}' \quad \text{where} \quad A_j = \begin{cases} E_i ; & j = i \\ X_j ; & j \neq i \end{cases}$$

$$\Rightarrow \mathcal{M} \subseteq \mathcal{M}(\mathcal{F}')$$

REMARKS.

① The argument above generalises for a separable metric spaces.

② If $(X_i, \mathcal{M}_i)_{i\in\Lambda}$, and Λ is COUNTABLE then again, $X = \prod X_i$, $\mathcal{M} = \prod \mathcal{M}_i$ generated by $\{\pi_i^{-1}(E) : E \in \mathcal{M}_i, i \in A\}$ is also generated by sets of the form

$$\left(\prod_{i\in\Lambda} E_i\right), \quad E_i \in \mathcal{F}_i.$$

MEASURE

(Measure)

Def^n. Suppose $(X, \mathcal{M})$ is a measure space, i.e., $\mathcal{M}$ is a σ-algebra on X. A measure on X is a map $\mu : \mathcal{M} \longrightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) if $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, then

$$\mu\Big(\underbrace{\bigcup_{i=1}^{\infty} E_i}_{\in \mathcal{M}}\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

By abuse, we shall interchangeably call $(X, \mathcal{M})$ or $(X, \mathcal{M}, \mu)$ a measure space.

EXAMPLES.

(1) $X = \{x_1, x_2, \ldots\}$ is countable. Suppose $p_i \geq 0$ are reals s.t. $\sum_{i=1}^{\infty} p_i = 1$. Let $\mathcal{M} = \mathcal{P}(X)$ and define $\mu : \mathcal{M} \longrightarrow [0,1]$ as

$$\mu(E) = \sum_{i:\, x_i \in E} p_i.$$

(2) $(X, \mathcal{M})$ be s.t. $\mathcal{M}$ is the countable-cocountable σ-alg. s.t. X itself is uncountable

Define

$$\mu(E) := \begin{cases} 0 & ; E \text{ is countable} \\ 1 & \ldots \end{cases}$$

, E is uncountable

Prop^n. Suppose $(X, \mathcal{M}, \mu)$ is a measure space.
Then,

① $E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$

② $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$ (μ is "sub-additive")

(sub-additive, subadditive, sub additive)

③ If $E_i \uparrow$ (i.e., $E_1 \subset E_2 \subset \cdots$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mu(E_i).$$

Proof. ① & ② are trivial

③ Define $F_i = E_i \setminus E_{i-1}$ for $i \geq 2$.
$F_1 = F_1$

Then, $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. Also, $F_i \in \mathcal{M}$ for each i.
($n = \infty$ as well) Moreover, $F_i \cap F_j = \phi$ for $i \neq j$.

Thus, $\mu\left(\bigcup E_i\right) = \mu\left(\bigcup F_i\right) = \sum_{i=1}^{\infty} \mu(F_i)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{\infty} E_i.$

Def^n. ① A null set in a measure space $(X, \mathcal{M}, \mu)$ is a set E s.t. $E \subseteq F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$.
($E \in \mathcal{M}$ NOT necessary.)

(Null set)

② Given a measure space $(X, \mathcal{M}, \mu)$, the completion of $\mathcal{M}$, denoted $\bar{\mathcal{M}}$ is the collection of all sets of the form $F \cup N$ where $F \in \mathcal{M}$ and N is a null set.

(Completion)

Prop^n. ① If $(X, \mathcal{M}, \mu)$ is a measure space, then $\overline{\mathcal{M}}$ is a σ-alg.

② Moreover, there exists a unique measure

$$\bar{\mu} : \overline{\mathcal{M}} \longrightarrow [0, \infty] \quad \text{s.t.}$$

$$\bar{\mu}\big|_{\mathcal{M}} = \mu .$$

(That is, there is a unique extension of μ to a measure $\bar{\mu}$ on $\overline{\mathcal{M}}$.)

Lecture 2

10 January 2021 14:33

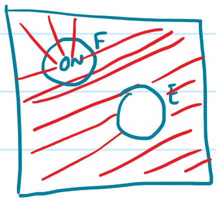
Prop. ① If $(X, \mathcal{M}, \mu)$ is a measure space, then $\bar{\mathcal{M}}$ is a σ-alg.
② Moreover, there exists a unique measure

$$\bar{\mu} : \bar{\mathcal{M}} \longrightarrow [0, \infty] \quad \text{s.t.}$$
$$\bar{\mu}\big|_{\mathcal{M}} = \mu .$$

Proof. ① To show that $\bar{\mathcal{M}}$ is a σ-algebra, we need to show:

(i) $A \in \bar{\mathcal{M}} \Rightarrow A^c \in \bar{\mathcal{M}}$

(ii) $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bar{\mathcal{M}}$.



(i) $A = E \cup N, \quad N \subset F, \quad \mu(F) = 0, \quad E, F \in \mathcal{M}$

$A^c = E^c \cap N^c$

Now, $E \cup F \in \mathcal{M}$ and hence, $E^c \cap F^c$.

Note that $E^c \cap N^c = \underbrace{(E^c \cap F^c)}_{\in \mathcal{M}} \cup \underbrace{(F \setminus N)}_{\subseteq F \text{ and } \mu(F) = 0} \in \bar{\mathcal{M}}$.

Now, if $\{A_i\}_{i=1}^{\infty} \subseteq \bar{\mathcal{M}}$, we can write

$$A_i = E_i \cup N_i , \quad N_i \subseteq F_i , \quad \mu(F_i) = 0, \quad E_i, F_i \in \mathcal{M}.$$

$$\bigcup A_i = \underbrace{\left(\bigcup E_i\right)}_{=:E} \cup \underbrace{\left(\bigcup N_i\right)}_{=:N}$$

Note $E \in \mathcal{M}$. Also, put $F = \bigcup F_i$. Then, $F \in \mathcal{M}$.
Moreover, $N \subseteq F$ & $\mu(F) = \leq \sum \mu(F_i) = 0$.

$\therefore \bigcup A_i = E \cup N$, in the desired form.

(ii) Define $\bar{\mu} : \mathcal{M} \to [0, \infty]$ as

$$\bar{\mu}(E \cup N) := \mu(E).$$

· To show: $\bar{\mu}$ is well-defined.

Suppose $E_1 \cup N_1 = E_2 \cup N_2$.

$\Rightarrow E_1 \subset E_2 \cup N_2 \subseteq E_2 \cup F_2$

$\Rightarrow \mu(E_1) \leq \mu(E_2) + \mu(F_2) = \mu(E_2)$.

$\|^{ly}$ $\mu(E_2) \leq \mu(E_1)$. $\therefore \mu(E_1) = \mu(E_2)$.

Thus, $\bar{\mu}$ is well-defined

(Note that this was also the only way to define $\bar{\mu}$. Thus, uniqueness will follow.)

· To show: $\bar{\mu}$ is a measure.

① $\bar{\mu}(\phi) = 0$ is trivial since $\bar{\mu}(\phi) = \bar{\mu}(\phi \cup \phi) = \mu(\phi) = 0$.

② Suppose $\{E_i \cup N_i\}_{i=1}^{\infty}$ are pairwise disjoint.

$$\bar{\mu}\left(\bigcup (E_i \cup N_i)\right) = \bar{\mu}\left(\left(\bigcup E_i\right) \cup \left(\bigcup N_i\right)\right) = \mu\left(\bigcup E_i\right) = \sum \mu(E_i)$$

$$= \sum \bar{\mu}(E_i \cup N_i) \blacksquare$$

same logic earlier

OUTER MEASURE

Def^n An outer measure on X is a map (Outer measure)

$$\mu^* : \mathcal{P}(X) \to [0, \infty] \text{ satisfying}$$

(i) $\mu^*(\phi) = 0$,

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$,

(iii) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Motivation for μ^* comes from the intuitive idea that knowing areas enclosed by rectangles, we "approximate" areas bounded by arbitrary sets by covering these by <u>countable union</u> of rectangles.

Prop^n. Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{F} \to [0, \infty]$ s.t.

(i) $\phi, X \in \mathcal{F}$

(ii) $\rho(\phi) = 0$.

For $E \in \mathcal{P}(X)$, define

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^{\infty} \rho(E_i) \;:\; E_i \in \mathcal{F},\ E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

Then, μ^* is an outer measure.

Proof. We need to show μ^* is well-defined ← This follows because $X \in \mathcal{F}$. and it satisfies

(i) $\mu^*(\phi) = 0$ ⟵ trivial since $\mu^*(\phi) \geq 0$ since inf over non-ve reals

Also, $\phi \subseteq \phi$ & $\rho(\phi) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

Any cover of B is also a cover of A.
$\therefore$ We are taking inf over a larger set & hence, it will be smaller.
$\Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) Suppose $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{P}(X)$. Fix $\varepsilon > 0$.

For each E_i, let $\{A_j^{(i)}\}_{j=1}^{\infty} \subseteq \mathcal{F}$ be a cover of E_i

$\therefore \mu^*(E_i) \geq \sum^{\infty} \rho(A^{(i)}) \ldots$

with $\mu^*(E_i) \geq \sum_{j=1}^{\infty} \rho(A_j^{(i)}) - \frac{\varepsilon}{2^i} \quad \forall i$

Then, $\bigcup_i \bigcup_j A_j^{(i)}$ covers $\bigcup E_i$.

Thus, $\mu^*(\bigcup E_i) \leq \sum_{i,j} A_j^{(i)} \leq \left(\sum_{i=1}^{\infty} \mu^*(E_i) + \frac{\varepsilon}{2^i} \right)$

$$= \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$$

$$\Rightarrow \mu^*\left(\bigcup E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$$

Since $\varepsilon > 0$ is arbit, this is completes the proof. ☒

Def^n. Given an outer measure μ^*, we say that a set $A \subseteq X$ is μ^*-measurable if for all $E \in \mathcal{P}(X)$,

(\mu^*-measurable)

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Def^n. A measure μ on $(X, \mathcal{M})$ is complete if for all $F \in \mathcal{M}$ with $\mu(F) = \phi$, we have $\mathcal{P}(F) \subseteq \mathcal{M}$.

(That is, all null sets are in $\mathcal{M}$.)

(Complete measure)

Thm. (CARATHÉODORY) (Carathéodory, Caratheodory)

Let μ^* be an outer measure on X.
Let $\mathcal{M} := \{E \subseteq X : E$ is μ^*-measurable$\}$.
Then,

(i) $\mathcal{M}$ is a σ-algebra.
(ii) μ^* restricted to $\mathcal{M}$ is a complete measure.

Proof. (i) $\mathcal{M}$ is closed under $(\ )^c$ since the def^n of μ^*-meas. is symmetric under $(\ )^c$.

Now, if $A, B \in \mathcal{M}$ and $E \subset X$,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\
&\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^c \cap B^c) \\
&= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)
\end{aligned}$$

Of course, we also have $\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \leq \mu^*(E)$.

$\therefore \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$.

Thus, $\mathcal{M}$ is closed under finite unions. To show that it is a σ-alg, it suffices to show closure under disjoint unions.

(Why? Given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, consider $F_1 = E_1$ and $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right)$ for $n \geq 2$.

Note that $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and that each $F_n \in \mathcal{M}$.)

Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint collection in $\mathcal{M}$. Put $B_n = \bigcup_{i=1}^{n} A_i \in \mathcal{M}$ and $B = \bigcup_{i=1}^{\infty} A_i$.

$$\left(\begin{aligned} B_n \cap A_n &= \bigcup_{i=1}^{n} (A_n \cap A_i) = A_n \\ B_n \cap A_n^c &= \bigcup_{i=1}^{n} (A_n^c \cap A_i) = \bigcup_{i=1}^{n-1} A_i = B_{n-1} \end{aligned} \right)$$

Then, for any $E \subset X$,

$$\begin{aligned}
\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\
&= \mu^*(E \cap A_n) + \underbrace{\mu^*(E \cap B_{n-1})}_{= \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2})} \\
&= \sum_{i=1}^{n} \mu^*(E \cap A_i)
\end{aligned}$$

$$\Rightarrow \mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \left[\sum_{i=1}^{n} \mu^*(E \cap A_i) \right] + \mu^*(E \cap B_n^c)$$

$(B_n \subseteq B \Rightarrow B^c \subseteq B_n^c)$

$$\geq \left[\sum_{i=1}^{n} \mu^*(E \cap A_i) \right] + \mu^*(E \cap B^c)$$

take $n \to \infty$

(take $n \to \infty$)

$$\Rightarrow \mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)$$

(*)
$$\geq \mu^*\left(\bigcup (E \cap A_i)\right) + \mu^*(E \cap B^c)$$

$$= \mu^*\left(E \cap \left(\bigcup A_i\right)\right) + \mu^*(E \cap B^c)$$

$$= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).$$

Thus, we have equality throughout giving

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$$ & hence, $B \in \mathcal{M}$.

Moreover, taking $E = B$; the (*) equation gives

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) = \sum_{i=1} \mu^*(A_i).$$

Thus, μ^* is countably additive on $\mathcal{M}$.

Thus, μ is a measure on $\mathcal{M}$.

To show completeness: Let $F \in \mathcal{M}$ be s.t. $\mu^*(F) = 0$ and $A \subseteq F$.

Then, $\mu^*(A) = 0$. Also, for any $E \subset X$, we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(A) + \mu(E \cap A^c)$$
$$= \mu^*(E \cap A^c) \leq \mu^*(E).$$

Thus, $A \in \mathcal{M}$. ∎

Defn. Suppose $\mathcal{F}$ is an algebra on X. (Pre-measure)
A map

$$\mu_0 : \mathcal{F} \to [0, \infty]$$

is called a pre-measure if

(i) $\mu_0(\emptyset) = 0$,
(ii) If $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint s.t. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$,

then
$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

(Note that the above also gives the thing for finite unions. guaranteed to be in $\mathcal{F}$.)

Prop^n. Suppose μ_0 is a premeasure on an algebra $\mathcal{F}$. Then, if μ^* is the outer measure as defined in the earlier proposition, then

(i) $\mu^*|_{\mathcal{F}} = \mu_0$
(ii) Every set in $\mathcal{F}$ is μ^*-measurable.

An immediate corollary:

Thm. Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$ is an algebra and suppose $\mathcal{M}$ is the σ-algebra generated by $\mathcal{F}$.
Let μ_0 be a premeasure defined on $\mathcal{F}$ and let μ^* be the outer measure as before. Then,

(i) $\mu^*|_{\mathcal{M}}$ is a measure on $(X, \mathcal{M})$. Put $\mu = \mu^*|_{\mathcal{M}}$.
(ii) If ν is any measure extending μ_0, then
$$\nu(E) = \mu(E)$$
whenever $\mu(E) < \infty$

Proof of Prop^n.

(i) For $E \in \mathcal{F}$, want to show, $\mu^*(E) = \mu_0(E)$.
Considering $E_1 = E$ and $E_i = \phi$ for $i \geq 2$ gives

$$\sum_{i=1}^{\infty} \mu_0(E_i) = \mu_0(E) \Rightarrow \mu^*(E) \leq \mu_0(E).$$

inf over all covers

To show $\geq$: let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ be a cover for E.

Let $F_i := E \cap \left(E_i \setminus \bigcup_{j=1}^{i-1} E_j \right)$.

Clearly (i) $F_i \in \mathcal{F}$ by closure properties
(ii) $F_i \cap F_j = \phi$ if $i \neq j$.
(iii) $\bigcup_{i=1}^{\infty} F_i = E$.

If $\{E_i\}$ is a cover s.t. $\sum_{i=1}^{\infty} \mu_0(E_i) \leq \mu^*(E) + \varepsilon$

(such a cover does exist)

Note $\mu_0(E) = \mu_0\left(\bigcup_{i=1}^{\infty} F_i \right) = \sum^{\infty} \mu_0(F_i) \leq \mu^*(E) + \varepsilon$

μ_0 is a premeasure

$$\Rightarrow \mu_0(E) \leq \mu^*(E) + \varepsilon \quad \forall \varepsilon > 0$$
$$\Rightarrow \mu_0(E) \leq \mu^*(E).$$

This establishes (i).

(ii) To show: Every $A \in \mathcal{F}$ is μ^*-measurable.
Let $E \subset X$. Let $\{E_n\}$ be a cover for E s.t.

$$\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon$$

Then, $\mu^*(E) + \varepsilon \geq \sum_n \mu_0(E_n)$ $E_n, A, A^c \in \mathcal{F}$ and μ_0 is pre-meas.

$$= \sum [\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c)]$$

$$= \sum_n \left[\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c)\right]$$

$$= \left[\sum_n \underbrace{\mu_0(E_n \cap A)}_{\text{cover for } E\cap A}\right] + \left[\sum_n \underbrace{\mu_0(E_n \cap A^c)}_{E\cap A^c}\right]$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

$$\Rightarrow \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad (E \text{ was arbit.})$$

$\leq$ is anyway true for any outer measure.

$$\Rightarrow \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Thus, A is μ^*-measurable.

Proof of Thm.

(i) Follows from Carathéodory and Prop^n. (Since $\mathcal{F}$ is μ^*-measurable and $\mathcal{M}$ is the smallest σ-alg containing $\mathcal{F}$.)

(ii) First note that if ν extends μ_0, then

for any $E \in \mathcal{M} = \mathcal{M}(\mathcal{F})$

if $E \subseteq \bigcup_n E_n$, $E_n \in \mathcal{F}$, then

$$\nu(E) \leq \sum_n \nu(E_n) = \sum_n \mu_0(E_n).$$

So if the cover is s.t. $\sum_n \mu_0(E_n) \leq \mu^*(E) + \varepsilon = \mu(E) + \varepsilon$,

this gives $\nu(E) \leq \mu(E) + \varepsilon$ for arbit ε. That is,

$$\nu(E) \leq \mu(E).$$

Now, if $\mu(E) < \infty$, we show $\geq$.

Let $\{E_i\}^{cf}$ be a cover for $E \in \mathcal{M}$ and let $A := \bigcup_n E_n \in \mathcal{M}$.

Note that $\nu(A) = \lim_{n\to\infty}\left(\nu\left(\bigcup_{i=1}^n E_i\right)\right) = \lim_{n\to\infty} \mu_0\left(\bigcup_{i=1}^n E_i\right)$

$$= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^n E_i\right)$$

$$= \mu(A)$$

Since $\mu(E) < \infty$ (by assumption), we can pick a cover $\{E_n\}$ s.t.

$$\underset{\mu(A\cap E) + \mu(A\setminus E)}{\overset{\shortparallel}{\mu(A)}} < \mu(E) + \varepsilon \text{ and thus, } \mu(A\setminus E) < \varepsilon.$$

Thus, $\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A\setminus E) \leq \nu(E) + \mu(A\setminus E)$

$$< \nu(E) + \varepsilon$$

This gives $\mu(E) \leq \nu(E)$, as desired.

Towards "Good" Borel Measures

The idea is to extend/define a measure μ on $\mathcal{B}(\mathbb{R})$ from the notion of length of bounded intervals. Whatever done so far leads to that.

(Half-interval, half interval)

Def^n A half-interval is a subset of $\mathbb{R}$ of the form:

(i) $(a, b]$ for $-\infty \leq a < b < \infty$, or

(ii) (a, ∞) for $-\infty \leq a < \infty$, or

(iii) ϕ.

Can be checked that the collection of fin unions of half-intervals is an algebra on $\mathbb{R}$

Prop^n. Let $\mathcal{F}$ be the algebra consisting of finite unions of half-intervals.
Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function.

$$\left(\lim_{\delta \downarrow 0} F(x+\delta) = F(x) \quad \forall x \in \mathbb{R} \right)$$

Define
$$\mu_0\left(\bigsqcup_{i=1}^{n} (a_j, b_j] \right) := \sum_{j=1}^{n} \left(F(b_j) - F(a_j) \right),$$

and $\mu_0(\phi) := 0$.
Then, μ_0 is a pre measure on $\mathcal{F}$.

Remarks (1) Note that $\mathcal{F}$ above actually generates $\mathcal{B}(\mathbb{R})$ as seen in Lec 1.

(2) If we take $F(x) = x$, then $F(b) - F(a) = b - a$
$=$ "length" of $(a, b]$.

So, the above extends the notion of measure arising from lengths of intervals onto the Borel σ-field.

(3) Why right-continuity?

Suppose μ is a finite Borel measure. Let
$$F(x) := \mu((-\infty, x]).$$
Then, if $x_n \downarrow x$, then
$$\lim_{n\to\infty} \mu((-\infty, x_n]) = \mu\left(\bigcap_{n=1}^{\infty} (-\infty, x_n] \right)$$
$$\lim_{n\to\infty} F(x_n) = \mu((-\infty, x]) = F(x)$$

Thus, this F above is right-continuous.
Note that closure on right

Proof of the Prop^n. First, we need to check that μ_0 is well-defined.

Let $\{(a_j, b_j]\}$ $(j = 1, \dots, n)$ be pairwise disjoint and let

$$\dot{\bigcup_{j=1}^{n}} (a_j, b_j] = (a, b].$$

Then, by re-arranging indices, if necessary, it follows that

$a = a_1 < b_1 = a_2 < b_2 = \cdots = a_n < b_n = b$ and in this case

$$\mu_0((a, b]) = F(b) - F(a)$$

$$\mu_0\left(\dot{\bigcup_{j=1}^{n}} (a_j, b_j]\right) = \sum F(b_j) - F(a_j) = F(b) - F(a)$$

Thus, μ_0 is well-defined in this case.

More generally, if $\{I_i\}_{i=1}^{n}$ and $\{J_j\}_{j=1}^{m}$ are s.t.

$$\dot{\bigcup_{i=1}^{n}} I_i = \dot{\bigcup_{j=1}^{m}} J_m, \quad \text{then}$$

$$\sum_i \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_j \mu_0(J_j)$$

since intersection of half-intervals is again a half-interval, for which the def^n is consistent.

This shows that μ_0 is well-defined on $\mathcal{F}$.

It now remains to show that μ_0 is indeed a premeasure on $\mathcal{F}$.

Lecture 3

13 January 2021 21:31

To show μ_0 is a pre-measure, we need to prove:

If $A_i \in \mathcal{F}$ are pairwise disjoint s.t. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, then

$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

Note the μ_0 being finitely additive is clear.
Suppose $\{I_j\}_{j=1}^{\infty}$ is a collection of p-wise disjoint half intervals and

$$\bigsqcup_j I_j = (a, b] = I, \text{ say}.$$

Case i: $-\infty < a < b < \infty$.

$$\mu\left(\bigsqcup_{j=1}^{\infty} I_j\right) = \mu_0\Big(\underbrace{\bigsqcup_{j=1}^{n} I_j}_{\in \mathcal{F}}\Big) + \mu_0\Big(\underbrace{I \setminus \bigcup_{j=1}^{n} I_j}_{\in \mathcal{F}}\Big)$$

$$\geq \mu_0\left(\bigcup_{j=1}^{n} I_j\right)$$

$$= \sum_{j=1}^{n} \mu_0(I_j)$$

and this holds $\forall n \in \mathbb{N}$. Thus, $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$.

For the other side, let $\varepsilon > 0$ be arbitrary.

We know (by def^n of μ_0) that $\mu_0(I) = F(b) - F(a)$.
(Write $I_j = (a_j, b_j]$.)
By right continuity, let $\delta > 0$ be s.t. $F(a+\delta) - F(a) < \delta$.
Similarly, let $\delta_j > 0$ be s.t.

$$F(b_j+\delta_j) - F(b_j) < \frac{\varepsilon}{2^j} \quad \forall j.$$

Note that $\{(a_j, b_j+\delta_j)\}_{j=1}^{\infty}$ covers $[a+\delta, b]$.
Since $[a+\delta, b]$ is compact, there is a finite subcover.
By removing those intervals that are contained in larger intervals, we assume WLOG that

(i) $(a_1, b_1+\delta_1), \dots, (a_N, b_N+\delta_N)$ cover $[a+\delta, b]$.

(ii) For each j, $b_j+\delta_j \in (a_{j+1}, b_{j+1}+\delta_{j+1})$ for $j=1,\dots,N-1$.

$$\mu_0(I) = F(b) - F(a) = F(b) - F(a+\delta) + \underbrace{F(a+\delta) - F(a)}_{<\varepsilon, \text{ by choice}}$$

$$F(b) - F(a+\delta) \le F(b_N+\delta_N) - F(a_1) \qquad (\text{since } F\uparrow)$$

$$= F(b_N+\delta_N) - F(a_N) + \sum_{j=1}^{N-1}[F(a_{j+1}) - F(a_j)]$$

$a_{j+1} < b_j+\delta_j$ & $F\uparrow$

$$\le F(b_N+\delta_N) - F(a_N) + \sum_{j=1}^{N-1}[F(b_j+\delta_j) - F(a_j)]$$

$$\le \underbrace{F(b_N) - F(a_N)}_{\mu_0(I_N)} + \sum_{j=1}^{N-1}\underbrace{[F(b_j) - F(a_j)]}_{\mu_0(I_j)} + \varepsilon$$

So, $\mu_0(I) \le \sum_{j=1}^{N}\mu_0(I_j) + 2\varepsilon \le \sum_{j=1}^{\infty}\mu_0(I_j) + 2\varepsilon.$

Since $\varepsilon>0$ was arbitrary, $\mu_0(I) \le \sum_{j=1}^{\infty}\mu_0(I_j)$.

Thus, $$\mu_0(I) = \sum_{j=1}^{\infty}\mu_0(I_j).$$

Case ii. $a=-\infty$, i.e., $I = (-\infty, b]$
$b<\infty$

In this case, $(a_j, b_j+\delta_j)$ cover $[-M, b]$ for M large
By the previous part, we have

By the previous part, we have

$$F(b) - F(-M) \leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\varepsilon.$$

As $M \to \infty$, the LHS $\to \mu_0(I)$.
(Note that the other inequality did not need $a > -\infty$.)

Case iii. If $b = \infty$, we use a similar argument with
$F(M) - F(a) \leq \sum_1^{\infty} \mu_0(I_j) + 2\varepsilon$. ∎

To summarise, we have:

Thm. Suppose $G, F : \mathbb{R} \to \mathbb{R}$ is $\uparrow$ and right continuous. Then there is a unique Borel measure $\mu = \mu_F$ s.t.
$\mu(a,b] = F(b) - F(a)$.
Furthermore
(i) $\mu_F = \mu_G \iff F - G =$ constant.
(ii) Conversely, if μ is a Borel measure s.t.
$\mu(a,b] < \infty$ whenever $|a|, |b| < \infty$ and we define

$$F(x) = \begin{cases} \mu(0, x] & ; \quad x > 0 \\ 0 & ; \quad x = 0 \\ -\mu(-x, 0] & ; \quad x < 0, \end{cases}$$

then F is $\uparrow$, right cts and $\mu = \mu_F$.

Proof. The "immediate corollary" theorem from last lecture shows that extension of pre measure is unique on sets with finite measures. Actually true in more generality. First a def^n.

Def^n. A measure μ on $(X, \mathcal{M})$ is called σ-finite if there exist sets $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ s.t.

there exist sets $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ s.t.

$$\mu(E_j) < \infty \ \forall j \quad \text{and} \quad X = \bigcup_{j=1}^{\infty} E_j.$$

(\sigma-finite)

The last theorem (extension of outer measure measure from a pre-measure on $\mathcal{F}$ to a measure on $\mathcal{M}(\mathcal{F})$) has an additional statement:

Thm. If ν is a measure extending μ_0 and suppose μ (coming from outer measure) is σ-finite, then $\nu = \mu$.

Proof. Let $\{A_i\}_{i=1}^{\infty}$ → p-wise disj. s.t. $\bigcup_{i=1}^{\infty} A_i = X$ & $\mu(A_i) < \infty$

(Can always disjointify the E_i in def^n of σ-finite.)

For any $E \in \mathcal{M}$

$$\mu(E) = \mu\left(\bigcup_{i=1}^{\infty}(E \cap A_i)\right) = \sum \mu(E \cap A_i)$$

$$= \sum \nu(E \cap A_i)$$

by earlier thm & $\mu(E \cap A_i) < \infty$

$$= \nu\left(\bigcup_{i=1}^{\infty} E \cap A_i\right)$$

$$= \nu(E).$$

☒

Remark. The measure μ_F is called the Lebesgue - Stieltjes measure associated with F.

(Lebesgue-Stieltjes measure)

Prop^n. Let F be ↑ and right-continuous. Let μ be the corresponding measure, $\mathcal{M} = \mathcal{M}_F = \mathcal{M}_{\mu_F}$ (is complete).

For any $E \in \mathcal{M}$,

$$\mu(E) = \inf\left\{\sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\right\}.$$

Proof. Let us denote

$$\inf\left\{\sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\right\} =: \nu(E).$$

Want to show $\mu(E) = \nu(E) \quad \forall E \in \mathcal{M}$.

First, note that $(a_j, b_j) = \bigcup_{i=1}^{\infty} I_j^{(i)}$ for some half-intervals $I_j^{(i)}$ and in particular

$$E \subset \bigcup_{i,j} I_j^{(i)} \quad \text{and hence,} \quad \mu(E) \leq \sum_{i,j} \mu(I_j^{(i)})$$
$$= \sum_i \mu(a_i, b_i)$$

$$\Rightarrow \quad \mu(E) \leq \nu(E).$$

For the other inequality, let $\varepsilon > 0$ be arbit.

Let $\{(a_j, b_j]\}_{j=1}^{\infty}$ be s.t.

$$\sum_1^{\infty} \mu(a_j, b_j] \leq \mu(E) + \varepsilon. \qquad (*)$$

As before, let $\delta_j > 0$ be s.t.

$$F(b_j + \delta_j) - F(b_j) < \varepsilon/2^j$$

$$\sum_1^{\infty} \mu(a_j, b_j + \delta_j) \leq \sum_1^{\infty} \mu(a_j, b_j] + \sum_1^{\infty} \mu(b_j, b_j + \delta_j]$$
$$= \sum_1^{\infty} \mu(a_j, b_j] + \sum_1^{\infty} [F(b_j + \delta_j) - F(b_j)]$$
$$< \sum_1^{\infty} \mu(a_j, b_j] + \varepsilon \qquad \text{use } (*)$$
$$\leq \mu(E) + 2\varepsilon$$

$$\Rightarrow \quad \nu(E) \leq \mu(E) + 2\varepsilon \qquad \forall \varepsilon > 0.$$

$$\Rightarrow \nu(E) \leq \mu(E)$$
$$\Rightarrow \nu(E) = \mu(E).$$

Thm. Suppose $F: \mathbb{R} \to \mathbb{R}$ is $\uparrow$ and right continuous. Let $\mu = \mu_F$. Then,

$$\mu(E) = \inf\{\mu(U) : E \subset U, U \overset{\subseteq\mathbb{R}}{} \text{ is open}\}$$
$$= \sup\{\mu(K) : K \subset E, K \overset{\subseteq\mathbb{R}}{} \text{ is compact}\}$$

Proof. (i) Suppose $U \supseteq E$, then $\mu(U) \geq \mu(E)$ and hence
$$\inf\{\quad\} \geq \mu(E).$$

Now, let $\varepsilon > 0$ be arbitrary.

By the previous proposition,

$$\mu(E) = \inf\left\{\sum_1^\infty \mu(a_i, b_i) : E \subset \bigcup_1^\infty (a_i, b_i)\right\}.$$

Then, $\exists \{(a_i, b_i)\}_{i=1}^\infty$ s.t.

$$\sum_1^\infty \mu(a_i, b_i) < \mu(E) + \varepsilon.$$

Let $U = \bigcup_{i=1}^\infty (a_i, b_i)$. Then, $\mu(U) < \sum_1^\infty \mu(a_i, b_i)$.

Then, $\mu(U) < \mu(E) + \varepsilon$. But U is open.

Thus, $\mu(E) = \inf\{\mu(U) : E \subseteq U \leftarrow \text{open}\}$.

(ii) First assume that E is bounded.

- If E is closed, then E is compact and the result is trivially true.

Consider the closure $\bar{E}$ $\leftarrow$ this is closed and bounded.

By the previous part, $\exists U$ open s.t.

$$U \supseteq \bar{E}\setminus E \quad \text{and} \quad \mu(U) < \mu(\bar{E}\setminus E) + \varepsilon.$$

↳ measurable since $\bar{E} \in \mathcal{B}_{\mathbb{R}}$ and $E \in \mathcal{M}$

Let $K = \bar{E}\setminus U$. Note $K \subseteq E$ and

$$\mu(K) = \mu(E) - \mu(E\cap U) = \mu(E) - \big(\mu(U) - \mu(U\setminus E)\big)$$

$$= \mu(E) - \mu(U) + \mu(U\setminus E)$$

$$\geq \mu(E) - \mu(U) + \mu(\bar{E}\setminus E) > \mu(E) - \varepsilon$$

$\Rightarrow \quad \mu(K) > \mu(E) - \varepsilon.$

$\Rightarrow \quad \mu(E) = \sup\{\mu(K) : E \supseteq K \to \text{compact}\}.$

Now, if E is unbounded, define

$$E_j = E \cap (j, j+1] \qquad \text{for } j \in \mathbb{Z}.$$

By the above argument, for any $\varepsilon > 0$, $\exists$ compact $K_j \subset E_j$ with

$$\mu(K_j) \geq \mu(E_j) - \frac{\varepsilon}{2^{|j|}}.$$

Let $H_n = \bigcup_{j=-n}^{n} K_j$. Then, H_n is compact, $H_n \subset E$ and

$$\mu(H_n) \geq \mu\Big(\bigcup_{j=-n}^{n} E_j\Big) - \varepsilon.$$

Since $\mu(E) = \lim_{n\to\infty} \mu\Big(\bigcup_{j=-n}^{n} E_j\Big)$, the result follows.

Cor. Suppose $F : \mathbb{R} \to \mathbb{R}$ is $\uparrow$ and right continuous. Let $\mu = \mu_F$.
TFAE:

(i) E is measurable,

(ii) $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$,

(iii) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

(iii) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

$F_\sigma \equiv \bigcup_{i=1}^{\infty} F_i$ (F_i closed) ; $G_\delta \equiv \bigcap_{i=1}^{\infty} G_i$ (G_i open)

Proof. (i) $\Rightarrow$ (ii), (iii)

Suppose $\mu(E) < \infty$. Then, $\exists$ open U_j and compact K_j s.t.

$$\mu(U_j) - \frac{1}{2^j} \leq \mu(E) \leq \mu(K_j) + \frac{1}{2^j}$$

with $K_j \subseteq E \subseteq U_j$.

Let $V = \bigcap_j U_j$ and $H = \bigcup_j K_j$. Then,

$$H \subset E \subset V \quad \text{and} \quad \mu(H) = \mu(E) = \mu(V) \quad \text{and}$$

so, $\mu(V \setminus E) = 0 = \mu(E \setminus H)$

If $\mu(E) = \infty$, put $E_j := E \cap (j, j+1]$ for $j \in \mathbb{Z}$.
Note that each E_j has finite measure. Thus,
$E_j = V_j \setminus N_j^{(1)} = H_j \cup N_j^{(2)}$. Then,

$$E = \left(\bigcup H_j\right) \cup \left(N_j^{(2)}\right)$$

still F_δ still null

$$= \left(\bigcup V_j\right) \setminus \left(\bigcup N_j^{(1)}\right)$$

still G_δ

(ii), (iii) $\Rightarrow$ (i) is obvious because $\mathcal{M}_{\mu_F}$ is complete and contains F_σ, G_δ sets. ☒

Def^n. Consider $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = x$.
Then, $m = \mu_F$ is called the Lebesgue measure and

we shall denote μ_F by $\mathcal{L}$, the set of Lebesgue measurable sets.

(Lebesgue measure)

Remark $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L} \subsetneq \mathcal{P}(\mathbb{R})$

$\mathcal{B}(\mathbb{R})$: not complete (why?)

$\mathcal{L}$: complete measure

Thm. For any $x \in \mathbb{R}$ and $E \in \mathcal{L}$, define

$$x + E := \{x + y : y \in E\} \text{ and}$$
$$x \cdot E := \{xy : y \in E\}.$$

Then, $x + E, x \cdot E \in \mathcal{L}$ and

$$m(x + E) = m(E), \text{ and}$$
$$m(x \cdot E) = |x| \cdot m(E).$$

Proof Consider the field $\mathcal{F}$ of disjoint union of half-intervals. Since $\mathcal{F}$ is invariant under translations and dilations, it follows that $\mathcal{B}(\mathbb{R})$ follows the same.

Let ν_1 and ν_2 be measures defined by

$$\nu_1(E) := m(x + E) \quad \text{and} \quad \nu_2(E) := m(x \cdot E).$$

Note that ν_1 and ν_2 are pre-measures on $\mathcal{F}$ and

$$\nu_1(I) = m(I) \quad \text{and} \quad \nu_2(I) = |x| \cdot m(I)$$

for intervals. Thus, by uniqueness of extension of pre-measures, it follows that

$$\nu_1(E) = m(x + E), \quad \nu_2(E) = |x| \cdot m(E) \quad \forall E \in \mathcal{B}_{\mathbb{R}}.$$

Thus, null sets remain null under ν_1 and ν_2.
Thus, the general result follows. ∎

REMARKS. (1) $\mathcal{L} \neq \mathcal{P}(\mathbb{R})$

(2) All countable sets have measure zero.
However, there are sets which are uncountable and have measure zero.

EXAMPLE (CANTOR SET)

Let $C_0 = [0,1]$. $C_1 = [0,1] \setminus (1/3, 2/3)$.

$C_2 = C_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \right), \ \ldots$

Define $$C = \bigcap_{n=0}^{\infty} C_n.$$

Facts:

(i) C is compact.

(ii) C consists of all those $x \in [0,1]$ which have a ternary expansion have no 1s.
In particular, C is uncountable.

(iii) $m(C_0) = 1$ and $m(C_n) = \frac{2}{3} m(C_{n-1}) \quad \forall n \geq 1$

$$= \left(\frac{2}{3}\right)^n$$

Thus, $m(C) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$

(iv) Since $C \in \mathcal{L}$ and $m(C) = 0$, every subset of C is in $\mathcal{L}$.

Thus, $|\mathcal{L}| \geq 2^{|C|} = 2^{\mathfrak{c}}$. $\mathfrak{c} \rightarrow |\mathbb{R}|$

Of course, $\mathcal{L} \subset \mathcal{P}(\mathbb{R}) \Rightarrow |\mathcal{L}| \leq 2^{\mathfrak{c}}$.

Thus, $|\mathcal{L}| = 2^{\mathfrak{c}}$.

On the other hand, $|\mathcal{B}(\mathbb{R})| = \mathfrak{c}$. (why?)

Lecture 4

19 January 2021 15:31

Integration on Measure Spaces

Draw backs of Riemann integrable

① Consider $f : \mathbb{R} \to \mathbb{R}$ defined as

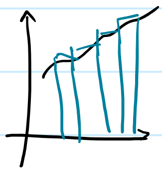
$$f(x) = \begin{cases} 1 & ; \ x \in \mathbb{Q} \\ 0 & ; \ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

It "ought" to be integrable since it is "essentially" 0.
(It is 1 only on a set of measure 0.)

② The theory of Riemann integration does not admit suitable convergence theorems.

If $f_n \to f$ pointwise, then $\int f_n \to \int f$ is not necessary.
(Uniform continuity helps but that's more restrictive)

Recall Riemann integration : We knew area of "rectangles".



Now we know more "areas" (via measure).

Defn Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measure spaces.
A function $f : X \to Y$ is called measurable
or $(\mathcal{M}, \mathcal{N})$-measurable, if

$$f^{-1}(N) \in \mathcal{M} \text{ for all } N \in \mathcal{N}.$$

Remark. Composition of measurable functions is measurable.
That is, given $(X, \mathcal{M})$, $(Y, \mathcal{N})$, and $(Z, \mathcal{O})$ and

$$\begin{tikzcd} X \arrow[r, "f"', "{\substack{(\mathcal{M},\mathcal{N})\\ \text{meas}}}"' {yshift=-12pt}] \arrow[rr, bend right=60] & Y \arrow[r, "g"', "{\substack{(\mathcal{N},\mathcal{O})\\ \text{meas.}}}"' {yshift=-12pt}] & Z \end{tikzcd}$$

$g \circ f$ is $(\mathcal{M}, \mathcal{O})$-measurable

Prop^n 1. Suppose $\mathcal{N}$ is generated by $\mathcal{F}$. Then,
$f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable iff $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{N}$.

Proof. $(\Rightarrow)$ Obvious.
$(\Leftarrow)$ Define $\mathcal{N}' = \{E \in \mathcal{N} : f^{-1}(E) \in \mathcal{M}\}$.
By hypothesis, $\mathcal{F} \subset \mathcal{N}'$.

It is easy to see that $\mathcal{N}'$ is a σ-alg.
Thus, $\mathcal{N} \subset \mathcal{N}'$. Thus, f is measurable. ∎

Cor. If X and Y are topological spaces, then any continuous function $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.

Cor. Suppose $(X, \mathcal{M})$ is a measure space and $f: X \to \mathbb{R}$ is a function. TFAE:
(i) f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}))$ measurable,
(ii) $f^{-1}((a, \infty)) \in \mathcal{M}$,
(iii) $f^{-1}([a, \infty)) \in \mathcal{M}$,
(iv) $f^{-1}((-\infty, a]) \in \mathcal{M}$.

Given a collection $\{(X_i, \mathcal{M}_i)\}_{i=1}^{n}$ of measure spaces and X is an arbitrary set.
Suppose $f_i: X \to X_i$ is a map for each i.

Then, consider the smallest σ-alg. $\mathcal{M}$ on X_i wrt which f_i are measurable. That is,

$$\mathcal{M} \text{ is generated by } \{ f_i^{-1}(E_i) \mid E_i \in \mathcal{M}_i,\ i = 1, \dots, n \}.$$

In particular, if $X = \prod X_i$ and $f_i = \pi_i$, then the above $\mathcal{M}$ is simply the product σ-alg as in the previous lectures.

Def^n. We define Borel sets in $\bar{\mathbb{R}}$ as

$$\mathcal{B}(\bar{\mathbb{R}}) = \{ E \subset \bar{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \}.$$

The above agrees with the Borel field of $\bar{\mathbb{R}}$ as a topological or metric space. ($\rho(x,y) = |\arctan x - \arctan y|$.)
Moreover, it is generated by the rays $(a, \infty]$ or $[-\infty, a)$. $(a \in \mathbb{R})$

$f : X \to \bar{\mathbb{R}}$ is measurable if it is $(\mathcal{M}, \mathcal{B}(\bar{\mathbb{R}}))$ measurable.

Def^n. $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be
- Borel measurable if it is $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ measurable,
- Lebesgue measurable if it is $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$ measurable.

Likewise for $f : \mathbb{R} \to \mathbb{C}$.

Remark If we talk of Lebesgue measurability, we have to be more careful.
If $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable, it is not necessary that $f \circ g$ is Lebesgue measurable.
The problem is in terminology, by $f : \mathbb{R} \longrightarrow \mathbb{R}$ being measurable,

we mean that it is $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$ measurable.
If $f^{-1}(E)$ is null, then $g^{-1}(f^{-1}(E)) \in \mathcal{L}$ is not guaranteed EVEN if g is continuous.

Prop^n 2. Suppose $(X, \mathcal{M})$ is a measure space and suppose $f, g : X \to \mathbb{R}$, $f_i : X \to \overline{\mathbb{R}}$ are measurable. $(i = 1, 2, \ldots)$
Then,

① $f+g$, fg are measurable (wrt Borel)

② g_1 defined as $g_1(x) := \sup_i f_i(x)$,
g_2 $\qquad g_2(x) := \inf_i f_i(x)$
g_3 $\qquad g_3(x) := \limsup_i f_i(x)$
g_4 $\qquad g_4(x) := \liminf_i f_i(x)$

are all measurable.

Proof. Define $F : X \to \mathbb{R} \times \mathbb{R}$, $\quad F(x) := (f(x), g(x))$
Recall $\pm : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\quad \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Then, $\quad f \pm g = \pm \circ F$.

measurable, by previous obs. of product spaces

cts

hence, measurable

Thus, $f \pm g$ and fg are measurable.

For g_1: $\quad g_1(x) = \sup_j f_j(x)$

$$\{x : g_1(x) > a\} = g_1^{-1}(a, \infty]$$

$$= \bigcup_{j=1}^{\infty} f_j^{-1}(a, \infty] \in \mathcal{M}.$$

By the previous prop^n, we are done since $(a, \infty]$ generated $\mathcal{B}(\overline{\mathbb{R}})$.

∞

g_2 : $g_2^{-1}([-\infty, a)) = \bigcup_{j=1} f_j^{-1}([-\infty, a))$.

More generally, $h_k(x) = \sup_{j>k} f_j(x)$ is measurable

Thus, $g_3 = \inf_k h_k$ is measurable. Similarly for g_4.

Cor. 3. If $\lim_j f_j(x)$ exists $\forall x$, then $f = \lim_j f_j$ is measurable.

Proof. $f = g_3 = g_4$.

Cor. 4. $f, g \to$ measurable $\Rightarrow$ $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$ are measurable.

Proof. $f \wedge g = \inf\{f, g\}$; $f \vee g = \sup\{f, g\}$

Cor. 5. $f \to$ measurable $\Rightarrow$ $|f|$ measurable

Proof. $|f| = f \vee -f$. Or : $|\cdot|$ is continuous.

Note : $f \wedge g = \frac{(f+g) - |f-g|}{2}$.

Def^n. Let $(X, \mathcal{M})$ be a measurable space
Suppose $E_1, \dots, E_n \in \mathcal{M}$ are p-wise disjoint.
A simple function is a function of the form

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}, \qquad a_i \in \mathbb{R}.$$

Prop^n. 6. Simple functions are (Borel) measurable.

Proof. Suffices to show $\mathbb{1}_E$ is measurable $\forall E \in \mathcal{M}$.
Note that, for $B \in \mathcal{B}_{\mathbb{R}}$,

$$\mathbb{1}_E^{-1}(B) = \begin{cases} \phi & ; B \cap \{0,1\} = \phi \\ E & ; B \cap \{0,1\} = \{1\} \end{cases}$$

$$\mathbb{1}_E^{-1}(B) = \begin{cases} \phi & ; \quad B \cap \{0,1\} = \phi \\ E & ; \quad B \cap \{0,1\} = \{1\} \\ E^c & ; \quad B \cap \{0,1\} = \{0\} \\ X & ; \quad B \cap \{0,1\} = \{0,1\} \end{cases}$$

Thus, $\mathbb{1}_E$ is measurable $\iff E \in \mathcal{M}$ ∎

Thm. 1 Let $(X, \mathcal{M})$ be a measure space.
If $f : X \to [0, \infty]$ is a measurable function (Borel), there is a sequence $\{\phi_n\}$ of simple functions s.t.

$$\phi_n \uparrow f \quad \text{pointwise.}$$

Moreover $\phi_n \to f$ UNIFORMLY on any set where f is bounded.

Proof.

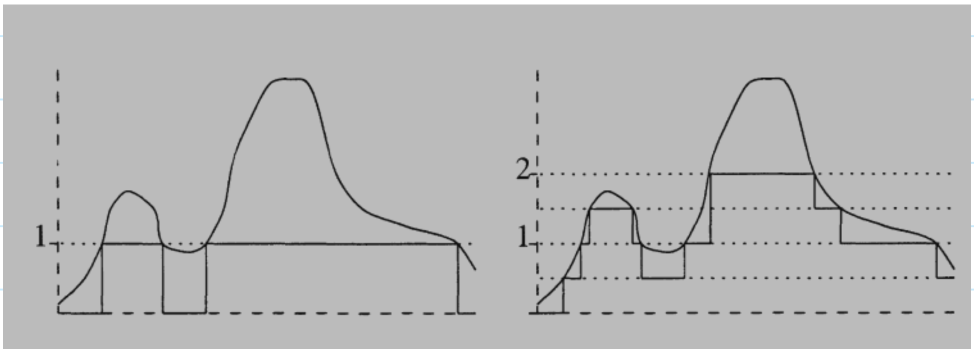


Fig. 2.1 The functions ϕ_0 (left) and ϕ_1 (right) in the proof of Theorem 2.10a.

Idea ↑

For $n \in \mathbb{N} \cup \{0\}$, and $0 \leq k \leq 2^{2n} - 1$, define

$$E_n^k := f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \quad \text{and} \quad F_n := f^{-1}((2^n, \infty]).$$

Note $E_n^k, F_n \in \mathcal{M}$ since f is measurable.
Moreover, all the sets appearing in the following eqⁿ are disjoint:

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k\, 2^{-n}\, \mathbb{1}_{E_n^k} + 2^n \mathbb{1}_{F_n}.$$

Then, $\phi_n \le \phi_{n+1}$ $\forall n$ and $0 \le f - \phi_n \le 2^{-n}$.
Thus both parts follow. ∎

Remark. When dealing with complete measures, one has to be careful as the following statement shows.

Prop^n. The following hold iff μ is complete:

(a) If f is measurable and $f = g$ μ a.e., then

(μ a.e. $\equiv$ μ almost everywhere $\equiv$ $\{f(x) \neq g(x)\} \subset N$ with $\mu(N) = 0$.

(If μ is complete, then, $\mu\{f \neq g\} = 0$.))

g is measurable.

(b) If $(f_n)_{n=1}^{\infty}$ are all measurable and suppose $f_n \to f$ μ a.e., then f is measurable.

Proof. (a) Suppose μ is complete, f is meas. and $f = g$ μ a.e..

To show: $g^{-1}(A) \in \mathcal{M}$ for any measurable A.

Let $N = \{x : f(x) \neq g(x)\}$. Note $N \in \mathcal{M}$ and $\mu(N) = 0$.

Note
$$g^{-1}(A) = (\underset{\in \mathcal{M}}{f^{-1}(A)} \cap \underset{\in \mathcal{M}}{N^c}) \sqcup (\underbrace{g^{-1}(A) \cap N}_{\subseteq N \text{ and hence null } \therefore \in \mathcal{M} \text{ since } \mathcal{M} \text{ is complete}})$$

$$\Rightarrow g^{-1}(A) \in \mathcal{M}$$

Conversely, if μ is not complete, pick E, N s.t. $E \subset N$ and $\mu(N) = 0$. (Exists since not complete.)

Define $f : X \to \mathbb{R}$ as constant 0.

$g: X \to \mathbb{R}$ as $g = \mathbb{1}_E$.

Then, $f = g$ μ a.e. but g is not measurable.

(b) Exercise. ∎

Prop 9. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $(X, \bar{\mathcal{M}}, \bar{\mu})$ is its completion. Then, for any $\bar{\mathcal{M}}$-measurable function f, there is an $\mathcal{M}$ measurable function g s.t.

$$f = g \quad \bar{\mu} \text{ a.e.}$$

Proof. If $f = \mathbb{1}_E$ for $E \in \bar{\mathcal{M}}$, then it is trivial. (Take $E = F \cup N$ for $F \in \mathcal{M}$ and $\mu(N) = 0$. Then, $g = \mathbb{1}_F$ works.)

(Thus, true for all simple functions.)

In general, write $f = f^+ - f^-$ where

$$f^+ = \max(f, 0) \quad \text{and} \quad f^- = \max(-f, 0).$$

Then, $f^+, f^- \geq 0$ are measurable. We get sequences ϕ_n^+ and ϕ_n^- converging p.wise to f^+ and f^-.

Then, $\phi_n = \phi_n^+ - \phi_n^- \longrightarrow f^+ - f^- = f$ p-wise.

For each n, let ψ_n be $\mathcal{M}$-measurable simple s.t. $\psi_n = \phi_n$ except on some $E_n \in \mathcal{M}$ with $\bar{\mu}(E_n) = 0$.

Choose $N \in \mathcal{M}$ s.t. $\mu(N) = 0$ and $N \supset \bigcup_{1}^{\infty} E_n$.

Put $g = \lim \mathbb{1}_{X \setminus N} \psi_n$. Then, g is μ-meas. and $g = f$ on N^c.

Remark The above shows that we can actually always approximate any measurable f pointwise using simple functions. (Not just non-negative ones.)

just non-negative ones.)

Integrals for Non-Negative Valued Functions

Def^n. Suppose $(X, \mathcal{M})$ is a measure space.
Define
$$\mathcal{L}^+ = \{ f : X \to [0,\infty] \mid f \text{ is } \mathcal{M}\text{-measurable}\}.$$

If $\phi = a \mathbb{1}_E \in \mathcal{L}^+$, define $(E \in \mathcal{M})$
$$\int \phi \, d\mu := a\,\mu(E).$$

For $\phi = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i} \in \mathcal{L}^+$ where $E_i \in \mathcal{M}$ are pairwise disjoint,

define
$$\int \phi \, d\mu := \sum_{i=1}^{n} a_i \, \mu(E_i).$$
($0 \cdot \infty = 0$, by convention)

For $A \in \mathcal{M}$, $\displaystyle\int_A \phi \, d\mu = \int \phi \cdot \mathbb{1}_A \, d\mu.$

(Note that ϕ simple $\Rightarrow$ $\phi \cdot \mathbb{1}_A$ is simple.)

Prop^n 12. Let ϕ, ψ be simple in $\mathcal{L}^+$.

(i) For $c \geq 0$, $\int c\phi \, d\mu = c \int \phi \, d\mu$,

(ii) $\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$,

linearity

(iii) If $\varphi \leq \psi$, then $\int \varphi \, d\mu \leq \int \psi \, d\mu$,

(iv) The map $A \mapsto \int_A 1 \, d\mu$ is a measure on $\mathcal{M}$.

Remark We would have had to talk about $\int$ being well-defined.
Note that a function $f: X \to \mathbb{R}$ is simple iff it is measurable and $f(X)$ is finite. In this case, we define the standard representation as

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i} \qquad \text{where} \quad f(X) = \{a_1, \dots, a_n\} \text{ and } E_i = f^{-1}(\{a_i\}).$$

Then, $\int f \, d\mu := \sum a_i \mu(E_i)$.
With this, all the above can be proven.

Proof. (i) trivial. (ii) Let $\varphi = \sum_i a_i \mathbb{1}_{E_i}$, $\psi = \sum_j b_j \mathbb{1}_{F_j}$ be std. reps.

Then, $$(\varphi + \psi) = \sum_{i,j} (a_i + b_j) \mathbb{1}_{E_i \cap F_j}$$

$$\Rightarrow \int (\varphi + \psi) \, d\mu = \sum_{i,j} (a_i + b_j) \mu(E_i \cap F_j)$$

$$= \sum_{i,j} a_i \mu(E_i \cap F_j) + \sum_{i,j} b_j \mu(E_i \cap F_j)$$

$$= \sum_i a_i \mu(E_i) + \sum_j b_j \mu(F_j) = \int \varphi \, d\mu + \int \psi \, d\mu.$$

Def^n. Suppose $f: X \to [0, \infty]$ is in L^+. Define

$$\int f \, d\mu := \sup \left\{ \int \phi \, d\mu : 0 \le \phi \le f, \ \phi \to \text{simple} \right\}.$$

Note that the def^n above coincides with that for simple f^n as earlier.
Also, it is easy to see from this def^n that if $f \le g$,

then

$$\int f\,d\mu \leq \int g\,d\mu \quad \text{and}$$

$$\int cf\,d\mu = c\int f\,d\mu \quad \text{for any } c \geq 0.$$

Thm. 11. (Monotone Convergence Theorem (MCT))

If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ s.t. $f_n \leq f_{n+1}$ $\forall n \in \mathbb{N}$, then

$$f = \lim_{n\to\infty} f_n$$

is in $\mathcal{L}^+$ and

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$

(Note that existence of $\int$ or limit is not an issue since we work in $\overline{\mathbb{R}}$.)

Proof. First, observe that $\left(\int f_n\,d\mu\right)_{n=1}^{\infty} \uparrow$ and thus, does have a limit.

Moreover, $f_n \leq f \Rightarrow \int f_n \leq \int f \quad \forall n$

$$\Rightarrow \lim_{n\to\infty} \int f_n\,d\mu \leq \int f\,d\mu.$$

For the other inequality, pick an arbitrary $\alpha \in (0,1)$ and let φ be a simple function with $0 \leq \varphi \leq f$.

Define

$$E_n = \{x : f_n(x) \geq \alpha\varphi(x)\}.$$

Then, $E_n \in \mathcal{M}$, $E_n \subset E_{n+1}$ and

$$\bigcup_n E_n = X.$$

(Since $\alpha\varphi(x) \leq \alpha f(x) < f(x)$ if $f(x) \neq 0$)

Hence, $\int f_n \geq \int_{E_n} f_n \geq \int_{E_n} \alpha \varphi = \alpha \int \varphi \mathbb{1}_{E_n}$.

(Ex.) $\int \phi \mathbb{1}_{E_n} \uparrow \int \phi$

Hence, $\int f_n \geq \alpha \int \varphi \mathbb{1}_{E_n}$

$$\Rightarrow \lim_{n\to\infty} \int f_n \geq \alpha \int \varphi \quad \forall \alpha \in (0,1)$$

$$\Rightarrow \lim_{n\to\infty} \int f_n \geq \int \varphi \quad \forall\ 0 \leq \varphi \leq f$$

$$\Rightarrow \lim_{n\to\infty} \int f_n \geq \int f. \qquad \blacksquare$$

Lecture 5

24 January 2021 18:12

A Quick Consequence of MCT:

Recall $$\int f \, d\mu = \sup \left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq f, \ \varphi \text{ simple} \right\}$$

possibly uncountable

The MCT allows us to take a sequence $(\varphi_n)_n$ of simple functions increasing to f and computing the limit there.
(Sequences are easier to work with.)

$f_n \uparrow f$ is a necessary condition. (At least, without any other extra hypothesis.)
Else, consider $f : [0, \infty) \to \mathbb{R}$ as
$$f_n = \mathbb{1}_{[n, n+1)} \quad \text{for} \quad n \geq 0.$$

Clearly, $\lim_{n \to \infty} f_n =: f = 0$. ← zero function

Then, $\int_{\mathbb{R}} f_n \, dm = 1 \quad \forall n$ but $\int f = 0 \neq 1$.

Lebesgue measure

Prop^n. For $f, g \in \mathcal{L}^+$,
$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

More generally, if $f = \sum_{n \geq 1} f_n$, with $f_n \in \mathcal{L}^+$,

then $$\int f \, d\mu = \sum_{n \geq 1} \int f_n \, d\mu.$$

Proof. Let $\varphi_n \uparrow f$, $\psi_n \uparrow g$ be sequences of simple functions.
Then, $\varphi_n + \psi_n \uparrow f + g$. By MCT

$$\lim_{n\to\infty} \int (\varphi_n + \psi_n)\, d\mu \underset{MCT}{=} \int (f+g)\, d\mu$$

$\Vert$ had shown for simple functions

$$\lim_{n\to\infty} \int \varphi_n\, d\mu + \lim_{n\to\infty} \int \psi_n\, d\mu \underset{MCT}{=} \int f\, d\mu + \int g\, d\mu$$

By induction, for finite sums, it follows than

$$\sum_{i=1}^{N} \int f_i\, d\mu = \int \sum_{i=1}^{N} f_i\, d\mu$$

Let $N \to \infty$ and we MCT on partial sums to get

$$\sum_{i=1}^{N} \int f_i\, d\mu = \int f\, d\mu.$$ ∎

Prop^n. If $f \in \mathcal{L}^+$, then $\int f\, d\mu = 0 \iff f = 0$ μ a.e.

Proof. The statement is obvious for simple functions
($\Leftarrow$) Suppose $f = 0$ μ a.e. and $0 \le \varphi \le f$.
↳ simple

Then, $\varphi = 0$ μ a.e. and thus, $\int \varphi\, d\mu = 0$.
Since $\int f\, d\mu$ is the sup over all such $\int \varphi\, d\mu$,
we get

$$\int f\, d\mu = 0.$$

($\Rightarrow$) Suppose $\int f\, d\mu = 0$.

$$\underset{\Vert}{\{x : f(x) > 0\}} = \bigcup_{n \ge 1} \underset{\Vert}{\{x : f(x) > \tfrac{1}{n}\}}$$

$$\{x : f(x) > 0\} = \bigcup_{n \geq 1} \{x : f(x) > \tfrac{1}{n}\}$$

$$\underset{E}{\shortparallel} \qquad\qquad \underset{E_n \in \mathcal{M}}{\shortparallel}$$

If $\mu(E) > 0$, then $\mu(E_n) > 0$ for some $n \in \mathbb{N}$.

But then, $\int f \geq \int \frac{1}{n} \cdot \mathbb{1}_{E_n} = \frac{1}{n}\mu(E_n) > 0$. $\rightarrow\leftarrow$ ∎

Cor. If $f_n \in \mathcal{L}^+$ and $f_n \uparrow f$ a.e., then $\lim\limits_{n\to\infty} \int f_n = \int f$.

Proof. Let $E = \{x : \lim\limits_{n\to\infty} f_n(x) = f(x)\}$. measurable since $g = \lim f_n$ is measurable and then, so is $g - f$

Then, $\mu(E^c) = 0$.

Thus, $f - f\mathbb{1}_E = 0$ a.e. and hence,

$$(\ast) \qquad \int f \, d\mu \underset{\text{above prop}^n}{=} \int f \mathbb{1}_E \, d\mu.$$

Also, $f_n - f_n \mathbb{1}_E = 0$ a.e. and hence,

$$(\ast) \qquad \int f_n \, d\mu = \int f_n \mathbb{1}_E \, d\mu \quad \forall n \in \mathbb{N}.$$

Note that $f_n \mathbb{1}_E \uparrow f \mathbb{1}_E$. Thus, by MCT,

$$\lim_{n\to\infty} \int f_n \mathbb{1}_E \, d\mu = \int f \mathbb{1}_E \, d\mu$$

$$\shortparallel (\ast) \qquad\qquad \shortparallel (\ast)$$

$$\lim_{n\to\infty} \int f_n \, d\mu \qquad\qquad \int f \, d\mu$$

Recall that we needed $f_n \uparrow f$ in MCT, as seen in example. However, the following does hold.

Thm. Fatou's Lemma

If $\{f_n\} \subseteq \mathcal{L}^+$ (any sequence in $\mathcal{L}^+$), then

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$$

Proof. Consider for each $k \in \mathbb{N}$,

$$\inf_{n \geq k} f_n(x) \leq f_j(x) \quad \forall j \geq k$$

$$\Rightarrow \int \inf_{n \geq k} f_n \, d\mu \leq \int f_j \, d\mu \quad \forall j \geq k$$

$$\Rightarrow \int \inf_{n \geq k} f_n \, d\mu \leq \liminf_{j \geq k} \int f_j \, d\mu \quad (*)$$

Note that $\left(\inf_{n \geq k} f_n \right)_k$ is a sequence in $\mathcal{L}^+$ increasing to $\liminf_{n \to \infty} f_n$.

By the MCT, taking $k \to \infty$ in $(*)$ gives

$$\int \liminf_{n \to \infty} f_n \, d\mu \leq \lim_{k \to \infty} \left(\liminf_{j \geq k} \int f_j \, d\mu \right)$$

$$\leq \liminf_{j \to \infty} \int f_j . \quad \blacksquare$$

Integrability of All Kinds of Functions

(Not necessarily non-negative.)

For $f : X \to \mathbb{R}$, define $f^+(x) := \begin{cases} f(x) & ; f(x) \geq 0, \\ 0 & ; f(x) < 0, \end{cases}$

and $f^-(x) := \begin{cases} -f(x) & ; f(x) \leq 0, \\ 0 & ; f(x) > 0. \end{cases}$

Hence,
$$f = f^+ - f^-.$$

Note that $f^+, f^- \geq 0$. $f^+ = \max\{f, 0\}$ and
$f^- = \min\{f, 0\}$.

Thus, if f is measurable, so are f^+, f^-. Moreover, $f^+, f^- \in \mathcal{L}^+$.

Def^n. (Integrable)
We say that f is integrable on $(X, \mathcal{M}, \mu)$ if both
$$\int f^+ \, d\mu < \infty \quad \text{and} \quad \int f^- \, d\mu < \infty.$$

(More generally, one can ask for at least one to be finite but we shall stick with the above.)

Def^n. (L^1)
$$\mathcal{L}^1(X, \mu) := \{ f \mid f \text{ is integrable on } (X, \mu) \}.$$
Shall also write $\mathcal{L}^1(\mu)$ or $\mathcal{L}^1$ if context is clear.

Note that $|f| = f^+ + f^-$.
So,
$$\mathcal{L}^1(\mu) = \left\{ f \;\middle|\; \int |f| \, d\mu < \infty \right\}.$$

Def^n. If f is integrable, i.e., $f \in \mathcal{L}^1$, then
$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu.$$

Prop^n. $\mathcal{L}^1(\mu)$ is a vector space and $\int$ is linear on this space. Furthermore, for $f \in \mathcal{L}^1$,

pointwise operations

$$\left|\int f\,d\mu\right| \leq \int |f|\,d\mu.$$

Proof. For $f, g \in \mathcal{L}^1$ and $\alpha, \beta \in \mathbb{R}$, we have

have finite integrals

$$|\alpha f + \beta g| \leq |\alpha|\,|f| + |\beta|\,|g|.$$

Thus, $\alpha f + \beta g \in \mathcal{L}^1$! To show linearity:

Let $h = f + g$, note

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$

or

$$h^+ + f^- + g^- = h^- + f^+ + g^+$$

and thus, (each function is in $\mathcal{L}^+$)

$$\int h^+\,d\mu + \int f^-\,d\mu + \int g^-\,d\mu = \int h^- + \int f^+ + \int g^+.$$

Rearrange to conclude.

Also,

$$\int \alpha f = \alpha \int f \quad \text{is easy to check.}$$

(Take $\alpha \geq 0$ and $\alpha < 0$.)

For the last part:

$$\left|\int f\,d\mu\right| = \left|\int f^+\,d\mu - \int f^-\,d\mu\right| \leq \left|\int f^+\,d\mu\right| + \left|\int f^-\,d\mu\right|$$

$$= \int f^+\,d\mu + \int f^-\,d\mu$$

$$= \int f^+ d\mu + \int f^- d\mu$$

$$= \int (f^+ + f^-) d\mu$$

$$= \int |f| d\mu. \qquad \blacksquare$$

Thm. (Dominated Convergence Theorem, DCT)

Suppose (f_n) is a sequence in $\mathcal{L}^1(\mu)$ and $f_n \to f$ a.e.
If furthermore $\exists g \in \mathcal{L}^1$ s.t.

$$|f_n| \le |g| \quad \text{a.e., then}$$

(i) $f \in \mathcal{L}^1$, and

(ii) $$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

part of thm that limit does exist

Proof. (i) It follows from Prop^n 8 and 9 from Lecture 4 that f is measurable after a possible redefinition on a null set. Also, $|f| \le |g|$ and thus, $f \in \mathcal{L}^1$.

(ii) Note that $g + f_n \ge 0$ and $g - f_n \ge 0$ a.e.
By Fatou,

$$\int g + \int f \le \liminf_n \int g + f_n = \int g + \liminf_n \int f_n$$

$$\Rightarrow \int f \le \liminf \int f_n \qquad \text{— (1)}$$

and $$\int g - \int f \le \liminf_n \int g - f_n = \int g - \limsup_n \int f_n$$

$$\Rightarrow \int f \ge \limsup \int f_n \qquad \text{— (2)}$$

From (1) and (2), we are through. ∎

Lecture 6 (25-01-2021)

25 January 2021 14:01

Consequences Of DCT

Prop^n. Suppose (f_i) is a sequence in $\mathcal{L}^1(\mu)$ such that

$$\sum_{i=1}^{\infty} \int |f_i| < \infty. \quad \text{Then,}$$

$\sum_{i=1}^{\infty} f_i$ converges a.e. to a function in $\mathcal{L}^1$, and

$$\int \sum f_i = \sum \int f_i.$$

Proof. Define $g := \sum_{i=1}^{\infty} |f_i|$. As a consequence of MCT and hypothesis, $g \in \mathcal{L}^1$.

Thm. (Differentiating + Taking limits within integral)

Suppose that $f : X \times [a, b] \to \mathbb{R}$ and $f(\cdot, t) \to \mathbb{R}$ is integrable for each $t \in [a, b]$.

Let

$$F(t) = \int_X f(x, t)\, d\mu(x).$$

(a) Suppose that there exists $g \in \mathcal{L}^1(\mu)$ such that $|f(x, t)| \le g(x)$ for all x, t.

If

$$\lim_{t \to t_0} f(x, t) = f(x, t_0) \quad \text{for every } x,$$

$t \to t_0$ ... for every x,

the $\lim_{t \to t_0} F(t) = F(t_0)$;

in particular, if $f(x, \cdot)$ is continuous for each x, then F is continuous.

(b) Suppose that $\frac{\partial f}{\partial t}$ exists and $\exists g \in \mathcal{L}^1(\mu)$ s.t.

$$\left| (\partial f / \partial t)(x, t) \right| \leq g(x) \qquad \forall x, t.$$

Then, F is differentiable and

$$F'(x) = \int (\partial f / \partial t)(x, t) \, d\mu(x).$$

Proof. (a) Let $t_n \to t_0$ and let $F_n(x) = F(x, t_n)$. By hypothesis, $|F_n(x)| \leq g(x)$ and $F_n(x) \to f(x, t_0)$. So, we are done, by DCT.

(b) Let $t_n \to t_0$ be an arbitrary sequence. $(t_n \neq t_0 \ \forall n)$
Let

$$h_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}.$$

h_n are measurable and $\lim_{n \to \infty} h_n(x) = \frac{\partial f}{\partial t} h(x, t_0)$.

So, $\frac{\partial f}{\partial t}(x, t_0)$ is measurable. Moreover

$$|h_n(x)| \underset{\text{mean value thm}}{\leq} \sup_{t \in [a,b]} \left| \frac{\partial f}{\partial t}(t, x) \right| \leq g, \text{ by assumption.}$$

Thus, we are again through, by DCT.

(Riemann integrability vs Lebesgue Integrability)

Riemann Integrability vs Lebesgue Integrability

Q. If f is Riemann integrable on $[a, b]$, is it (Lebesgue)-integrable? If yes, are the integrals equal?

If $\mathcal{P}$ is a partition

$$a = t_0 < t_1 < \dots < t_n = b,$$

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}), \qquad m_i = \inf_{[t_i, t_{i-1}]} f$$

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}), \qquad M_i = \sup_{[t_i, t_{i-1}]} f$$

Notation: $\int_a^b f(x)\,dx$ ← Riemann integral

Consider, $$G_{\mathcal{P}} := \sum M_i \mathbb{1}_{(t_{i-1}, t_i]},$$

$$g_{\mathcal{P}} := \sum m_i \mathbb{1}_{(t_{i-1}, t_i]}.$$

Note $\quad g_{\mathcal{P}} \le f \le G_{\mathcal{P}} \quad \forall \mathcal{P}.$

If $f \in \mathcal{R}[a, b]$, then $\exists$ a sequence of refined partitions $(\mathcal{P}_n)$ s.t.

$$L(f, \mathcal{P}_n) \uparrow \int_a^b f(x)\,dx \quad \text{and} \quad U(f, \mathcal{P}_n) \downarrow \int_a^b f(x)\,dx.$$

Let $g_n := g_{\mathcal{P}_n}$ and $G_n := G_{\mathcal{P}_n}$.

Consider

$$g(x) := \lim_{n\to\infty} g_n(x) \quad \text{and} \quad G(x) := \lim_{n\to\infty} G_n(x).$$

Furthermore, since $g_{\varphi_n} \le f \le G_{\varphi_n}$, it follows that

$g \le f \le G$. Furthermore since f

is bdd and $[a,b]$ is a bounded interval, by DCT

$$\lim_{n\to\infty} \int g_n \, dm = \int g \, dm$$

↪ Lebesgue measure

Also, $\lim_{n\to\infty} \int G_n =$

Remark The above is for proper Riemann integrals.
Consider $f: \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := \begin{cases} \frac{\sin x}{x} & ; \; x \neq 0, \\ 1 & ; \; x = 1. \end{cases}$$

$\int_{-\infty}^{\infty} f(x)\,dx$ exists, as an improper Riemann integral but

$\int |f| \, dm = \infty$ and hence, f is not Lebesgue integrable

MODES OF CONVERGENCE (Modes of Convergence)

Suppose we have a sequence of functions $\{f_n\}$ on say $[0,1]$. Want to talk about limit.

POINTWISE. For each x, $\{f_n(x)\}$ must be convergent.
Not very good with overall functional properties, as we have seen.

NOTIONS OF CONVERGENCE = 'NEW' METRIC SPACES.

UNIFORM CONVERGENCE: $\{f_n\}$ converges uniformly on X
$\Leftrightarrow$ given $\varepsilon > 0$, $\exists N(\varepsilon)$ s.t. $\forall x$ $\forall n \geq N(\varepsilon)$:

$$|f_n(x) - f(x)| < \varepsilon.$$

$$C[0,1] := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

$C[0,1]$ is a metric space w.r.t. sup norm metric,

$$d(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)|.$$

(exists since $f-g$ is cts.)

Now, uniform convergence (of continuous functions) is precisely convergence in this metric.
Moreover, continuous functions uniformly converge to cts functions.
Thus, $C[0,1]$ is a complete metric function.

Thm. Suppose $(X, \mathcal{M}, \mu)$ is a complete measure space.

$$\mathcal{L}^1(\mu) = \{ f : X \to \mathbb{R} \mid \int |f| d\mu < \infty \}.$$

$\mathcal{L}^1$ is a $\mathbb{R}$-vector space. Moreover

1. $\int af\, d\mu = a \int f\, d\mu$ for $a \in \mathbb{R}$, $f \in \mathcal{L}^1$.
2. $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$ for $f, g \in \mathcal{L}^1$.

NORMED SPACES

(Normed spaces)

Defⁿ. Suppose V is an $\mathbb{R}$-vector space. A **norm** is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying

- $\|x\| = 0 \iff x = 0$,
- $\|\alpha x\| = |\alpha| \cdot \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

Example: $V = \mathbb{R}^n$, $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_n^2}$

$\|x\|_1 := |x_1| + \cdots + |x_n|$

are norms.

FACT: If V has a norm $\|\cdot\|$, then this induces a metric.
$d(x, y) := \|x - y\|$.

Consider a complete measure space $(X, \mathcal{M}, \mu)$.
For $f \in \mathcal{L}^1(\mu)$, define

$$\|f\|_1 := \int |f| \, d\mu.$$

↳ doesn't satisfy norm axioms, $\int |f| d\mu \not\Rightarrow f \equiv 0$

Consider an equivalence relation : $f \sim g$ if $f \neq g$ only on a set of measure 0.

We know that if $f = g$ μ a.e., then

$$\int f \, d\mu = \int g \, d\mu.$$

Thus, $\int$ makes sense on equivalence classes of $\sim$.
With that, $\mathcal{L}^1(\mu)/\sim$ becomes a normed space.
By abuse of notation, we shall continue denoting it $\mathcal{L}^1(\mu)$.

Thm. $L^1(\mu)$ is a complete metric space w.r.t. the metric induced by $\|\cdot\|_1$.

(Special case of Riesz-Fischer Theorem.)

Def^n. A normed vector space which is complete w.r.t. the metric induced by the norm is called a Banach space.

(Thus, $\mathcal{L}^1$ is a Banach space)

Proof. Want to show: Cauchy sequence converge in $\mathcal{L}^1$!

Suppose $\{f_n\}$ is Cauchy, i.e., given $\varepsilon > 0$, $\exists N(\varepsilon)$ s.t.

$\forall m, n \geq N(\varepsilon)$

$$\|f_n - f_m\|_1 < \varepsilon.$$

WTS: $\exists f \in \mathcal{L}^1(\mu)$ s.t. $\{f_n\}$ converges to f.

$\hookrightarrow$ in $\|\cdot\|$

Get a sequence $\{n_i\}_i$ s.t.

$$\|f_{n_{i+1}} - f_{n_i}\|_1 < \frac{1}{2^i} \quad \text{for } i \geq 1.$$

(Follows straightforwardly from Cauchy.)

Define $g_k : X \to \mathbb{R}$ by

$$g_k := \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \quad \text{and}$$

$$g := \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|.$$

$\|g_k\|_1 < 1 \quad \forall k.$ $\qquad g(x) = \lim_{k\to\infty} g_k(x).$

So, by Fatou's Lemma

$$\int \liminf g_n \leq \liminf \int g_n \leq 1$$

$$\int \liminf g_n \leq \liminf \int g_n \leq 1$$

$$\int g \, d\mu$$

Since $g \geq 0$, we get $g \in \mathcal{L}^1$. ($g < \infty$ μ a.e.)

Let
$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{i=1}^{\infty} \left(f_{n_{i+1}}(x) - f_{n_i}(x) \right) & \text{when the sum } < \infty \\ 0 & \text{otherwise} \end{cases}$$

f is our candidate for the limit.

$$F_N := f_{n_1} + \sum_{i=1}^{N} f_{n_{i+1}} - f_{n_i}$$
$$= f_{n_{N+1}}$$

Thus, $f_{n_{N+1}}$ converges μ a.e. as $N \to \infty$

To complete the proof, $\underline{\underline{\text{WTS}}}$: $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Note that $|f - f_n| = \liminf_N |f_{n_{N+1}} - f_n|$

So again, by Fatou

$$\|f - f_n\|_1 = \int |f - f_n| \, d\mu \leq \liminf_N \underbrace{\int |f_{n_{N+1}} - f_n| \, d\mu}_{= \|f_{n_{N+1}} - f_n\|_1}$$

Now, given $\varepsilon > 0$, pick $N_0(\varepsilon)$ large enough so that $\forall$ $m, n \geq N_0(\varepsilon)$, $\|f_m - f_n\|_1 \leq \varepsilon$.
Picking N large gives the result.

Picking N large gives the result.

This shows convergence in $\mathcal{L}^1$! $\blacksquare$

$\hookrightarrow f \in \mathcal{L}^1$ follows since $\int |f| \leq \underbrace{\int |f - f_n|}_{\downarrow\, 0} + \underbrace{\int |f_n|}_{< \infty}$

Def^n. (Convergence in measure, Cauchy in measure)

Given a sequence $(f_n)_n$ of measurable functions in $(X, \mathcal{M}, \mu)$, we say that (f_n) is Cauchy in measure if for any $\varepsilon > 0$,

$$\mu\{x : |f_n(x) - f_m(x)| > \varepsilon\} \to 0 \quad \text{as } m, n \to \infty.$$

$f_n \to f$ in measure if $\forall \varepsilon > 0$,

$$\mu\{x : |f_n(x) - f(x)| > \varepsilon\} \to 0 \quad \text{as } n \to \infty$$

Lecture 7 (28-01-2021)

28 January 2021 14:06

Another perspective of convergence in measure:

Consider the sequence in $\mathbb{R}$: $X_n = \begin{cases} 1 ; & n \text{ is a square} \\ 0 ; & \text{otherwise} \end{cases}$

Clearly, (X_n) does not converge. However, X_n is "mostly" 0. The squares are "sparse" in $\mathbb{N}$. We would want the sequence to converge to 0. We don't want irregularities on sparse sets to make a difference.

Some remarks:

① $f_n \to f$ and $f_n \to g$ in measure.

Then, $f = g$ μ a.e.

$\{x : |f(x) - g(x)| > \varepsilon\} \subseteq \{x : |f(x) - f_n(x)| > \varepsilon/2\}$
$\cup \{x : |g(x) - f_n(x)| > \varepsilon/2\}$.

② $f_n \to f$ a.e. $\not\Rightarrow$ $f_n \to f$ in measure or $f_n \to f$ in $\mathcal{L}^1$

ⓐ $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}(\mathbb{R})$, $\mu = m$.

$f_n(x) = \mathbb{1}_{(n, n+1]}$. Clearly $f_n \to 0$ pointwise everywhere.

But $\int f_n \, dm = 1$. Thus, $f_n \not\to 0$ in $\mathcal{L}^1$.

Also, $\mu\{x : |f_n(x) - 0| > 1/2\} = \mu(n, n+1] = 1 \quad \forall n$.

Thus, $f_n \not\to f$ in measure.

ⓑ Consider: $f_1 = \mathbb{1}_{[0,1]}$

$f_2 = \mathbb{1}_{[0,1/2]}$; $f_3 = \mathbb{1}_{[1/2,1]}$

$f_4 = \mathbb{1}_{[0,1/4]}$; $f_5 = \mathbb{1}_{[1/4, 2/4]}$; ...

Note:

For any $\varepsilon > 0$, we have $\mu\{x : |f_n(x)| > \varepsilon\} \to 0$.

Thus, $f \to 0$ in measure.

Also, $\int f_n \to 0$. Thus, $f_n \to 0$ in $\mathcal{L}^1$.

However, for any $x \in [0,1]$, $\{f_n(x)\}_n$ does NOT converge!

Prop^n. $f_n \to f$ in $\mathcal{L}^1(\mu)$, then $f_n \to f$ in measure.

(Variant of Markov inequality)

Proof. Fix $\varepsilon > 0$. For $n \geq 1$, define

$$E_n := \{x : |f_n(x) - f(x)| > \varepsilon\}.$$

Then,

$$\underbrace{\int |f_n - f| \, d\mu}_{\downarrow \, 0} \geq \int_{E_n} |f_n - f| \, d\mu \geq \varepsilon \int_{E_n} 1 \, d\mu = \varepsilon \mu(E_n) \geq 0$$

$$\therefore \mu(E_n) \longrightarrow 0. \qquad \blacksquare$$

Converse is not true! Consider the earlier example modified:

$$f_1 = \mathbb{1}_{[0,1]}, \quad f_2 = 2\,\mathbb{1}_{[0,1/2]}, \quad f_3 = 2\,\mathbb{1}_{[1/2,1]},$$

$$f_4 = 4\,\mathbb{1}_{[0,1/4]}, \quad \ldots$$

(Make all integrals 1.) $f_n \to 0$ in measure but not in $\mathcal{L}^1$!

Thm. ① Suppose (f_n) is Cauchy in measure, then $\exists f$ measurable s.t. $f_n \to f$ in measure.

② There is a subsequence (f_{n_j}) s.t. $f_{n_j} \to f$ a.e.

Proof. Pick $n_1 < n_2 < \cdots$ s.t. $g_i = f_{n_i}$ satisfies the

$$\mu(E_i) \leq 2^{-i} \text{ where } E_i := \{x : |g_i(x) - g_{i+1}(x)| \geq 2^{-i}\}.$$

(Can pick using Cauchy.)

Let $E^{(k)} = \bigcup_{j \geq k} E_j$. Then, $E^{(k)} \leq \frac{1}{2^{k-1}}$.

Thus, if $x \in E^{(k)}$, then $x \in E_j$ $\forall j \geq k$.

In particular, for $m > n \geq k$,

$$|g_m(x) - g_n(x)| \leq \sum_{j=n}^{m-1} |g_{j+1}(x) - g_j(x)|$$

$$\leq \sum_{j=n}^{\infty} \frac{1}{2^j} \leq \frac{1}{2^{n-1}}$$

Thus, for any $x \notin E^{(k)}$, $\{g_n(x)\}$ is Cauchy.

Now, if we set $E = \bigcap_{k \geq 1} E^{(k)}$, then

(a) $\mu(E) = 0$

(b) For $x \notin E$, $\{g_n(x)\}$ converges.

Define

$$f(x) = \begin{cases} \lim g_n(x) & ; \ x \notin E \\ 0 & ; \ x \in E \end{cases}$$

Then, f is measurable and $g_j \to f$ a.e.

Since $\mu(E^{(k)}) \to 0$, it follows $g_j \to f$ in measure.

It also follows that $f_n \to f$ in measure.

$$\mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} \leq \mu\{x : |f_n(x) - g_j(x)| \geq \varepsilon/2\}$$
$$\cup\ \mu\{|g_j - f| \geq \varepsilon/2\}. \ \blacksquare$$

Thm. (Egoroff's Theorem) Suppose $\mu(X) < \infty$ and suppose measurable $f_n \to f$ μ a.e.

Then, given any $\varepsilon > 0$, $\exists E \in X$ s.t. $\mu(E) < \varepsilon$ s.t.

$f_n \to f$ UNIFORMLY on E^c.

Proof. (Sketch) We may assume $f_n \to f$ on X.

For $n, k \in \mathbb{N}$, define

$$E_{n,k} := \left\{x : |f_m(x) - f(x)| > \frac{1}{k} \text{ for some } m \geq n\right\}.$$

Know: $f_n(x) \to f(x)$ $\forall x \in X$

Thus, for a fixed k, we have

$$\bigcap_{n\in\mathbb{N}} E_{n,k} = \phi.$$

Since $\mu(X) < \infty$, we have $\mu(E_{n,k}) < \infty$.

This gives us that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$ for each fixed k.

$(E_{1,k} \supseteq E_{2,k} \supseteq \cdots)$

Pick n_k large so that

$$\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k} \quad \text{and let}$$

$$E = \bigcup_{k\geq 1} E_{n_k}(k).$$

Then, $\mu(E) < \varepsilon$ and for $x \notin E$, $|f_n(x) - f(x)| < \frac{1}{k}$

whenever $n \geq n_k$.

In particular, the convergence is uniform on E^c.

Product Measures

(Product measure)

We will now use $\otimes$ instead of Π for product of σ-fields.

Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \mu)$ are measure spaces.

Had defined $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$ as

$$\langle E \times F : E \in \mathcal{M}, F \in \mathcal{N} \rangle.$$

Def^n A set of the form $E \times F$ for $E \in \mathcal{M}$, $F \in \mathcal{N}$ is called a rectangle.

Want a product measure on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$.

Would want $\rho(E \times F) := \mu(E)\nu(F)$ on rectangles.

Consider $\mathcal{A}$ = finite disjoint unions of rectangles

$\mathcal{A}$ is indeed an algebra.

Extend ρ as:

- $\rho\left(\bigsqcup_{i=1}^{n} E_i \times F_i\right) := \sum_{i=1}^{n} \mu(E_i)\nu(F_i)$; $\rho(\phi) = 0$.

So, if ρ is a pre-measure on $\mathcal{A}$, then the general theory from before allows us to extend ρ to a measure on $M \otimes N$. (Since $\langle \mathcal{A} \rangle = \mathcal{M} \otimes \mathcal{N}$.)

Moreover, if μ, ν are σ-finite extensions, then the extension is unique.

Thus, we now wish to show that ρ is a pre-measure.

Suppose $A \times B$ is a rectangle and

$$A \times B = \bigsqcup_{i=1}^{\infty} (A_i \times B_i)$$

Q. Is $\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)$?

Consider $f(x,y) = \mathbb{1}_A \cdot \mathbb{1}_B$ on $X \times Y$.

Then, $\mathbb{1}_{x \in A} \mathbb{1}_{y \in B} = \sum_i \mathbb{1}_{x \in A_i} \mathbb{1}_{x \in B_i}$

Fix y; this is integrable, so

$$\int \mathbb{1}_{x \in A} \mathbb{1}_{y \in B} \, d\mu(x) = \mathbb{1}_{y \in B} \, \mu(A)$$

By MCT, $\mathbb{1}_{y \in B} \mu(A) = \sum_i \mathbb{1}_{y \in B_i} \mu(A_i)$

Again, by MCT, and integrating w.r.t. y

$$\mu(A) \int \mathbb{1}_{y \in B} \, d\nu(y) = \sum \mu(A_i) \int \mathbb{1}_{y \in B_i} \, d\nu(y)$$

$$\mu(A)\nu(B) \qquad\qquad \sum_i \mu(A_i)\nu(B_i)$$

Thus, we get the product measure, denoted $\mu \otimes \nu$.

Def^n. Let $E \subseteq X \times Y$, $x \in X$, $y \in Y$ with $E \in \mathcal{M} \otimes \mathcal{N}$.
x-section of E $(E_x) := \{y \in Y : (x, y) \in E\} \subseteq Y$,
y-section of E $(E^y) := \{x \in X : (x, y) \in E\} \subseteq X$.

Prop^n. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$.

Proof. Let $\mathcal{R} \subseteq \mathcal{P}(X \times Y)$ be the set of all those $E \subseteq X \times Y$ s.t. $E_x \in \mathcal{N}$ and $E_y \in \mathcal{M}$.
Clearly, all rectangles are in $\mathcal{R}$.

$$(A \times B)_x = \begin{cases} B & ; x \in A \\ \phi & ; x \notin A \end{cases}$$

Moreover, $\mathcal{R}$ is a σ-algebra since $\left(\bigcup_1^\infty E_j\right)_x = \bigcup_1^\infty (E_j)_x$
and $(E^c)_x = (E_x)^c$
and likewise for y.
Thus, $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{R}$, as desired.

Cor. Given $f : X \times Y \to Z$ measurable, we can define the slice functions $f_x : Y \to Z$ for $x \in X$ and
$f^y : X \to Z$ for $y \in Y$ by

$$f_x(y) = f(x,y) = f^y(x).$$

Then, f_x is $\mathcal{N}$-measurable for all $x \in X$ and
f^y is $\mathcal{M}$-measurable for all $y \in Y$.

Lecture 8 (01-02-2021)

01 February 2021 13:58

Recall:

- Suppose $(f_n)_n$ is a seq. of meas. f" which is Cauchy in measure. Then:

  ① $\exists f$ measurable s.t. $f_n \to f$ in measure.

  ② $\exists (n_k)_k$ s.t. $f_{n_k} \to f$ μ a.e.

Idea of proof: Get an f s.t. ② holds. Show ① for that f.

How? Get $n_1 < n_2 < \dots$ s.t. $g_k = f_{n_k}$ satisfy:

$$\mu\left\{x : |g_{k+1}(x) - g_k(x)| \geq \frac{1}{2^k}\right\} \leq \frac{1}{2^k}.$$

- If $f_n \to f$ μ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure. Moreover, given $\varepsilon > 0$, $\exists E$ s.t. $\mu(E) < \varepsilon$ and on E^c, $f_n \to f$ uniformly.

Fubini's Theorem

$$\lambda = \mu \otimes \nu$$

$$\int_{X \times Y} f \, d\lambda = \int_X \left(\int_Y f(x,y) \, d\nu(y) \right) d\mu(x)$$

"WISHFUL THINKING" THEOREM

The above is not true, in general. Even with infinite sums.

1	0	0	...
-1	1	0	...
0	-1	1	...
0	0	-1	...
⋮	⋮	0 ⋮	

Each column sum $= 0$.

First row sum $= 1$, later ones $= 0$.

Σ (column sums) $= 0 \neq 1 = \sum$ (row sums)

Prototype for Fubini (Tonelli's Theorem) :

Suppose $f \geq 0$, $f \to$ measurable on $\mathcal{M} \otimes \mathcal{N}$

$$\int f \, d\lambda \overset{?}{=} \int_X \underbrace{\int_Y f(x,y) \, d\nu(y)}_{\text{function of } x} \, d\mu(x)$$

(Recall from last lec: $f(x, -)$ was measurable for all x.)

Is this $\mathcal{M}$ measurable?

Let us make the following assumption: $\mu(X) < \infty$, $\nu(Y) < \infty$.

Prop^n. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. Then, for any $E \in \mathcal{M} \otimes \mathcal{N}$:

1. The functions $g = (x \mapsto \nu(E_x))$ and $h = (y \mapsto \mu(E^y))$ are measurable. ($\mathcal{M}$ and $\mathcal{N}$ meas., resp.)
2. $(\mu \otimes \nu) E = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y)$.

In particular, we have proven Fubini for $f = \mathbb{1}_E$.

Proof. Let $\mathcal{F} = \{ E \in \mathcal{M} \otimes \mathcal{N} : ① \text{ and } ② \text{ hold for } E \}$.

Suppose $\mu(X) < \infty$ and $\nu(Y) < \infty$.

Note that if $E = A \times B$ for $A \in \mathcal{M}$ and $B \in \mathcal{N}$, then $E \in \mathcal{F}$. Thus, all rectangles are in $\mathcal{F}$.

$g(x) = \mathbb{1}_{x \in A} \cdot \nu(B)$ is clearly $\mathcal{M}$-measurable and

$$\int g(x) \, d\mu(x) = \nu(B) \int \mathbb{1}_{x \in A} \, d\mu(x) = \mu(A)\nu(B).$$

$$= (\mu \otimes \nu)(A \times B).$$

Similarly for y.

Similarly for y.

Now, it suffices to show that $\mathcal{F}$ is a σ-algebra.
Unfortunately, showing that is not too easy.
Suppose $E_n \in \mathcal{F}$ and $E_n \uparrow$. (That is, $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$)
Let g_n and h_n be the corresponding slice functions:

$$g_n(x) = \nu((E_n)_x) \text{ and}$$
$$h_n(y) = \mu((E_n)^y).$$

It is easy to see that $g_n \uparrow$. $(g_n(x) \le g_{n+1}(x) \ \forall x)$.
Similarly, $h_n \uparrow$.
Let $g = \lim g_n$ and $h = \lim h_n$.

Put $E := \bigcup_{n \geq 1} E_n$. Easy to see that $g(x) = \nu(E_x)$ and $h(y) = \mu(E^y)$.

Moreover, g and h are measurable. Thus, ① holds for E.

For ②: By MCT,

$$\int \mu(E^y)\, d\nu \underset{\text{MCT}}{=} \lim_{n\to\infty} \int \mu((E_n)^y)\, d\nu$$

$E_n \in \mathcal{F}$ and thus ② is true for E_n.

$$= \lim_{n\to\infty} (\mu \otimes \nu)(E_n) \underset{E_n \uparrow E}{=} (\mu\otimes\nu)(E).$$

Similarly, we have the other equality for x.
Thus, $E \in \mathcal{F}$ and hence, $\mathcal{F}$ is closed under increasing unions.

(To be continued.)

Def^n. ① A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a monotone class if $\mathcal{C}$ is closed under countable increasing unions and countable decreasing intersections.

② Given an algebra $\mathcal{A}$, $\mathcal{C}(\mathcal{A})$ is the smallest monotone class containing $\mathcal{A}$; called the monotone class generated by $\mathcal{A}$.

(Monotone class)

Lemma (Monotone class Lemma)

If $\mathcal{A}$ is an algebra on X, then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

That is, the monotone class generated by $\mathcal{A}$ is the σ-alg. generated by $\mathcal{A}$.

(Continuing) We showed that $\mathcal{F}$ is closed under $\uparrow$ ctble unions.

Now suppose $\{E_n\} \subseteq \mathcal{F}$ and $E_n \downarrow E$.

Note again that
$$g(x) = \nu(E_x) = \lim_{n\to\infty} \nu((E_n)_x)$$

and
$$h(y) = \lim_{n\to\infty} \mu((E_n)^y)$$

General Q. If (f_n); $f_n \geq 0$ and $f_n \downarrow f$ in $(X, \mathcal{M}, \mu)$, is it true that $\int f_n \, d\mu \longrightarrow \int f \, d\mu$?

NO. Not in general.

Stuff like this requires finiteness of measure.

Here, we assumed that μ, ν are finite.

So, $g_n \leq g_1$ and DCT helps us.

$\hookrightarrow$ measurable with finite

Hence, if μ and ν are finite, then $\mathcal{F}$ is a monotone class and $\mathcal{F}$ contains all rectangles. Moreover, (by linearity) $\mathcal{F}$ contains all finite disjoint unions of rectangles.

Thus, the usual algebra $\mathcal{A}$ which generates $\mathcal{M} \otimes \mathcal{N}$.

Thus, by monotone class lemma, $\mathcal{F} = \mathcal{M} \otimes \mathcal{N}$ and thus, we have Fubini for functions which are indicators of measurable sets.

For σ-finites split $X = \bigcup X_i$ and $Y = \bigcup Y_j$ and conclude.

Lecture 9 (04-02-2021)

04 February 2021 14:03

Thm. Let $\mathcal{A}$ be an algebra on X. Then, if $\mathcal{C} = \mathcal{C}(\mathcal{A})$ is the monotone class generated by $\mathcal{A}$, and $\mathcal{M} = \mathcal{M}(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$, then $\mathcal{C} = \mathcal{M}$.

Proof. Note that if $\mathcal{C}$ itself is an algebra, then we are through.
(Countable unions will follow from finite unions since $\mathcal{C}$ is a monotone class.)
Thus, we only need to show that $\mathcal{C}$ is an algebra.
For $E \in \mathcal{C}$, define

$$\mathcal{C}(E) := \{ F \in \mathcal{C} : E \cap F,\ E \setminus F,\ F \setminus E \ \in \mathcal{C} \}.$$

Note that:
- $E \in \mathcal{C}$
- $\emptyset \in \mathcal{C}$
- $E \in \mathcal{C}(F) \Leftrightarrow F \in \mathcal{C}(E)$ for $E, F \in \mathcal{C}$.

(Note $\emptyset \in \mathcal{C}$ since $\emptyset \in \mathcal{A} \subset \mathcal{C}(\mathcal{A}) = \mathcal{C}$.)

Suppose $E \in \mathcal{A}$. Then, for any $F \in \mathcal{A}$, then $F \in \mathcal{C}(E)$.
[Since $E \cap F,\ E \setminus F,\ F \setminus E \in \mathcal{A} \subset \mathcal{C}$.]
Thus, $\mathcal{A} \subseteq \mathcal{C}(E)$. — $(*)$
Verify: $\mathcal{C}(E)$ is a monotone class.
Then, $\mathcal{C} = \mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(E)$ [by $(*)$]

$$\Rightarrow \quad \mathcal{C} = \mathcal{C}(E) \quad \forall E \in \mathcal{A}$$

In particular, for $F \in \mathcal{C}$, $F \in \mathcal{C}(E)$. (if $E \in \mathcal{A}$.)
By reciprocity: $\forall E \in \mathcal{A}\ \forall F \in \mathcal{C} : E \in \mathcal{C}(F)$.

$$\Rightarrow \mathcal{A} \subseteq \mathcal{C}(F) \quad \forall F \in \mathcal{C}$$
$$\Rightarrow \mathcal{C} = \mathcal{C}(F) \quad \forall F \in \mathcal{C}$$

Since $X \in \mathcal{A} \subseteq \mathcal{C}$, for any $E \in \mathcal{C}$

$X\setminus E,\quad X\cap E,\quad X\cup E \in \mathcal{C}$; in particular, $X\setminus E \in \mathcal{C}$.

For $E, F \in \mathcal{C}$, $E\cap F \in \mathcal{C}$.

Thus, $\mathcal{C}$ is closed under complement and finite intersection.

~~Hence,~~ hence, also ~~under~~ a ~~-alg~~ $\overset{\circ}{\bigcup}_{i}$. Thus, $\mathcal{C}$ is an algebra

$\therefore\ \mathcal{M} \subseteq \mathcal{C}$.

OTOH, $\mathcal{M}$ is itself a monotone class. Thus, $\mathcal{C} \subseteq \mathcal{M}$. ⊠

Thus, we are done.

Thm. (Fubini-Tonelli Theorem)

Suppose $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ are σ-finite measures.

(i) (Tonelli's Thm.) If $f \in \mathcal{L}^+(X\times Y)$, then the slice functions

$$g(x) = \int_Y f(x,y)\,d\nu(y) \quad \text{and}$$
$$h(y) = \int_X f(x,y)\,d\mu(x) \quad \text{are}$$

in $\mathcal{L}^+(X)$ and $\mathcal{L}^+(Y)$, resp.

Moreover,
$$\int f\,d(\mu\otimes\nu) = \int_X\left(\int_Y f(x,y)\,d\nu\right)d\mu$$

$$= \int_Y\left(\int_X f(x,y)\,d\mu\right)d\nu$$

(ii) (Fubini's Thm.) If $f \in \mathcal{L}^1(\mu\otimes\nu)$, then $f_x \in \mathcal{L}^1(\nu)$ a.e. x (w.r.t. μ) and $f_y \in \mathcal{L}^1(\mu)$ a.e. y (w.r.t. ν) and

$$\int f\,d(\mu\otimes\nu) = \int_X\left(\int_Y f(x,y)\,d\nu\right)d\mu$$

$$= \int_Y\left(\int_X f(x,y)\,d\mu\right)d\nu$$

Proof. The proposition from previous lecture shows that (i) holds if $f = \mathbb{1}_E$ for $E \in \mathcal{M}\otimes\mathcal{N}$.

By linearity, the statement holds for finite linear combinations

i.e., taking increasing limits in MCT, (i) follows.

In fact, the same proof shows that if $f \in \mathcal{L}^+$ and
$\int f \, d(\mu \otimes \nu) < \infty$, then $f < \infty$ a.e. w.r.t. $\mu \otimes \nu$.

Consequently, the functions $g < \infty$ μ a.e. and
$h < \infty$ ν a.e. and

Finally the same conclusion follows by arguing with f^+ and f^-. ∎

Remarks:

1. The above can be extended to products of finitely many measure spaces $\{(X_i, \mathcal{M}_i, \mu_i)\}_{i=1}^n$, if we already know what $\bigotimes_{i=1}^n \mathcal{M}_i$ is. Fubini holds there with $\bigotimes_{i=1}^n \mu_i$.

2. If f is not in $\mathcal{L}^1(\mu \otimes \nu)$, then Fubini may fail.
(Recall the array of 1s and -1s.)

3. In general, if μ and ν are not σ-finite, then it may not hold.

Q. Non σ-finite measures?
Take $Y = [0,1]$, $\mathcal{N} = \mathcal{B}(X)$ and $\nu \equiv$ counting measure.
(In some sense, this typifies non σ-finite measures.)
Let $X = Y$, $\mathcal{M} = \mathcal{N}$ and $\mu = m$. (Lebesgue measure)

Let $D = \{(x, x) : x \in [0,1]\} \subset X \times Y$. (Why is it measurable?) $\to$ closed in $X \times Y$ (Borel)
Put $f = \mathbb{1}_D$.

$$\int \Big(\int f \, d\mu \Big) d\nu = \int 0 \, d\nu = 0$$

$$\int_X \left(\int_Y f \, d\nu \right) d\mu = \int_X 1 \, d\mu = 1$$

But $(\mu \otimes \nu)(D) = \infty$.

If we consider any open $U \supseteq D$, then it contains a ball which contains a rectangle which have $\mu \otimes \nu = \infty$.

Thus, $(\mu \otimes \nu)(D) = \inf \{ (\mu \otimes \nu)(U) : U \text{ open}, U \supseteq D \}$
$= \infty$.

(There are some conditions such that it still goes through with σ-finite. Exercise in Folland.)

Digressive remarks:

Random variable : Measurable function on a probability space.
$(\mu(\Omega) = 1)$

- $(\Omega, \mathcal{M}, \mathbb{P})$, $X_i : \Omega \to \mathbb{R}$ r.v.s

What does it mean to say $X_1, \dots, X_n$ are indep.?

In general, for any $(X, \mathcal{M}, \mu)$, $f : X \to \mathbb{R}$ with
not necessarily finite
$f \in \mathcal{L}^1 \cap \mathcal{L}^+$, define

$$\nu(A) := \int f \cdot \mathbb{1}_A \, d\mu .$$

Then, ν is also a measure on X.
finite!

Something something marginals and constructing joint measure.

- Probabilities allow us to define infinite products in a non-trivial way.

The n-dimensional Lebesgue integral

Note. The product measure $\mu \otimes \nu$ is (usually) not complete even if μ and ν are.

Def^n By $\mathcal{L}^n$, we mean the completion of $\otimes \prod_{i=1}^{n} (\mathbb{R}, \mathcal{B}_i, m)$ on $\mathbb{R}^n$. We shall use m again to denote the measure.

$\hookrightarrow (\mathbb{R}^n, \mathcal{L}^n, m)$ is the Lebesgue measure

Prop^n. If $E \in \mathcal{L}^n$, then

(1) $m(E) = \inf \{ m(U) : U \text{ open}, U \supseteq E \}$
$= \sup \{ m(K) : K \text{ compact}, K \subseteq E \}$.

(2) Suppose $m(E) < \infty$. Then, $\forall \varepsilon > 0$, $\exists$ a finite collection $\{R_i\}_{i=1}^{n}$ of p.w. disjoint rectangles whose SIDES are open intervals s.t.

$$\mu\left(E \,\Delta\, \bigcup_{i=1}^{n} R_i\right) < 3\varepsilon.$$

Rectangle: $E = E_1 \times \cdots \times E_n$, E_i are called the sides of E.

Another important property of m:

Prop^n. m is translation invariant, i.e., for any $a \in \mathbb{R}^n$:
$E \in \mathcal{L}^n \Rightarrow (E+a) \in \mathcal{L}^n$ and $m(E+a) \in \mathcal{L}^n$.

Moreover, it is rotation invariant.

Lecture 10 (08-02-2021)

08 February 2021 14:02

Def^n. Let $\Omega \subseteq \mathbb{R}^n$ be open.
Suppose $G : \Omega \longrightarrow \mathbb{R}^n$. G is called a C^1-diffeomorphism if G is injective, $D_x G$ is continuous and $D_x G \in GL(n, \mathbb{R})$ $\forall x \in \Omega$.

Thm. Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. Let $f : G(\Omega) \to \mathbb{R}$.

(i) If f is Lebesgue m'ble on $G(\Omega)$, then $f \circ G : \Omega \to \mathbb{R}$ is m'ble.

(ii) If $f \geq 0$ or $f \in \mathcal{L}^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f \, dm = \int_{\Omega} (f \circ G)(x) \, |\det D_x G| \, dm$$

Proof. Suppose Q is a regular cube, i.e., Q is a rectangle whose sides are closed intervals. (Not necessarily of same length!)

We say that Q is centered at a if
$Q_a(h) := Q = \{ \underline{x} : \|\underline{x} - \underline{a}\| \leq h \}$ where for $\underline{x} \in \mathbb{R}^n$, we define $\|\underline{x}\| = \|\underline{x}\|_\infty = \max\limits_{1 \leq i \leq n} |x_i|$.

Suppose $G = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$. By MVT on each g_i, we have

$$g_i(\underline{x}) - g_i(\underline{a}) = \nabla g_i(\xi_i) \cdot (\underline{x} - \underline{a})$$

where ξ_i is on the line seg. joining $\underline{a}$ and $\underline{x}$.

In other words,

$$g_i(\underline{x}) - g_i(\underline{a}) = \sum_j \left(\frac{\partial g_i}{\partial x_j} \right)(\xi_i) \, (x_j - a_j)$$

So,

$$G(\underline{x}) - G(\underline{a}) = \begin{pmatrix} (\nabla g_1)(\xi_1) \\ \vdots \\ (\nabla g_n)(\xi_n) \end{pmatrix} (\underline{x} - \underline{a})$$

$$\begin{pmatrix} \vdots \\ (\nabla g_n)(s_n) \end{pmatrix}$$

Define for a matrix $T = (T_{ij})$,

$$\|T\| := \max_{1 \le i \le n} \sum_j |T_{ij}|.$$

Then, $\|Tx\|_\infty \le \|T\| \cdot \|x\|_\infty$.

Now, we get

$$\|G(x) - G(a)\|_\infty \le h \cdot \left(\sup_{y \in Q} \|D_y G\| \right) \quad \text{for } x \in \underset{Q_a(h)}{Q}.$$

Consequently, $G(Q)$ is contained in cube centered at $G(a)$ with side length $h \cdot \left(\sup_{y \in Q} \|D_y G\| \right)$.

Hence, $m(G(Q)) \le \left(\sup_{y \in Q} \|D_y G\| \right)^n \cdot m(Q)$. — $(*)$

Fix $\varepsilon > 0$: $D_x G$ is continuous on Q, so there exists $\delta > 0$ s.t. whenever $\|x - y\| < \delta$, then $D_x G \approx D_y G$. (??)

$$\left| \det\left((D_y G)^{-1} (D_x G) \right) \right| \le 1 + \varepsilon$$

We now prove the following Prop^n:

Prop^n. Suppose $T \in GL(n, \mathbb{R})$. Then, if $f : \mathbb{R}^n \to \mathbb{R}$ m'sble, so is $f \circ T$. If $f \ge 0$ or $f \in L^1(m)$, then

$$\int f \, dm = |\det T| \int (f \circ T) \, dm.$$

Proof. If the theorem holds for $S, T \in GL(n, \mathbb{R})$. Then it holds for $ST = S \circ T$.

Thus, we may only consider the elementary transformations:

1. $T_1(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, c x_j, \ldots, x_n)$; $c \ne 0$,
2. $T_2(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$,
3. $T_3(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, x_i + x_j, \ldots, x_j, \ldots, x_n)$.

All are simple consequences of Fubini.

$T_1, T_2 \to$ use the fact for 1-dim integral

All are simple consequence of Fubini.

$T_1, T_2 \rightarrow$ use the fact for 1-dim integral

$T_2 \rightarrow$ direct $\blacksquare$

We shall now use the above prop^n to prove the general theorem.

Lecture 11 (11-02-2021)

11 February 2021 14:06

Had seen: $$m(G(Q)) \leq \left(\sup_{y\in Q} \|D_y G\|\right)^n \cdot m(Q). \quad \text{--- } (*)$$

Since the theorem holds for linear maps, replacing G with $T^{-1}\circ G$ in $(*)$, we get

$$m(G(Q)) = |\det T| \, m((T^{-1}\circ G)\, Q) \quad (\text{thm. holds for } T)$$
$$\leq |\det T| \cdot \left(\sup_{y\in Q} \|T^{-1} D_y G\|\right)^n m(Q)$$

Let $\varepsilon > 0$ be given.

Since G is a C^1-diffeo ($D_y G$ is cont. on Ω), there exists $\delta > 0$ s.t. if $\|x - y\| < \delta$, then
$$\|(D_x G)^{-1}(D_y G)\|^n \leq 1+\varepsilon$$

Cut Q into small cubes of length $< \delta$, with centres $x_1, \dots, x_N$. Call these cubes $Q_1, \dots, Q_N$, resp.

Hence,
$$m(G(Q)) \leq \sum_1^N m(G(Q_i))$$
$$\leq \sum_1^N |\det D_{x_i} G| \left(\sup_{y\in Q_i} \|(D_{x_i}G)^{-1}(D_y G)\|\right)^n m(Q_i)$$
$$\leq (1+\varepsilon) \sum_1^N |\det D_{x_i} G| \, m(Q_i)$$

The last expression is the integral of $\sum_{i=1}^N |\det D_{x_i} G| \cdot \mathbb{1}_{Q_i}(x)$

But this last function converges to $\int_Q |\det D_x G| \, dm$ as $\varepsilon, \delta \to 0$.

Hence, we have
$$\boxed{m(G(Q)) \leq \int_Q |\det D_x G| \cdot dm} \quad \to \text{holds for cubes } Q$$

FACT. Any open set is a countable union of cubes whose interiors are pairwise disjoint.

Let U be open. $U = \bigcup_{i=1}^{\infty} Q_i$, where Q_i are cubes with p-wise disjoint interiors.

So,

$$m(G(U)) \leq \sum_{i=1}^{\infty} m(G(Q_i)) \leq \sum_{i=1}^{\infty} \int_{Q_i} |\det D_x G| \, dm$$

MCT

$$= \int_U |\det D_x G| \, dm$$

Now, suppose E is Borel, suppose $m(E) < \infty$. For the Lebesgue measure,

$$m(E) = \inf \{ m(U) : U \supseteq E, \ U \text{ open} \}.$$

In particular, $\exists \{U_j\}_j$ of $\downarrow$ open sets such that $E \subseteq U_j \ \forall j$ and $m\left(\bigcap_{j=1}^{\infty} U_j \setminus E\right) = 0$.

$$m(G(E)) \leq m(G(U_j)) \leq \int_{U_j} |\det D_x G| \, dm \qquad \forall j$$

Taking $j \to \infty$ gives

$$m(G(E)) \leq \lim_j \int_{U_j} |\det D_x G| \, dm$$

DCT

$$= \int_E |\det D_x G| \, dm$$

If $m(E) = \infty$, use σ-finiteness to conclude

$$m(G(E)) \leq \int_E |\det D_x G| \, dm \quad \text{for all borel sets } E.$$

Now, suppose $f = \sum_{j=1}^{N} a_j \mathbb{1}_{E_j}$, where E_j are Borel and $a_j > 0$. (f defined on $G(\Omega)$.)

$$\int_{G(\Omega)} f \, dm = \sum_j a_j m(E_j) \leq \sum_j a_j \int_{G^{-1}(E_j)} |\det D_x G| \, dm$$

$G(\Omega)$ $\quad j \quad\quad G^{-1}(E_j)$

$$\Rightarrow \int_{G(\Omega)} f\,dm \le \int_\Omega (f\circ G)(x)\,|\det D_x G|\,dm. \quad (\dagger)$$

for f simple

By MCT, $(\dagger)$ holds for all $f \geq 0$ measurable.
Since $(\dagger)$ holds for all f and all "nice" G, replace (f, G) in $(\dagger)$ with $(f\circ G, G^{-1})$ to get

$$\int_\Omega (f\circ G)(x)\,|\det D_x G|\,dm \le \int_{G(\Omega)} (f\circ G\circ G^{-1})(x)\,|\det D_x G|\,|\det D_x G^{-1}|\,dm$$

$$= \int_{G(\Omega)} f(x)\,dm \quad\quad (\ddagger)$$

$(\dagger)$ and $(\ddagger)$ together give the equality. If $f\in L^1$, do the same for f^+ and f^-. ∎

Example: $(r, \theta) \xrightarrow{G} (\underset{x}{\underset{\shortparallel}{r\cos\theta}}, \underset{y}{\underset{\shortparallel}{r\sin\theta}})$

$(0,\infty)\times S^1 \to \mathbb{R}^2$

unit circle, can think as $[0, 2\pi)$.

$$D_{(r,\theta)}G = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det D_{(r,\theta)}G = r$$

So, $$\int_\Omega f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}_{\text{what measure?}} = \int_{G(\Omega)} f(x,y)\,dm.$$

An induced measure on S^{n-1}:

$$\Phi : \mathbb{R}^n\setminus\{0\} \to (0,\infty)\times S^{n-1}$$

$$\underline{x} \mapsto (r, \underline{x}') \quad \text{where } r = |\underline{x}| = \|\underline{x}\|_2, \quad x' = \frac{\underline{x}}{r}.$$

$\Phi^{-1}(r, x') = rx'$. Φ is a homeomorphism.

This suggests that we can pull the Borel measure on $\mathbb{R}^n$ onto $(0,\infty) \times S^{n-1}$ via the pull back:

If $E \subseteq (0,\infty) \times S^{n-1}$ is Borel (use product topology), define

$$m_*(E) := m(\Phi^{-1}(E)).$$

It is easy to check that m_* is a measure on $(0,\infty) \times S^{n-1}$.

Thm. There exists a UNIQUE Borel measure σ on S^{n-1} such that $m_* = \rho \times \sigma$ where ρ on $(0,\infty)$ is defined by $\rho(E) = \int_E r^{n-1}\, dr$. If f is Borel on $\mathbb{R}^n$ and $f \geq 0$ or $f \in L^1(m)$, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{S^{n-1}} f(rx')\, r^{n-1}\, d\sigma(x')\, dr.$$

Recall that if $E \subseteq \mathbb{R}^n$ is Lebesgue meas., then for any $a > 0$, $m(aE) = a^n m(E)$.

Let $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$, $m(B) = C_n$

Hence, $B_r = \{x \in \mathbb{R}^n : |x| \leq r\}$ has $m(B_r) = C_n \cdot r^n$.

The shell at radius r and thickness $\delta < r$ has

Lebesgue vol $= C_n \cdot r^n - C_n (r-\delta)^n$

$= C_n \cdot n r^{n-1} + \delta(\quad)$

Proof (of thm.) We construct such a σ. Let $E \subset S^{n-1}$ be Borel.

For $a > 0$, let

$$E_a := \Phi^{-1}((0,a] \times E)$$

$m(E_1) \overset{\text{def}^n}{=} m_*((0,1] \times E)$ forced, if we want $m = \rho \times \sigma$

$= \rho(0,1] \times \sigma(E)$

$\overset{\text{def}^n}{=}$ $\rho(0,1] \times \sigma(E)$ want $m = \rho \times \sigma$

$\sigma(E)/n$

Thus, we define $\sigma(E) := n\, m(E_1)$.
Note that for any E Borel in S^{n-1}, $E \mapsto E_1$ is Borel-preserving and commutes under $\cup$, $\cap$, complements.
Thus, this <u>DOES</u> define a measure on S^{n-1}!

Now, we need to show $m_* = \rho \times \sigma$ on all Borel subsets of $(0,1) \times S^{n-1}$. For this, it suffices to show that for all $A \subseteq (0,\infty)$ Borel and $E \subseteq S^{n-1}$ Borel.

$$m_*(B) = (\rho \times \sigma)(B) \quad \text{—} (\#)$$

($B \subseteq (0,\infty) \times S^{n-1}$ is Borel.)

So far, we have:

(i) $\sigma(E)$ is defined

(ii) (#) holds if $A = (0,1]$

If $f = (0, a]$, then note since $(0,a] \times E = a \cdot ((0,1] \times E)$,

it follows that

$$m(E_a) = a^n \cdot m(E_1) = \left(\int_{(0,a]} r^{n-1}\, dr \right) \cdot \sigma(E).$$

So, (#) holds for $A = (0, a]$.

If $0 < a < b$,

$$m_*((a,b] \times E) = m(E_b \setminus E_a) = \frac{b^n}{n}\sigma(E) - \frac{a^n}{n}\sigma(E)$$

$\Rightarrow$ (#) for $(a,b] \times E$ as well.

Let $\mathcal{A}_E$ = finite disj. union of $(a,b] \times E$

↳ algebra on $(0,\infty) \times E$

Generates $\mathcal{M}_E = \{A \times E : A \in \mathcal{B}_{(0,\infty)}\}$.

We have $\sigma_* = \rho \times \sigma$ on $\mathcal{A}_E$ and hence, on $\mathcal{M}_E$.

$\bigcup_{E \in \mathcal{B}(S^{n-1})} \mathcal{M}_E$ = set of Borel rect.s on $(0,\infty) \times S^{n-1}$

Thus, $m_* = \rho \times \sigma$ on all Borel sets. $\blacksquare$

Cor. If $g : \mathbb{R}^n \to \mathbb{R}$ is radial, i.e., $|x| = |y| \Rightarrow g(x) = g(y)$, then

$$\int_{\mathbb{R}^n} g(x)\, dm = \sigma(S^{n-1}) \int_0^\infty g(r)\, r^{n-1}\, dr.$$

Ex. $\sigma(S^{n-1}) = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$, where $\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\, dx$.

Lecture 12 (15-02-2021)

15 February 2021 14:05

Differentiation

Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Suppose $f \in L^+$.

Consider $\nu : \mathcal{M} \to [0, \infty)$ defined by

$$\nu(E) = \int_E f \, d\mu = \int f \cdot \mathbb{1}_E \, d\mu.$$

Easy to see that ν is also a measure.

Not all measures can be obtained this way though.

Ex On $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, put the pre-measure

$$\rho(\{k\}) := \begin{cases} 1/2^k & \text{for } k \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

ρ gives rise to a measure ρ (by abuse of notation) which is "singular w.r.t. m".

Let $E = \mathbb{N}$ and $F = \mathbb{R} \setminus E$.

$m(E) = 0 = \rho(F)$.

Another suggestive point: If $f \in L^1$ (not necessarily ≥ 0), we can still define

$$\nu(E) := \int_E f \, d\mu \quad \text{is a function}$$

$$\nu : \mathcal{M} \to \mathbb{R} \quad \text{satisfying}$$

① $\nu(\phi) = 0$,

② If $\{E_i\}_{i=1}^{\infty}$ is a countable disjoint collection of sets in $\mathcal{M}$, then $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$. (From DCT)

Signed measures

Let $(X, \mathcal{M})$ be a set with a σ-alg. Then, a signed measure is a map $\nu : \mathcal{M} \to [-\infty, \infty]$ satisfying

1. $\nu(\emptyset) = 0$,
2. ν takes at most one of $-\infty$, ∞,
3. If $\{E_j\}_{j=1}^{\infty}$ is a countable disjoint collection, then $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$. Moreover, if this is finite, then the series converges absolutely.

Remark We shall, by "measure", mean the usual measures, which we sometimes call "positive measures".

① A more general example for signed measures goes thus. Suppose $f : X \to \mathbb{R}$ is s.t. at least one of $\int f^+$ or $\int f^-$ is finite. Then again,

$$\nu(E) = \int_E f \, d\mu \text{ defines a signed measures.}$$

② Suppose μ_1, μ_2 are two (positive) measures and suppose at least one of $\mu_1(X), \mu_2(X) < \infty$.

Define

$$\nu(E) := \nu_1(E) - \nu_2(E).$$

Then, ν is another example.

① and ② are "effectively" the only ones.

Thm Jordan decomposition theorem

If ν is a signed measure, then there exist UNIQUE positive measures ν^+ and ν^- s.t.

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-.$$

Def^n. We say that (signed) measures μ, ν are singular with respect to each other if there exist E and F s.t.
$E \cup F = X$ and E is null for μ and F for ν.
$(E \cap F = \phi)$

$(\forall A \subseteq E, \ \mu(A) = 0.)$

This is denoted $\mu \perp \nu$. $(A, E, F \in \mathcal{M})$

Def^n. Suppose ν is a signed measure on $(X, \mathcal{M})$.
A set $E \in \mathcal{M}$ is positive for ν, if for all $A \subseteq E$ with $A \in \mathcal{M}$, $\nu(A) \geq 0$.
Similarly, we define negative sets.

(Positive set, negative set)

Thm. Hahn Decomposition Theorem

If ν is a signed measure, there exist $P, N \in \mathcal{M}$ s.t. P is positive and N is negative for ν with $P \cap N = \phi$ and $P \cup N = X$.
If (P', N') is another such pair, then $P \Delta P'$ and $N \Delta N'$ are null for ν.

Proof. Wlog, suppose ν does not take $-\infty$.
Let $D \in \mathcal{M}$ be s.t. $\nu(D) \leq 0$.
Claim. $\exists A \subseteq D$ s.t. A is negative for ν.

Proof. Let $A_0 = D$.
Having defined $A_n \subset A_{n-1} \subset \cdots \subseteq D$, let
$$t_n = \sup\{\nu(B) : B \in \mathcal{M}, \ B \subseteq A_n\}.$$
$t_n \geq 0$ since $\phi = B$ is here.

let $B_n \subseteq A_n$ be s.t. $\mu(B_n) \geq \min\{\frac{t_n}{2}, 1\}$

Set $A_{n+1} = A_n \setminus B_n$ and finally consider

$$A = D \setminus \left(\bigcup_0^\infty B_n \right).$$

First, note that $\{B_n\}$ are disjoint. Thus, by defⁿ of measure,

$$\nu(A) = \nu(D) - \sum_0^\infty \nu(B_n).$$

We now show that A is negative.

If not, suppose $A_0 \subseteq A$ with $\nu(A_0) > 0$.

Then, note that $t_n \geq \nu(A_0)$ or

$$\frac{t_n}{2} \geq \frac{\nu(A_0)}{2} \leftarrow \text{fixed}.$$

But then, $\nu(A) = \nu(D) - \sum_0^\infty \nu(B_n) = -\infty$.

But this was ruled by assumption.

This proves the Claim.

We now construct a negative set N. Let $N_0 = \phi$.

Having constructed N_n, let

$$S_n := \inf \{ \nu(D) : D \in \mathcal{M}, \ D \subseteq X \setminus N_n \}.$$

As before $S_n \leq 0$. $\because \phi$ ↺

Again as before, let $D_n \subseteq X \setminus N_n$ s.t.

$$\nu(D_n) \leq \max \left\{ \frac{S_n}{2}, -1 \right\} \leq 0.$$

Let $A_n \subseteq D_n$ be as in prev. part, i.e., $A_n \subset D_n$ is negative à la the previous construction.

Set $N_{n+1} = N_n \cup A_n$ and let $N = \bigcup_0^\infty A_n$.

Note that $\mu(A_n) \leq 0$, and moreover A_n are p-wise disjoint.

If $E \subseteq N$, $E \in \mathcal{M}$, $E = \bigcup_0^\infty (E \cap A_n)$ and so,

$$\nu(E) = \sum_{0}^{\infty} \nu(E \cap A_n) \leq 0.$$

$\Rightarrow$ N is negative.

Let $P = X \setminus N$. We now claim that P is positive w.r.t. ν.

Suppose not. Let $E \subseteq P$ be s.t. $\nu(E) < 0$.

Again, in the def^n of the inf of s_n, E is candidate set. Thus, $s_n \leq \mu(E)$ $\forall n$.

Now, $$\nu(N) = \sum_{0}^{\infty} \nu(A_n) \leq \sum_{0}^{\infty} \max\left(\frac{\nu(E)}{2}, -1\right) = -\infty.$$

$\rightarrow \leftarrow$

Hence, P is positive.

Finally, if (P', N') is another such pair for ν, then note that $P \cap N'$, $P' \cap N$ are both +ve & -ve sets. Thus, ν is null on these sets and hence on their union, which is $P \Delta P'$. ☒

Corollary. Jordan decomposition. (Given ν: $\exists\, \nu^+, \nu^-$ s.t. $\nu = \nu^+ - \nu^-$.)

Proof. Given ν, get P and N as above.

For $E \in \mathcal{M}$, define $\nu^+(E) := \nu(E \cap P)$ and
$\nu^-(E) := -\nu(E \cap N)$.

Easy to see that ν^+, ν^- are (positive) measures and that for any E, $\nu(E) = \nu^+(E) - \nu^-(E)$.

Note that if μ^+ and μ^- are s.t. $\nu = \mu^+ - \mu^-$, then their positive-negative sets give another Hahn decomposition (P', N') for X, and so by the (essential) uniqueness of Hahn-decomposition, the uniqueness for ν^+, ν^- follows.

Finally, we need to check $\nu^+ \perp \nu^-$ but that follows directly. ∎

Def^n. Given signed measures μ and ν on $(X, \mathcal{M})$, we say that ν is absolutely continuous with respect to μ if $\mu(E) = 0 \Rightarrow \nu(E) = 0$. This is denoted as

$$\nu \ll \mu.$$

- Recall the signed meas. $\nu(E) := \int_E f \, d\mu$.

  For this, we have $\nu \ll \mu$.

Thm. Radon-Nikodym Theorem

Let ν be a σ-finite signed measure and suppose μ is σ-finite positive on $(X, \mathcal{M})$. Then, there exist σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ s.t.

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, there exists $f : X \to \mathbb{R}$ which is μ-integrable (in the extended sense) s.t.

$$\rho(E) = \int_E f \, d\mu.$$

Lastly, this decomposition is unique μ a.e.

Lecture 13 (18-02-2021)

18 February 2021 14:05

A digressive remark: Consider $([0,1], \mathcal{L}, m)$. It is obvious than an arbitrary union of $\mathcal{L}$-null sets need not be null? But what about an arbitrary increasing family? Is their union null?

Ans. Not necessary.

Proof. Suppose not. Suppose every chain of null sets has null union. Consider

$$\mathcal{N} = \{E \subseteq [0,1] : E \text{ is null}\}.$$

Then our assumption gives us that $\mathcal{N}$ has a maximal element E_0. (Zorn's Lemma)

This is absurd. ($E_0 \neq [0,1]$ clearly but then $\{x\} \cup E \supsetneq E$ for $x \in [0,1] \setminus E$.)

Prop^n. Suppose μ, ν are finite (positive) measures on $(X, \mathcal{M})$. Then either $\nu \perp \mu$ or $\exists E \in \mathcal{M}$ and $\epsilon > 0$ s.t.

$$\nu \geq \epsilon \mu \text{ on } E, \text{ and } \mu(E) > 0.$$

Proof. For $n \in \mathbb{N}$, consider $\nu_n := \nu - \frac{1}{n}\mu$. Let (P_n, N_n) be the Hahn-Decomposition for ν_n.

Let $N = \bigcap_{n=1}^{\infty} N_n$ and $P = N^c$.

Note that for $\overset{\in \mathcal{M}}{E} \subseteq N$, $\left(\nu - \frac{1}{n}\mu\right) E \leq 0 \quad \forall n$

and thus, $\nu(E) \leq 0$ but ν is a pos. measure and hence, $\nu\big|_N \equiv 0$.

So, if $\mu(P) = 0$, then $\nu \perp \mu$.

Else, $\mu(P) > 0$

Since $P_n \uparrow$, $\mu(P_n) > 0$ for some n.

$\Rightarrow P_n$ is a +ve set for ν_n. ∎

Proof (of Radon-Nikodym)

Assume first that ν and μ are finite, positive.

Consider

$$\mathcal{F} := \left\{ f : X \to [0,\infty] : \int_E f\,d\mu \le \nu(E)\ \forall E \in \mathcal{M} \right\}.$$

$\mathcal{F} \neq \emptyset$ since $f \equiv 0$ is in $\mathcal{F}$.

FACT. If $f, g \in \mathcal{F}$, then $f \vee g := \max\{f, g\}$ is also in $\mathcal{F}$.

Proof. Let $A := \{x : f(x) > g(x)\}$

$$\int_E (f\vee g)\,d\mu = \int_{E\cap A} f\vee g + \int_{E\cap \bar{A}} f\vee g = \int_{E\cap A} f + \int_{E\cap\bar{A}} g$$

$$\le \nu(E\cap A) + \nu(E\cap\bar{A}) = \nu(E).$$

Let $a = \sup\left\{ \int f\,d\mu : f \in \mathcal{F} \right\} \le \nu(X) < \infty.$

$\left(\int f\,d\mu \le \nu(X) \quad \forall f \in \mathcal{F} \right)$

Let $\{f_n\} \subseteq \mathcal{F}$ s.t. $\int f_n\,d\mu \longrightarrow a$.

Let $g_n := f_1 \vee \dots \vee f_n$. Then,

① $g_n \in \mathcal{F}$

② $g_n \uparrow$ pointwise to $f := \sup f_n$.

Note that $\int g_n\,d\mu \longrightarrow a$ by Sandwich. By MCT,

$$\int f\,d\mu = \lim_n \int g_n\,d\mu = a.$$

(In particular, $f < \infty$ μ. a.e.)

Define $\rho(E) := \int_E f\,d\mu$ and let $\lambda = \mu - \rho$.

Note $\lambda \geq 0$ by construction of $\mathcal{F}$.

Claim. $\lambda \perp \mu$.

Proof. Suppose not. Then, by the prev. prop^n, $\exists E \in \mathcal{M}$, $\varepsilon > 0$ s.t. $\mu(E) > 0$ and $\lambda \geq \varepsilon\mu$ on E.

Claim Consider the function $g := f + \varepsilon \mathbb{1}_E$. Then $g \in \mathcal{F}$ and

$$\int g \, d\mu = \int f \, d\mu + \varepsilon \mu(E) > a,$$ giving a contradiction.

Proof. To show $g \in \mathcal{F}$: $\forall F \in \mathcal{M}$, want to show:

$$\int_F g \, d\mu \leq \nu(F).$$

By def^n of E, $\lambda(E) = \nu(E) - \int_E f \, d\mu$ satisfies $\lambda \geq \varepsilon\mu$ and so,

$$\nu(E) \geq \int_E f \, d\mu + \varepsilon.$$

The conclusion now follows by splitting the integral over $F \cap E$ and $F \cap E^c$. ⊠

Thus, $\lambda \perp \mu$. ⊠

UNIQUENESS Suppose

$$\nu = \lambda + f\,d\mu$$
$$= \lambda' + f'\,d\mu \quad \text{s.t.}$$

$\lambda, \lambda' \perp \mu$ and f, f' are integrable.

But note that if $\lambda, \lambda' \perp \mu$, then so is $\lambda - \lambda' \perp \mu$. (1)

But $\lambda - \lambda' = (f' - f)\,d\mu$.

$\Rightarrow \lambda - \lambda' \ll \mu$ — (2)

(1) and (2) $\Rightarrow \lambda = \lambda' \Rightarrow f = f'$ μ a.e.

This establishes Radon-Nik. for $\mu, \nu \geq 0$ and finite.

Now, suppose $\mu, \nu \geq 0$ with μ, ν σ-finite.

Get a sequence $\{A_j\}$ of p-wise disjoint sets s.t.

$$\bigsqcup_j A_j = X \text{ and both } \mu, \nu < \infty \text{ on } A_j \; \forall j.$$

Define $\nu_j := \nu|_{A_j}$ and $\mu_j = \mu|_{A_j}$. Then, by the earlier part, gives f_j s.t.

$$\nu_j = \lambda_j + f_j \, d\mu_j \quad \forall j.$$

Take $\lambda = \sum \lambda_j$ and $f = \sum_j f_j$ and check that it works. The main points here are the following:

1. If $\{\lambda_j\} \to$ +ve measures and $\mu \geq 0$ and $\lambda_j \perp \mu$, so is $\sum_{j=1}^{\infty} \lambda_j$.

2. $\lambda_j \ll \mu \;\; \forall j \;\; \Rightarrow \;\; \sum \lambda_j \ll \mu$.

Finally, if ν is signed, write $\nu = \nu^+ - \nu^-$ from the Jordan decomposition and argue individually.

Notation. If $\lambda \ll \mu$, then $\lambda(E) = \int_E f \, d\mu$ for some f which is μ- integrable.

We shall denote this f by $\dfrac{d\lambda}{d\mu}$.

This is called a Radon- Nikodym derivative.

(Radon-Nikodym derivative)

Remark. If (say) ν is not σ- finite, the theorem is not true.
Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}[0,1]$, $\mu = m$ (Lebesgue)
$\nu \equiv$ counting

Then, clearly $m \ll \nu$ but $m \neq f \, d\nu$ for any f which is ν integrable

Prop^n. Suppose ν is σ-finite, signed and suppose μ, λ are σ- finitive +ve measures. Suppose $\nu \ll \mu$, $\mu \ll \lambda$. Then,

1. if $g \in L^1(\nu)$, then
$$\int g \, d\nu = \int g \, \frac{d\nu}{d\mu} \cdot d\mu$$

$$\int g \, d\nu = \int g \frac{d\nu}{d\mu} \cdot d\mu$$

② $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}.$$ (Chain rule)

DIFFERENTIATION Á LA LEBESGUE

We shall restrict ourselves to the Lebesgue measure m on $(\mathbb{R}^n, \mathcal{B})$ (or $(\mathbb{R}^n, \mathcal{L})$).

Suppose $\nu \ll m$. So, $\frac{d\nu}{dm}$ exists.

1-dim case. $F(x) = \int_a^x f(t)\,dt \leftarrow$ think of this as $\nu([a,x])$

$$\lim_{x \to a} \frac{\nu((a,x])}{m((a,x])} = f(a).$$

2-dim?

Lecture 14 (04-03-2021)

04 March 2021 14:00

$f \in L^1_{loc}(m) \rightarrow$ Read def^n after thm

Thm. Suppose $x \in \mathbb{R}^n$ and ~~$f \in L^1(m)$.~~ Define

$$(A_r f)(x) := \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dm.$$

$$\left(B(x,r) := \{ y \in \mathbb{R}^n : \|x - y\|_2 < r \}. \right)$$

Then, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} (A_r f)(x) = f(x).$$

Note that asking $f \in L^1(m)$ is too much. Don't need $\int_{\mathbb{R}^n} f$ to exist.

Def^n. $L^1_{loc}(m) = \{ f \text{ m'sble s.t. } \int_K |f| < \infty \text{ for all bounded } K \}$.

Remark Suppose f is continuous at x. Let $\varepsilon > 0$ be arbit.
Then $\exists \delta > 0$ s.t. whenever $\|x - y\| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Note

$$|(A_\delta f - f)(x)| \leq \frac{1}{m(B(x,\delta))} \int_{B(x,\delta)} |f(y) - f(x)| \, dy$$

$$\leq \varepsilon$$

This suggests the following:

Prop^n. Given $f \in L^1(m)$, given any $\varepsilon > 0$, $\exists$ g continuous s.t.

$$\int |f - g| \, dm < \varepsilon.$$

Proof Exercise. ☒

The main tool for proving the theorem is an estimate due to <u>Hardy - Littlewood</u>.

Def^n. Maximal function

Given $f \in \mathcal{L}^1_{loc}(m)$,

$$(Hf)(x) := \sup_{r>0} (A_r|f|)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| \, dm.$$

Thm. (Maximal theorem) Given any $\alpha > 0$, there exists an absolute constant $c = C_\alpha > 0$ s.t. for all $f \in L^1(m)$,

$$m\{x : (Hf)(x) > \alpha\} \le \frac{c}{\alpha} \int |f| \, dm = \frac{c}{\alpha} \|f\|_1.$$

One immediate consequence is that $Hf < \infty$ a.e.

Q. Why is Hf measurable?

$$\{x : (Hf)(x) > \alpha\} = \bigcup_{r>0} (A_r|f|)^{-1}(\alpha, \infty).$$

The measurability of Hf follows from the observation that the function $(r,x) \mapsto A_r f(x)$ is continuous in both variables.

Recall $$(A_r f)(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f \cdot dm$$

Note that $m(B(x,r)) = c \cdot r^n$ where $c = m(B(0,1))$.

$$\therefore \quad (A_r f)(x) = \frac{1}{C\cdot r^n}\int_{B(x,r)} f\cdot dm = \frac{1}{Cr^n}\int_{\mathbb{R}^n} f\cdot \mathbb{1}_{B(x,r)}\, dm.$$

Now, if $(x_n, r_n) \longrightarrow (x_0, r_0)$, then

$$f\cdot \mathbb{1}_{B(x_n,r_n)} \longrightarrow f\cdot \mathbb{1}_{B(x_0,r_0)} \quad \text{outside } \underset{m=0}{\partial B(x_0,r_0)}$$

The continuity now follows from DCT.

Proof of thm 1 using Maximal Thm.

WLOG, we may assume $f \in \mathcal{L}^1(m)$. Given $\varepsilon > 0$, get g continuous s.t.

$$\int |f-g|\,dm < \varepsilon.$$

lim sup

$$\overline{\lim_{r\to 0}}\, |(A_r f - f)(x)| = \overline{\lim_{r\to 0}}\, |A_r(f-g)(x) + (A_r g - g)(x) + (g-f)(x)|$$

$$\leq \overline{\lim_{r\to 0}}\, |A_r(f-g)(x)| + \overline{\lim_{r\to 0}}\, |\overset{\to 0}{(A_r g - g)(x)}| + |(g-f)(x)|$$

$$\leq H(f-g)(x) + |(f-g)(x)|.$$

Suppose $\alpha > 0$. let

$$E_\alpha := \{x : \overline{\lim_{r}}\, |A_r f - f| > \alpha\} \quad \text{and}$$

$$F_\alpha := \{x : H(f-g)(x) > \alpha\}.$$

Then, $E_\alpha \subseteq F_{\alpha/2} \cup \{x : |f(x) - g(x)| > \alpha/2\}$.

Note $m(\{x : |f(x) - g(x)| > \alpha/2\}) \leq \frac{2}{\alpha}\|f-g\|_1 < \frac{2\varepsilon}{\alpha}$.

Now, $m(F_{\alpha/2}) \leq \frac{2c}{\alpha}\|f-g\|_1 = \frac{2c}{\alpha}\varepsilon$.

$\therefore \quad m(E_\alpha) \leq \dfrac{2\varepsilon}{\alpha}(c+1).$ Note c is constant.

Since $\varepsilon > 0$ is arbit, $m(E_\alpha) = 0$.

Hence, $E = \bigcup_{n \in \mathbb{N}} E_{1/n}$ has measure 0.

On E^c, we have $\lim_{r \to 0} A_r f(x) = f(x)$. ☒

Lemma. (Vitali-Covering Type Argument) Suppose $U \subseteq \mathbb{R}^n$ is an open set and let $\mathcal{C}$ be an open cover for U by open balls.
If $m(U) > c > 0$, then there exists k and balls $B_1, \ldots, B_k \in \mathcal{C}$ s.t.

1. The B_i s are p-wise disjoint.
2. $\sum_{i=1}^{k} m(B_i) \geq c/3^n$.

Proof. Since $U \subseteq \mathbb{R}^n$ is open, we know

$$m(U) = \sup\{m(K) : K \subseteq U, K \text{ compact}\}$$

In particular, let $K \subseteq U$ be compact s.t.

$$m(K) > c.$$

Since $\mathcal{C}$ is an open cover for K, it admits a finite subcover $\mathcal{B} = \{B_{i_1}, \ldots, B_{i_N}\}$. Assume they are listed in decreasing radii.

Let $A_1 = B_{i_1}$. Assume $A_1, \ldots, A_j$ chosen. Let A_{j+1} be with largest radius in $\mathcal{B}$ disjoint from $A_1, \ldots, A_j$.
This stops after a point. Suppose $A_1, \ldots, A_k$ are picked.

For any $B_{i_j} \notin \{A_1, \ldots, A_k\}$, let A_i be the ball of largest radius s.t. $B_{i_j} \cap A_i \neq \emptyset$.
From our picking criteria, it follows that radius of B_{i_j} is $\leq$ rad (A_i).
In particular,

$A_i^* \equiv$ centered at center of A_i, but radius $= 3\,\text{rad}(A_i)$

contains B_{ij}.

Hence, $B_{ij} \subseteq \bigcup_{i=1}^{k} A_i^*$.

The result now follows.

Proof of the maximal thm.

TS: $\exists c > 0$ s.t. $\forall f \in L^1(m)$

$$m\{x : Hf(x) > \alpha\} \leq \frac{c}{\alpha}\|f\|_1.$$

$(Hf)(x) = \sup_{r>0} A_r|f|(x)$

Let $E_\alpha = \{x : Hf(x) > \alpha\}$.

For $x \in E_\alpha$, $\exists r_x > 0$ s.t.

$(A_{r_x}|f|)(x) > \alpha$. In particular, let $\mathcal{C} = \{B(x, r_x) : x \in E_\alpha\}$.

Then, $\mathcal{C}$ is an open cover for E_α with open balls.

By earlier lemma, there A_i in $\mathcal{C}$ $(1 \leq i \leq k)$ s.t. if

$c < m(E_\alpha)$, then

$$\frac{c}{3^n} < \sum_{i=1}^{k} m(A_i).$$

Note that $A_r|f|(x) = \frac{1}{m(B(x,r))}\int_{B(x,r)} |f|\, dm > \alpha$

$$\Downarrow$$

$$m(B(x,r)) < \frac{1}{\alpha}\int_{B(x,r)} |f|\, dm.$$ "This"

"This" holds for each A_i in the preceeding line.

So, $\frac{c}{3^n} < \sum_{i=1}^{k} m(A_i) < \frac{1}{\alpha}\sum_{i=1}^{k}\int_{B_i} |f|\, dm$

$\leq \frac{1}{\alpha}\sum^{k}\int |f|\, dm = \frac{\|f\|_1}{\cdot}$

$$\leq \frac{1}{\alpha} \sum_{i=1}^{k} \int_{\mathbb{R}^n} |f| \, dm = \frac{\|f\|_1}{\alpha}$$

$$\Rightarrow \quad c < \frac{3^n}{\alpha} \|f\|_1.$$

$\left(\frac{3^n}{\alpha} \text{ is the } c\right)$ So, taking $c \uparrow m(E_\alpha)$ we are through. ▣

We have shown:

Thm. If $f \in L^1_{loc}(m)$, then for a.e. $x \in \mathbb{R}^n$

$$\lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(x,r)} (f(y) - f(x)) \, dy = 0.$$

In fact, more is true:

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

Def^n. For $f \in L^1_{loc}$, the Lebesgue set of f is defined as

$$L_f := \left\{ x : \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0 \right\}.$$

(Lebesgue set)

Stronger thm. For $f \in L^1_{loc}(m)$, $m(L_f^c) = 0$.

Proof. For any $c \in \mathbb{R}$, Consider $g_c(x) := |f(x) - c|$.
Applying the theorem to g_c, we get a measure zero set E_c s.t. $\forall x \notin E_c$:

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - c| \, dy = |f(x) - c|.$$

Let $E = \bigcup_{c \in \mathbb{Q}} E_c$ so that $m(E) = 0$. For any $x \notin E$

and $\epsilon > 0$, pick $c \in \mathbb{Q}$ s.t. $|f(x) - c| < \epsilon$.

$\Rightarrow \quad |f(y) - f(x)| \leq |f(y) - c| + \epsilon$

$$\Rightarrow \overline{\lim_{r \to 0}} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dx \leq \frac{1}{m(\ )} \int_{B(\ )} |f(y) - c| \, dy + \epsilon$$

$\leq 2\epsilon.$ ∎

Lecture 15 (08-03-2021)

08 March 2021 14:10

Def. (Shrink nicely) A family $\{E_r\}_{r>0}$ of Borel sets is said to shrink nicely to x if

① $E_r \subseteq B(x, r) \quad \forall r$

② $\exists\, \alpha > 0$ s.t. $m(E_r) \geq \alpha\, m(B(x, r))$.

Note that $x \in E_r$ is not necessary.

Thm. (Lebesgue differentiation theorem for nicely shrinking sets)

Suppose $f \in L^1_{loc}$. Then, for $x \in L_f$,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\, dy = 0.$$

(Where $\{E_r\}_{r>0}$ is any family shrinking nicely to x.)

Consequently, $\displaystyle \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\, dy = f(x).$

Proof. Let $\alpha > 0$ be as in the defn for $\{E_r\}_{r>0}$ to be nicely shrinking.

$$\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\, dy \leq \frac{1}{m(E_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy$$

$$\leq \frac{1}{\alpha} \cdot \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy \xrightarrow{r \to 0} 0. \quad \blacksquare$$

Defn. ① A Borel measure $\nu \geq 0$ on $\mathbb{R}^n$ is called regular if

(i) $\nu(K) < \infty \quad \forall\, K$ compact,

(ii) $\nu(E) = \inf\{\nu(U) : E \subseteq U,\ U \text{ open}\}$.

(regular)

② A signed measure ν is regular iff $|\nu|$ is regular.

$\left(\text{If } \nu = \nu^+ - \nu^-; \ \nu^+, \nu^- \text{ are +ve measures, then } |\nu| = \nu^+ + \nu^- . \right)$

Thm. Suppose ν is a signed Borel measure on $\mathbb{R}^n$ and suppose $\nu = \lambda + f\,dm$ is its Radon–Nikodym decomposition w.r.t. m. Then, for a.e. $x \in \mathbb{R}^n$ (w.r.t. m), we have

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x) \quad \text{for any family } \{E_r\}_{r>0}$$

shrinking nicely to x.

Proof. ① $\nu = \lambda + f\,dm$

$\Rightarrow |\nu| = |\lambda| + |f|\,dm$ (Ex.)

② ν regular $\Rightarrow$ λ, $f\,dm$ are also regular.

So, it suffices to show $\displaystyle\lim_{r\to 0} \frac{\lambda(E_r)}{m(E_r)} = 0.$

Again, it is sufficient to work with $E_r = B(x, r)$.

Also, $\left| \dfrac{\lambda(E_r)}{m(E_r)} \right| \leq \dfrac{|\lambda|(E_r)}{m(E_r)}$. Thus, we may assume $\lambda \geq 0$.

So, it suffices to prove:

If $\lambda \geq 0$, regular and $\lambda \perp m$, then

$$\lim_{r \to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = 0.$$

Since $\lambda \perp m$, $\exists A$ $\xrightarrow{}$ Borel s.t. $m(A^c) = \lambda(A) = 0$.

Let $F_k = \left\{ x \in A : \overline{\lim_{r\to 0}} \frac{\lambda(B(x,r))}{m(B(x,r))} > 1/k \right\}$ for $k \in \mathbb{N}$.

We show each F_k and hence, $\bigcup_{k\geq 1} F_k$ has measure 0.

Now, let $\varepsilon > 0$ be given. By regularity of λ, $\exists\, U_\varepsilon \supseteq A$ s.t.
$\lambda(U_\varepsilon) \leq \varepsilon$. $(\lambda(A) = 0.)$

Also, each $x \in F_k$ is the center of some ball $B_x \subseteq U_\varepsilon$ s.t.

$$\lambda(B_x) > \frac{1}{k} m(B_x).$$

Let $V_\varepsilon = \bigcup_{x\in F_k} B_x$. By the covering lemma, $\exists\, x_1, \dots, x_N$ and

corresp. balls $B_1, \dots, B_N$ (p-wise disjoint) s.t. if $c < m(V_\varepsilon)$, then $\sum_{i=1}^{n} m(B_i) > \frac{c}{3^n}$.

In particular,

$$c < 3^n \sum_1^N m(B_i) < 3^n k \sum_{i=1}^N \lambda(B_i) \leq 3^n k\, \lambda(U_\varepsilon) \leq 3^n k \varepsilon.$$

Since ε is arbitrary, $c = 0$. ☒

Functions of Bounded Variations and the
Fundamental Theorem of Calculus for Lebesgue Integrals

Recall: Distribution function for regular Borel measures on $\mathbb{R}$.

μ is a (positive) regular measure: define for $x \in \mathbb{R}$,

$$F(x) := \mu(-\infty, x].$$

F is $\uparrow$ and right continuously. Conversely, ...

· Suppose ν is a signed measure on $\mathbb{R}$.

$$\nu = \nu^+ - \nu^-$$

Each ν^+, ν^- are +ve measures on $\mathbb{R}$.

$\nu^+ \longleftrightarrow F^+$; $F \longleftrightarrow \nu$. What is F? $F = F^+ - F^-$?

$\nu^- \longleftrightarrow F^+$

Let us assume that ν is a finite signed measure on $\mathbb{R}$

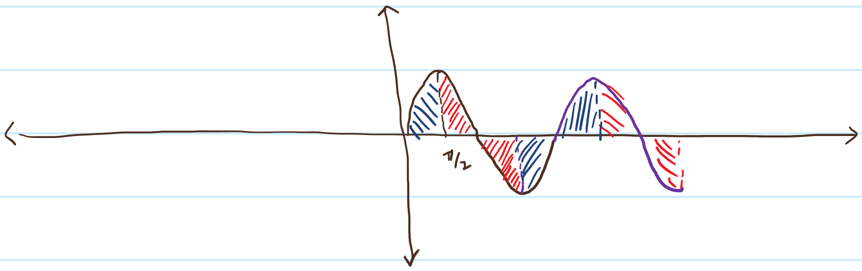
$$\nu(-\infty, x] = \nu^+(-\infty, x] - \nu^-(-\infty, x]$$
$$= F^+(x) - F^-(x) = F(x).$$

Q. Which F correspond to regular ν.

"Can be written as a diff of $\uparrow$ right cts. functions" is not good.

Example. $F(x) = \sin x$.

Is F the distribution of some regular ν?



Def^n For $F : \mathbb{R} \to \mathbb{R}$, the total variation function of F (denoted T_F) is defined as

$$T_F(x) := \sup \left\{ \sum_1^n |F(x_i) - F(x_{i-1})| : x_0 < x_1 < \cdots < x_n = x \right\}.$$

(Total variation function, bounded variation)

A function $F : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation (denoted $F \in BV$) if $\lim_{x \to \infty} T_F(x) < \infty$.

Remarks (1) $T_F \uparrow$

(2) One can define $BV[a, b]$ for $F : [a, b] \to \mathbb{R}$ via the same manner. $F \in BV \Rightarrow F \in BV[a, b]$ $\forall [a, b]$

Furthermore, the total variation of F in $[a, b]$ is

$$T_F(b) - T_F(a).$$

③ If $F \in BV[a,b]$ $\forall [a,b]$, then F need not be in BV.

Ex. $F(x) = \sin x \notin BV$. (In fact, $T_F \equiv \infty$.)

But $F \in BV[a,b]$ $\forall [a,b]$.

Lemma. If $F \in BV$, then $T_F + F$ and $T_F - F$ are both increasing.

Pf. Suppose $x < y$. WTS: $(T_F + F)(x) \leq (T_F + F)(y)$.

Let $\varepsilon > 0$ be arbitrary. $\exists x_0 < \cdots < x_n = x$ s.t.

$$\sum_1^n |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon.$$

In particular,

$$\underbrace{|F(y) - F(x)| + \sum_1^n |F(x_i) - F(x_{i-1})|}_{x_0 < \cdots < x_n < x_{n+1} = y} \geq |F(y) - F(x)| + T_F(x) - \varepsilon$$

$$\Rightarrow T_F(y) \geq |F(y) - F(x)| + T_F(x) - \varepsilon$$

$$\Rightarrow T_F(y) + F(y) \geq \underbrace{|F(y) - F(x)| + F(y) - F(x)}_{\geq 0} + F(x) + T_F(x) - \varepsilon$$

$$\Rightarrow T_F(y) + F(y) \geq T_F(x) + F(x) - \varepsilon.$$ Let $\varepsilon \to 0^+$. ∎

(Similar for $T_F - F$.)

Now, note $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$.

both increasing

Thus, $F \in BV \Rightarrow F$ is difference of increasing functions.

Def^n (Normalised Bounded Variation (NBV))

$NBV := \{ F \in BV : F$ is right continuous and $F(-\infty) = 0 \}$.

Lemma. ① If $F \in BV$, then $T_F(-\infty) = 0$. (Note $T_F \geq 0$.)

② If F is right continuous, so is T_F.

Proof. ① Suppose $\epsilon > 0$. For $x \in \mathbb{R}$, choose $x_0 < x_1 < \cdots < x_n = x$ s.t.

$$\sum_1^n |F(x_i) - F(x_{i-1})| \geq T_F(x) - \epsilon.$$

But $F \in BV \Rightarrow F \in BV[x_0, x]$. In particular

$$T_F(x) - T_F(x_0) \geq \sum_1^n |F(x_i) - F(x_{i-1})| \geq T_F(x) - \epsilon.$$

$$\Rightarrow T_F(x_0) \leq \epsilon.$$

$\therefore$ For $y \in (-\infty, x_0]$, we have $T_F(y) \leq \epsilon$. ∎

② Suppose F is right continuous. Fix $x \in \mathbb{R}$.

Let $\alpha = T_F(x^+) - T_F(x)$, where

$$T_F(x^+) = \lim_{h \downarrow 0} T_F(x+h).$$ (Exists since $T_F \uparrow$.)

(TS: $\alpha = 0$)

By right continuity of F, get $\delta > 0$ s.t.

$$|F(x+h) - F(x)| < \epsilon \quad \forall \quad 0 < h < \delta.$$

For the same δ, we may assume

$$T_F(x+h) - T_F(x^+) < \epsilon \quad \forall \quad 0 < h < \delta.$$

Get $x = x_0 < x_1 < \cdots < x_n = x+h$ s.t.

$$\sum_1^n |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}\left(T_F(x+h) - T_F(x)\right) \geq \frac{3}{4}\alpha.$$

$$\Rightarrow \sum_2^n |F(x_i) - F(x_{i-1})| \geq \frac{3\alpha}{4} - |F(x_1) - F(x_0)| \geq \frac{3\alpha}{4} - \epsilon.$$

take x_1 sufficiently close

Let $x_0 = t_0 < t_1 < \cdots < t_m = x_1$, and

$$\sum_{1}^{n} |F(t_i) - F(t_{i-1})| \geq \frac{3}{4}\alpha \quad \text{(same reason)}$$

So, $\varepsilon + \alpha > T_F(x^+) - T_F(x) + T_F(x+h) - T_F(x^+)$

$$= T_F(x+h) - T_F(x) \geq \sum_{1}^{m} |F(t_i) - F(t_{i-1})| + \sum_{2}^{n} |F(x_i) - F(x_{i-1})|$$

$$\geq \frac{3}{4}\alpha + \frac{3}{4}\alpha - \varepsilon$$

$\Rightarrow \alpha < 4\varepsilon$ and $\varepsilon > 0$ was arbit. Thus, $\alpha = 0$. ∎

Lecture 16 (18-03-2021)

18 March 2021 14:21

Thm. If μ is a regular finite signed Borel measure and $F(x) := \mu(-\infty, x]$, then $F \in NBV$.

Conversely, for $F \in NBV$, $\mu_F(-\infty, x] := F(x)$ defines a regular, finite Borel signed measure.

Proof. Write $\mu = \mu^+ - \mu^-$ (Jordan decomposition), $\mu^+, \mu^- \geq 0$.

By what we have already seen, $F^\pm$ defined by
$$F^\pm(x) := \mu(-\infty, x]$$
satisfy

$F^\pm(-\infty) = 0$, $F^\pm$ are right-continuous.

Moreover $F = F^+ - F^- \in NBV$.

Conversely, given $F \in NBV$, note that we can write
$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F).$$

We have already seen that $T_F + F$, $T_F - F$ are $\uparrow$ and right continuous. Set $F^\pm = \frac{1}{2}(T_F \pm F)$, then both $F^\pm$

are $\uparrow$ and right continuous. Then $\exists\, \mu^\pm$ s.t. $\mu^\pm$ are regular, positive and described by $\mu^\pm(-\infty, x] = F^\pm(x)$.

Setting $\mu = \mu^+ - \mu^-$ gives the associated regular signed measure, corresponding to F. ∎

Prop^n. Suppose F is $\uparrow$. Let $G(x) = F(x^+)$.

① F is continuous on a countable set. (Saw in $\mathbb{R}$ Analysis)

② G is $\uparrow$, G is right continuous, G is diff. a.e. m.

③ F' exists a.e. m and $F' = G'$ a.e.

Proof. ① Exercise.

② G $\uparrow$ and right continuous is ex.

Thus, μ_G is a regular positive measure.

Moreover, $G(x) \neq F(x) \iff F$ is not right continuous at x.

Let $h > 0$.

$$\frac{G(x+h) - G(x)}{h} = \frac{\mu_G(x, x+h]}{m(x, x+h]}$$

(Analogous if $h < 0$.)

Now, $\{(x, x+h]\}_{h>0}$ shrinks nicely to x.

Thus, by the diff. thm, $\lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$ exists a.e. m.

③ Let $\{x_j\}_j$ be an enumeration of all the points where $F(x) \neq G(x)$. Let $H = G - F$. Then, $H \geq 0$ and $H(x) > 0$ iff $x = x_j$ for some $j \in \mathbb{N}$.

Define $\mu = \sum_j H(x_j) \mathbb{1}_{x_j}$, i.e., $\mu(\{x_j\}) = H(x_j) \; \forall j$.

Note that μ is finite on compact sets. Thus, μ is regular. So again,

$$\left| \frac{H(x+h) - H(x)}{h} \right| \leq \frac{H(x+h) + H(x)}{|h|} \leq 4 \left(\frac{\mu(x - 2|h|, x + 2|h|]}{4|h|} \right) \to 0$$

since $\mu \perp m$.

This establishes that H' exists a.e. and $H' = 0$ a.e.

$\implies F'$ exists a.e. and $F' = G'$ a.e. ∎

The above also shows that a function of BV has derivative a.e.

The following questions are now suggestive:

Suppose $F \in NBV$.

① When is $\mu_F \ll m$?

② When is $\mu_F \perp m$?

Prop^n. If $F \in NBV$, then $F' \in L^1(m)$. Moreover,

① $\mu_F \ll m$ iff $F(x) = \int_{-\infty}^{x} F'(t)\, dt$.

② $\mu_F \perp m$ iff $F' = 0$ a.e. m.

Proof. Note that $F \in NBV \Rightarrow F = F_1 - F_2$ with $F_1, F_2 \uparrow$.
Thus, F' exists a.e.

By Radon-Nikodym, $\mu_F = \lambda + f\, dm$, with $f \in L^1(m)$.

If $\mu_F \perp m$, then $f \equiv 0$ a.e. and if $\mu_F \ll m$, then $\lambda \equiv 0$.

Since μ_F is regular, the diff. thm implies that
$$F'(x) = \lim_{r \to 0} \frac{\mu_F(E_r)}{m(E_r)}$$
exists a.e., where

$$E_r = \begin{cases} (x, x+r] & ; \ r > 0, \\ (x+r, x] & ; \ r < 0. \end{cases}$$

(Recall: ν is regular, $\nu = \lambda + f\, dm \Rightarrow \lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ a.e. whenever E_r shrinks nicely $\to x$.)

The results now follow easily. ∎

Def^n. $F : \mathbb{R} \to \mathbb{R}$ is said to be *absolutely continuous* if given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. whenever $\{(a_i, b_i)\}_{i=1}^{N}$ is a collection of p.w. disjoint intervals s.t. $\sum_{i=1}^{N} |b_i - a_i| < \delta$, then
$$\sum_{i=1}^{N} |F(b_i) - F(a_i)| < \varepsilon.$$

Note that F is absolutely continuous $\Rightarrow$ F is uniformly continuous.

Prop^n $F \in NBV$. $\mu_F \ll m \Leftrightarrow F$ is absolutely continuous.

Proof. If $\mu_F \ll m$, to show the abs. continuity of F, we need to show : $\exists \delta > 0$ s.t. $\sum_{i=1}^{N} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{N} |F(b_i) - F(a_i)| < \epsilon$.

Note $|F(b_i) - F(a_i)| = |\mu(a_i, b_i]|$.

So, $\sum_{1}^{N} |F(b_i) - F(a_i)| = |\mu| \left(\bigcup_{i=1}^{N} (a_i, b_i] \right)$.

Stating $\mu \ll \nu$ in terms of $\epsilon - \delta$ gives the above condition.
(For finite measures)

That is, $\mu \ll \nu$ if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t.
$\nu(E) < \delta \Rightarrow |\mu|(E) < \epsilon$. $(\nu \overset{?}{\geq} 0)$

Conversely, suppose F is absolutely continuous. TS: $\mu_F \ll m$.
Suppose $m(E) = 0$. WTS: $\mu_F(E) = 0$.
Let $\epsilon > 0$ be given. Let $\delta > 0$ be as in the def^n of abs. continuity of F.
Since $F \in NBV$, μ_F is regular, so get $U_1 \supseteq U_2 \supseteq \cdots \supseteq E$ with all U_i open, $m(U_i) < \delta$, $\lim_j \mu_F(U_j) = \mu_F(E)$.

Each U_j is open in $\mathbb{R}$, so it is a countable disjoint union $\{(a_j^{(k)}, b_j^{(k)})\}_{k \geq 1}$ s.t. $U_j = \bigcup_{k \geq 1} (a_j^{(k)}, b_j^{(k)})$.

So for any N,
$$\sum_{i=1}^{N} |\mu_F(a_j^{(k)}, b_j^{(k)})| \leq \sum_{i=1}^{N} |F(b_j^{(k)}) - F(a_j^{(k)})| < \epsilon,$$
by def^n of δ.

(Since F is abs. continuous, $\mu_F(\{x\}) = 0 \ \forall x$.
$\therefore \mu_F(a, b) = \mu_F(a, b] = F(b) - F(a)$ above.)

The result follows by taking $N \to \infty$

Lecture 17 (22-03-2021)

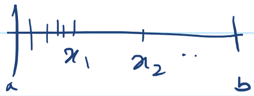
22 March 2021 14:00

We first make a simple observation.

Obs. If F is absolutely continuous on $[a, b]$, then $F \in BV[a, b]$.

Proof. Suppose $a = x_0 < x_1 < \cdots < x_N = b$. Want a bound for

$$\sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)| \quad \text{over all possible } N \text{ and } \{x_i\}.$$



From the def^n of abs. continuity, let $\delta_0 > 0$ be s.t. whenever $\sum (b_i - a_i) < \delta_0$, then

$$\sum |F(b_i) - F(a_i)| < 1.$$

It is now easy to see that

$$\sum_{i=1}^{N} |F(x_{i+1}) - F(x_i)| \leq \frac{b-a}{\delta_0} + 1 \quad \text{for any } x_0, x_1, \ldots, x_N.$$

Converse not true. It need not even be continuous.

Fundamental Theorem of Calculus for Lebesgue Integrals:

Thm. Suppose $F : [a, b] \to \mathbb{R}$. TFAE:

(1) F is absolutely continuous on $[a, b]$.

(2) $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a, b], m)$.

(3) F is diff a.e. on $[a, b]$ and $F' \in L^1([a, b], m)$ and moreover,

$$F(x) - F(a) = \int_a^x F'(t)\,dt.$$

Proof. WLOG, $F(a) = 0$.
Extend F from $[a, b]$ to $\mathbb{R}$ by defining

$$F(x) = \begin{cases} F(0) \ ; & x < a, \\ F(b) \ ; & x > b. \end{cases}$$

Extended F is now in NBV. The theorem now follows from earlier parts.

Remarks. LEBESGUE - STIELTJES INTEGRALS.
If $f \in NBV$, then for any g integrable w.r.t. μ_F, we denote

$$\int g \, d\mu_F =: \int g \, dF$$ and call this the

Lebesgue - Stieltjes integral w.r.t. F.

INTEGRATION BY PARTS:
Suppose $F, G \in NBV$ and one of them, say G, is continuous.
Then, for any $-\infty < a < b < \infty$, we have

$$\int_{(a,b]} f \, dG + \int_{(a,b]} G \, dF = F(b)G(b) - F(a)G(a)$$

Proof. WLOG, assume F, G are $\uparrow$. (Else $F = F_1 - F_2$, where $F_i \uparrow$.)
So, $\mu_F, \mu_G \geq 0$.
Consider the region

$$\Omega = \{ (x, y) : a < x \leq y \leq b \}.$$

So, $\mu_F \times \mu_G (\Omega) < \infty$, so by Fubini,

$$\mu_F \times \mu_G(\Omega) = \int_{x \in (a,b]} \left(\int_{y \in [x,b]} dG(y) \right) dF(x)$$

G continuous

$$= \int_{(a,b]} \left(\int_{(x,b]} dG(y) \right) dF(x)$$

$$= \int_{(a,b]} (G(b) - G(x))\, dF(x)$$

$$= G(b)(F(b) - F(a)) - \int_{(a,b]} G\, dF.$$

Similarly :
$$\mu_F \times \mu_G(\Omega) = \int_{(a,b]} \left(\int_{(a,y]} dF(x) \right) dG(y)$$

$$= \int_{(a,b]} [F(y) - F(a)]\, dG(y)$$

$$= \int_{(a,b]} F\, dF - F(a)\,(G(b) - G(a)).$$

Equate the two. ☒

× ×

L^p Spaces

Recall : $C[0, 1] = \{ f : [0, 1] \to \mathbb{R} \mid f \text{ continuous} \}$.

Note that $C[0,1]$ is an $\mathbb{R}$-vector space with pointwise operations

Any Borel regular measure μ on $\mathbb{R}$ gives a LINEAR FUNCTIONAL on $C[0, 1]$:

$$\Lambda_\mu f := \int f \, d\mu$$

$$\Lambda_\mu (f+g) = \Lambda_\mu(f) + \Lambda_\mu(g) ; \quad \Lambda_\mu(\alpha f) = \alpha \Lambda_\mu f.$$

(Not all linear maps need be "nice".)
This sets up some basic motivation to consider "nice" function spaces. (Here we shall assume $\mu \geq 0$.)

- Given $(X, \mathcal{M}, \mu)$ a measure space, we have already defined

$$L^1(\mu) := \{ f : X \to \mathbb{R} \mid f \text{ is integrable} \}.$$

(i.e., f is meas'ble and $\int |f| d\mu < \infty$.)

We had seen that $L^1(\mu)/\sim$ is a complete metric space with

$$d(f,g) := \int |f-g| \, d\mu.$$

Def^n. Given $(X, \mathcal{M}, \mu)$ a σ-finite measure space,

$$L^p(\mu) := \{ f : X \to \mathbb{R} \text{ s.t. } \int |f|^p d\mu < \infty \}$$

for $p \geq 1$.

EXAMPLES.

1. $X = \mathbb{R}^n$, $\mu \equiv m$. Denoted $L^p(m)$.
2. $X = \mathbb{Z}$, $\mu \equiv$ counting measure. Denoted ℓ^p or ℓ_p.

Def^n. Define for $f \in L^p(\mu)$,

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p}.$$

HÖLDER'S INEQUALITY Suppose $p, q > 1$ are s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

(q is called the conjugate of p.)
If $f \in L^p(\mu)$, $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

As an example, suppose $p = q = 2$. Hölder's inequality states

$$\int_X |f(x)\, g(x)|\, d\mu(x) \le \left(\int_X |f|^2 d\mu\right)^{1/2} \left(\int_X |g|^2 d\mu\right)^{1/2}$$

(Cauchy - Schwarz)

Proof. We need the following fact:

If $a, b > 0$ and $0 \le \theta \le 1$, then

$$a^\theta b^\theta \le \theta a + (1-\theta) b.$$

$$\left(\frac{a}{b}\right)^\theta \le \theta\left(\frac{a}{b}\right) + (1-\theta).$$

Consider $f_\theta(x) = x^\theta - \theta x - (1-\theta)$.

WTS: $f_\theta \le 0$. Note that $f_\theta'(x) = \theta x^{\theta-1} - \theta$.

Conclude.

Now, $a(x) := |f(x)|^p$, $b(x) := |g(x)|^q$ where p and q are conjugates. Take $\theta = 1/p$. $(1-\theta = 1 - 1/p = 1/q.)$

$$\text{FACT} \Rightarrow \left(|f(x)|^p\right)^{1/p} \left(|g(x)|^q\right)^{1/q} \le \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

$$\| $$

$$|f(x)|\,|g(x)|$$

Thus, $\int_X |fg| \le \frac{1}{p}\int_X |f|^p + \frac{1}{q}\int_X |f|^q < \infty.$

Now, we may assume $\|f\|_p = 1 = \|g\|_q$.

(If either is zero, then it is 0 a.e. and both are 0.)
Else, scale it.

Thus, the above becomes $\|fg\|_1 \le \frac{1}{p}\|f\|_p + \frac{1}{q}\|f\|_q = 1.$ ∎

Minkowski's Inequality

If $1 \le p < \infty$, $f, g \in L^p(\mu)$, then $f + g \in L^p(\mu)$ and moreover,

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \qquad \text{(Triangle inequality)}$$

Quick consequence: $\|\cdot\|_p$ is a NORM on $L^p(\mu)/\sim$.
This makes $L^p(X, \mathcal{M}, \mu)$ a metric space with

$$d_p(f, g) = \|f - g\|_p.$$

As we will see next, L^p is a complete normed vector space.
(Banach Space)

Proof. For $p = 1$, we know the statement.
Let $p > 1$. First, we wish to show

$$\int_X |f + g|^p \, d\mu < \infty.$$

Note that $\left| \frac{f(x) + g(x)}{2} \right|^p \le |f(x)|^p + |g(x)|^p$ holds for all x.

$$\Rightarrow \quad |f(x) + g(x)|^p \le 2^p |f(x)|^p + 2^p |g(x)|^p.$$

Now, integrate both sides to get $\int_X |f + g|^p \, d\mu < \infty$.

Now, consider

$$|f(x) + g(x)|^p \le |f(x)| \, |f(x) + g(x)|^{p-1} + |g(x)| \cdot |f(x) + g(x)|^{p-1}.$$

Since $p > 1$, p admits a conjugate q.
Note that

$$|f + g|^{p-1} \in L^q(\mu).$$

Proof

$$\int \left(|f + g|^{p-1} \right)^q d\mu = \int |f + g|^p \, d\mu < \infty$$

So, using Hölder's on each term gives

$$\|f+g\|_p^p \leq \|f\|_p \, \| |f+g|^{p-1} \|_q + \|g\|_p \cdot \| |f+g|^{p-1} \|_q .$$

Note $$\| |f+g|^{p-1} \|_q = \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} = \left(\int |f+g|^p \right)^{1-1/p}$$

Thus,

$$\|f+g\|_p^p \leq \left(\|f\|_p + \|g\|_p \right) \|f+g\|_p^{p-1}$$

$$\Rightarrow \quad \|f+g\|_p \leq \|f\|_p + \|g\|_p .$$ ∎

Lecture 18 (25-03-2021)

25 March 2021 14:09

We saw $L^p(\mu)$ is a Normed Linear Space, with

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}.$$

Thm. $L^p(\mu)$ is a BANACH SPACE, i.e., the metric induced by $\|\cdot\|_p$ is complete. (For $1 \leq p < \infty$.)

Proof. We had shown this for $p = 1$. The general proof is similar. Suppose $\{f_n\} \subseteq L^p(\mu)$ is Cauchy, i.e., given $\varepsilon > 0$ $\exists N_0 \in \mathbb{N}$ s.t.

$$\|f_m - f_n\|_p < \varepsilon \qquad \forall\ n, m > N_0.$$

By the Cauchyness, get a subsequence $(n_k)_{k\geq 1}$ s.t.

$$\|f_{n_{k+1}} - f_{n_k}\| \leq \frac{1}{2^k}.$$

Consider
$$f(x) := f_{n_1}(x) + \sum_{k=1}^{\infty}\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and

$$g(x) := |f_{n_1}(x)| + \sum_{k=1}^{\infty}\left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|,$$

which arise from the functions

$$(S_K f)(x) := f_{n_1}(x) + \sum_{k=1}^{K} f_{n_{k+1}}(x) - f_{n_k}(x) \quad \text{and}$$

similarly for $S_K g$.

By Minkowski's inequality

$$\|S_K g\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K} \|f_{n_{k+1}} - f_{n_k}\|_p$$

$$\leq \|f_{n_1}\|_p + \sum_{1}^{K} \frac{1}{2^k}$$

So, taking $k \to \infty$ and using MCT, it follows that

$$\int g^p \, d\mu < \infty.$$

In particular, f converges a.e. and $f \in L^p$.

It suffices now to show that $f_n \longrightarrow f$ in $L^p(\mu)$. We first show that $f_{n_k} \longrightarrow f$ in $L^p(\mu)$.

First observe that $S_K f = f_{n_{K+1}}$, so by what we have established, it follows that

$$f_{n_{k+1}} \longrightarrow f \quad \mu \text{ a.e.}$$

To show convergence in L^p, note that

$$\left| \frac{f - f_{n_{k+1}}}{2} \right|^p \leq \max\{ |f|^p, |f_{n_{k+1}}|^p \} \leq g^p$$

$$\Rightarrow \quad |f - f_{n_{k+1}}|^p \leq 2^p g^p.$$

Thus, DCT $\Rightarrow$ $\displaystyle\int_X |f - f_{n_{k+1}}|^p \, d\mu \longrightarrow 0$ as $k \to \infty$.

That $f_n \to f$ follows from general argument about sequences. ☒

A DIGRESSIVE REMARK:

Suppose X is a locally compact Hausdorff space.
One can define the Borel measure $\mathcal{B}_X$ and a measure μ on $(X, \mathcal{B}_X)$ is called

① Outer regular : if $\mu(E) = \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\}$.
② Inner regular : if $\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}$.
③ Regular : if both ① and ②.

Consider $C_c(X) \equiv$ Space of continuous functions with COMPACT SUPPORT on X.

For any regular measure μ, $f \in C_c(X)$

$$I(f) := \int_X f \, d\mu$$

defines a LINEAR, POSITIVE, FUNCTIONAL on $C_c(X)$.

Riesz Representation Theorem. Suppose I is a positive linear functional on $C_c(X)$. Then, there is a UNIQUE RADON MEASURE μ s.t.

$$I(f) = \int_X f \, d\mu.$$

SOME REMARKS

If $1 \le p_0 < p_1$, $\quad L^{p_0} \overset{?}{\underset{\supseteq}{\subseteq}} L^{p_1}$. Neither.

Eg. $X = \mathbb{R}$, $\mu = m$.
Fix $\alpha > 0$.

$$f_0(x) := \begin{cases} |x|^{-\alpha} & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

and

$$f_1(x) := \begin{cases} |x|^{-\alpha} & \text{if } |x| \ge 1, \\ 0 & \text{if } |x| < 1. \end{cases}$$

Then, $f_0 \in L^p$ iff $p\alpha < 1$ and $f_1 \in L^p(m)$ iff $p\alpha > 1$.

Thus, given $p < q$, choose α s.t. $p\alpha < 1$ and $q\alpha > 1$.

One can generalise this to $L^p(\mathbb{R}^d)$.

However, if the space has finite measure, then:

Prop^n. Suppose $\mu(X) < \infty$.
If $1 \le p_0 < p_1$, then $L^{p_1}(X) \subseteq L^{p_0}(X)$. Moreover,

$$\|f\|_{p_0} \le \frac{(\mu(X))^{1/p_0}}{(\mu(X))^{1/p_1}} \|f\|_{p_1}.$$

Proof. Suppose $f \in L^{p_1}$. WTS: $f \in L^{p_0}$.
Let $F = |f|^{p_0}$, $G \equiv 1$.
Let $p := \frac{p_1}{p_0} > 1$ and $q = p'$, the conjugate.

By Hölder,

$$\int |f|^{p_0}\, d\mu \le \left(\int F^p\right)^{1/p} \left(\int G^q\right)^{1-1/p}$$

$$\Rightarrow \|f\|_{p_0}^{p_0} \le \left(\int |f|^{p_1}\right)^{p_0/p_1} (\mu(X))^{1-\frac{p_0}{p_1}} < \infty.$$

Simplify. ∎

Prop^n. If $X = \mathbb{Z}$ and $\mu \equiv$ counting measure, then if $p_0 \le p_1$, then
$\ell^{p_0}(\mathbb{Z}) \subseteq \ell^{p_1}(\mathbb{Z})$.

Furthermore, $\|f\|_{\ell^{p_1}} \le \|f\|_{\ell^{p_0}}$.

Proof. Suppose $f = (f(n))_{n\in\mathbb{Z}}$.

$$\sum_{n\in\mathbb{Z}} |f(n)|^{p_0} = \|f\|_{\ell(p_0)}^{p_0} \text{ and hence,}$$

$$\sup_n |f(n)| \leq \|f\|_{\ell(p_0)}.$$

$$\Rightarrow \sum_n |f(n)|^{p_1} = \sum_n |f(n)|^{p_0} \cdot |f(n)|^{p_1 - p_0}$$

$$\leq \left(\sup_n |f(n)|^{p_1 - p_0}\right) \sum_n |f(n)|^{p_0}$$

$$\leq \|f\|_{\ell(p_0)}^{p_1 - p_0} \cdot \|f\|_{\ell(p_0)}. \qquad \boxtimes$$

$p = \infty$?

Def^n. $L^\infty(\mu) = \{f : f \text{ is "essentially finite"}\}$.

That is, $\exists\, 0 < M < \infty$ and E s.t. $\mu(E) = 0$ and
$$|f(x)| \leq M \quad \forall x \in E^c.$$

For $f \in L^\infty$, define

$$\|f\|_\infty := \inf\{M : |f(x)| \leq M \text{ outside a set of measure } 0\}.$$

essential supremum

Prop^n. $(L^\infty, \|\cdot\|_\infty)$ is a BANACH SPACE.

Proof Exercise. $\boxtimes$

SOME BASICS OF BANACH SPACES

Suppose $(\mathcal{B}, \|\cdot\|)$ is a Banach space.

Examples. ① $\mathbb{R}^n$, ② $L^p(X, \mathcal{M}, \mu)$ for $1 \leq p \leq \infty$, Examples. ① $\mathbb{R}^n$, ② $L^p(X, \mathcal{M}, \mu)$ for $1 \le p \le \infty$,

③ $C[0,1] \equiv$ cts f^n $[0,1] \to \mathbb{R}$.

$$\|f\| = \sup_{x \in [0,1]} |f(x)|.$$

Convergence w.r.t. above is uniform convergence. This shows the above is complete

④ $0 < \alpha \le 1$, $\Lambda^\alpha(\mathbb{R}) := \left\{ f : \sup_{t_1 \ne t_2} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha} < \infty \right\}$.

Lipschitz exponent α

$$\|f\|_{\Lambda^\alpha} := \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

$(\Lambda^\alpha(\mathbb{R}), \|\cdot\|_{\Lambda^\alpha})$ is also a Banach space.

Suppose $(\mathcal{B}, \|\cdot\|)$ is a Banach space.

A linear functional T is a map $T: \mathcal{B} \to \mathbb{R}$ which is continuous.

We say that T is bounded if $\exists M > 0$ s.t.

$$\|Tx\| \le M\|x\| \quad \forall x \in \mathcal{B}.$$

Prop^n Let T be a linear functional on a Banach space. Then, T is continuous $\iff$ T is bounded.

Proof. T is continuous $\iff$ T is continuous at 0. (By linearity)

($\Rightarrow$) Take $\delta > 0$ corresp. to $\varepsilon = 1$.

Then, $\|Tx\| < 1 \quad \forall \|x\| < \delta$.

Now, for any $0 \ne x$, consider $x' = \frac{\delta}{2} \frac{x}{\|x\|}$. Then, $\|x'\| < \delta$.

Then, $\|T x'\| < 1$ or $\|Tx\| < \frac{2}{\delta}\|x\|$.

($\Leftarrow$) $\|Tx\| \leq M\|x\| \;\forall x \Rightarrow T$ continuous at 0. ∎

Given a Banach space $(\mathcal{B}, \|\cdot\|)$, one defines the dual of $\mathcal{B}$ as

$$\mathcal{B}^* = \{T : \mathcal{B} \to \mathbb{R} \mid T \text{ is bounded and linear}\}.$$

$$\|T\| := \sup_{\|x\|=1} \|Tx\|.$$

Prop^n. $(\mathcal{B}^*, \|\cdot\|)$ is a Banach space.