∫(͡° ͜ʖ ͡°)dx

MA 406

General Topology

Notes By: Aryaman Maithani

Spring 2020-21

Lecture 1 (07-01-2021)

07 January 2021 15:04

Def^n. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties: (Topology)

(1) ϕ and X are in $\mathcal{T}$.
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(Open set)

Any $U \in \mathcal{T}$ is called an open set of X w.r.t. $\mathcal{T}$.
The pair $(X, \mathcal{T})$ or just the set X is called a topological space. (abuse of notat^n.)

Can reconcile the above with open sets in $\mathbb{R}$, or in general, any metric space X. That can be seen as a motivation for the definition.

Examples

(1) $X = \{a, b, c\}$
$\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ Can be seen (fairly easily) that this is a topology
$\mathcal{T}_2 = \{\phi, X\}$ trivial (pun intended, cf. next example)

(2) If X is any set, the collection of all subsets of X is a topology on X, it is called the discrete topology. ($\mathcal{T} = \mathcal{P}(X)$, that is) (Discrete topology)

The collection $\{\phi, X\}$ is also a topology on X called the indiscrete topology or trivial topology. (Indiscrete topology
Trivial topology)

(3) Let X be a set. Let

$$\mathcal{T}_f = \{ U \subseteq X : |X \setminus U| < \infty \} \cup \{\emptyset\}.$$

(Finite complement topology)

Then, $\mathcal{T}_f$ is a topology on X, called the finite complement topology on X.

- $\emptyset \in \mathcal{T}_f$ is clear. $X \in \mathcal{T}_f$ since $|X \setminus X| = 0 < \infty$.
- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be sets in $\mathcal{T}_f$. WLOG, $U_\alpha \neq \emptyset$ $\forall \alpha$.

Note $$X \setminus \left(\bigcup_\alpha U_\alpha \right) = X \cap \left(\bigcup_\alpha U_\alpha \right)^c$$
$$= \bigcap_\alpha (U_\alpha^c)$$

Note that each U_α^c is finite. ($U_\alpha \neq \emptyset$)
Thus, the above intersection is finite.

- Similarly, for finite unions, again reduce it to $\bigcup_{i=1}^{n} (U_i^c)$

and conclude as earlier.

(Here, if some U_i were $\emptyset$, then so would be the intersection.)

(If X is finite, the $\mathcal{T}_f = \mathcal{P}(X)$. Thus, we get discrete.)

(4) Let X be a set.
Let $\mathcal{T}_c$ be the collection of subsets such that $X \setminus U$ is either countable or all of X.
Called the co-countable topology. (Cocountable topology Co-countable topology)
(Generalising the previous)

Def^n. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X.
If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and that $\mathcal{T}$ is coarser than $\mathcal{T}'$!
If $\mathcal{T}' \supsetneq \mathcal{T}$, then the above is strictly finer and strictly ...

coarser, respectively. (Finer, coarser, strictly finer, strictly coarser)

(The above gives us a way to compare two topologies)

EXAMPLE We have the usual topology on $\mathbb{R}$. ← strictly coarser than this
We also have the discrete topology on $\mathbb{R}$.

Def^n. If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that (Basis)

(1) for each $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.
(2) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Note that in the above, $\mathcal{B}$ is just some collection of subsets of X satisfying (1) & (2). No topology is mentioned so far.

EXAMPLES

(1) $X = \mathbb{R}^2$, $\mathcal{B}$ is the collection of all discs w/o boundary.
(2) ——— " ——— " - rectangles — " —
(3) Any X. The singletons form a basis.

We now get a topology out of a basis:

Def^n. If $\mathcal{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows: (Topology generated)

A subset U of X is said to be open if for every $x \in U$, there exists $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

$$x \in B \subset U.$$

(By "open" in above, we mean element of ***T***. Same thing for what we see in the proof below.)

EXAMPLES (1) & (2) $\to$ gives standard topology on $\mathbb{R}^2$

(3) $\to$ gives discrete topology on X.

We still have to show that it is topology.

Proof:

- $\phi \in \mathcal{T}$ vacuously

  $X \in \mathcal{T}$ since given any $x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$.

  $B \subset X$ is by definition.

- Let $\{U_\alpha\}_{\alpha \in \Lambda \neq \phi}$ be open. Let $U := \bigcup_\alpha U_\alpha$.

  Fix $\alpha_0 \in \Lambda$.

  Let $x \in U$ be arbitrary. Then, $x \in U_{\alpha_0} \leftarrow$ open

  $\therefore \exists B \in \mathcal{B}$ s.t. $x \in B \subset U_{\alpha_0} \subset U$.

  $\therefore U \in \mathcal{T}$.

- Let U_1 and U_2 be open. Put $U := U_1 \cap U_2$.

  Let $x \in U$.

  Then $x \in U_1$ and $x \in U_2$

  $\Downarrow$ $\Downarrow$

  $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1 \subset U_1$ and $\exists B_2 \in \mathcal{B}$ s.t. $x \in B_2 \subset U_2$

  $\therefore \underbrace{x \in B_1 \cap B_2}_{\Downarrow} \subset U_1 \cap U_2$

  $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 = U$.

  $\Rightarrow U \in \mathcal{T}$.

By induction, any finite intersection is in $\mathcal{T}$. $\boxtimes$

$n \geq 3 \left(\bigcap_{i=1}^{n} U_i = U_n \cap \left(\bigcap_{i=1}^{n-1} U_i \right) \right.$.

Lecture 2 (11-01-2021)

11 January 2021 15:31

Lemma 1 Let $\mathcal{B}$ be a basis and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then, $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Note that $\emptyset$ is empty union.

Proof. Given $\{U_\alpha\} \subset \mathcal{B}$, it is clear that $\bigcup U_\alpha \in \mathcal{T}$ since $\mathcal{T}$ is a topology and U_α are open. (By def^n)

Conversely, let $U \in \mathcal{T}$. Given any $x \in U$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. (By def^n of $\mathcal{T}$.)

Thus,
$$\bigcup_{x \in U} B_x = U. \qquad \blacksquare$$

($\subseteq$) since $B_x \subset U$

($\supseteq$) Each $x \in U$ is in B_x.

(Note that if $U = \emptyset$, the last union is the empty union!)

[The above gives us a way of extracting a basis $\mathcal{B}$ if we are already given a topology $\mathcal{T}$. Namely, pick any subcollection $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{T}$ is precisely the collection of all unions of elements of $\mathcal{B}$.]

Lemma 2. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$. (recall this means $\mathcal{T} \subset \mathcal{T}'$)

(ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \in B$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and $B \in \mathcal{B}$ be arbitrary.

Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.

Thus $B \in \mathcal{T}'$. (by (i))

Note that B is open in $(X, \mathcal{T})$, i.e., $B \in \mathcal{T}$.
Thus, $B \in \mathcal{T}'$. (by (i))
Since B is open in $\mathcal{T}'$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.
(Def^n of top. generated.)

(ii) $\Rightarrow$ (i) Suppose $U \in \mathcal{T}$. We show that $U \in \mathcal{T}'$.
Let $x \in U$. By def^n of $\mathcal{T}$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
By (ii), $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B \subset U$.

Since x was arbit, we see that $U \in \mathcal{T}'$. (By def^n of $\mathcal{T}'$)
Thus, $\mathcal{T} \subset \mathcal{T}'$. ☒

Lemma 3. Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X s.t. for each open set $U \subset X$ and each $x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.
Then $\mathcal{C}$ is a basis for the topology.

Proof. • Showing $\mathcal{C}$ is a basis.

(i) Given any $x \in X$, X is an open set containing x.
Thus, by hypothesis, $\exists C \in \mathcal{C}$ s.t. $x \in C$.

(ii) Let $C_1, C_2 \in \mathcal{C}$ s.t. $x \in C_1 \cap C_2$.
Note that C_1, C_2 are open and hence, $C_1 \cap C_2$ is open.
By hypothesis, $\exists C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$.

Thus, $\mathcal{C}$ satisfies both properties of a topology.

• $\mathcal{C}$ generates the topology.
Let $\mathcal{T}$ denote the topology of X. Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$.

Let $U \in \mathcal{T}'$, then U is some union of elements of $\mathcal{C}$.

But elements of $\mathcal{C}$ are elements of $\mathcal{T}$ and thus, $U \in \mathcal{T}$.
($\mathcal{T}$ is topo.)

Thus, $\mathcal{T}' \subset \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. for each $x \in U$, $\exists C_x \in \mathcal{C}$ s.t.
$x \in C_x \subset U$.

As earlier,

$$U = \bigcup_{x \in U} C_x \in \mathcal{T}'$$

Thus, $\mathcal{T} \subset \mathcal{T}'$. ∎

Def^n: Let $\mathcal{B}$ be the collection of all bounded intervals.
That is,

$$\mathcal{B} = \{ (a, b) : -\infty < a < b < \infty \}.$$

$\mathcal{B}$ is a basis and the topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$. (Standard topology on \Bbb R)

If $\mathcal{B}'$ is the collection of all half open intervals of the form $[a, b)$, then $\mathcal{B}'$ is also a basis and the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$. (Lower limit topology on \Bbb R)

Lemma 4. The lower limit topology is strictly finer than the standard topology.

Proof. Let $\mathcal{T}$ denote the standard topology and $\mathcal{T}'$ the lower limit.

- $\mathcal{T} \subseteq \mathcal{T}'$. Let (a, b) be an arbit. basis element and let $x \in (a,b)$.
  Then, $[x, b)$ is a basis element for $\mathcal{T}'$ & $x \in [x, b) \subset (a, b)$.

Thus, $\mathcal{T} \subseteq \mathcal{T}'$, by Lemma 2.

- $\mathcal{T}' \neq \mathcal{T}$. Note that $[0, 1) \in \mathcal{T}'$.
  But given $0 \in [0, 1)$, there is no $(a,b) \ni 0$
  s.t. $(a, b) \subset [0, 1)$. ∎

Def^n: A subbasis $\mathcal{S}$ for a topology is a collection of subsets of X whose union is X. (Subbasis, sub basis)

(Note that no topology given so far. Similar to what we saw for basis.)

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

—

We need to show that the topology defined above is actually a basis.

Let $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{S}$. We show $\mathcal{B}$ is a basis. (This suffices. Why? Lemma 1.)

(i) Let $x \in X$. Then, $\exists S \in \mathcal{S}$ s.t. $x \in S$. $\left(\because \bigcup_{S \in \mathcal{S}} S = X \right)$
But $S \in \mathcal{B}$.

(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$.
But note that $B_1 \cap B_2 \in \mathcal{B}$. (Why?)

Thus, both the conditions are satisfied.

Remark: The standard topology of $\mathbb{R}$ is also called the order topology on $\mathbb{R}$, because of the order relation of $\mathbb{R}$.

(We will see this in general, later.)

Def^n Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.

(Product topology)

(Open in the respective topologies, i.e.)

Note that $\mathcal{B}$ is a basis because:

(i) $X \times Y$ is itself a basis element

(ii) $U \times V,\ U' \times V' \in \mathcal{B} \Rightarrow (U \times V) \cap (U' \times V') = \underbrace{(U \cap U')}_{} \times \underbrace{(V \cap V')}_{} \in \mathcal{B}$

intersection of open sets

Note $\mathcal{B}$ itself won't be the topology. (In general)

Thm 5. If $\mathcal{B}$ is a basis for a topology $\mathcal{J}_x$ on X, and $\mathcal{C}$ for $\mathcal{J}_y$ on Y, then the collection

$$\mathcal{D} = \{ B \times C : B \in \mathcal{B},\ C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We check that the hypotheses of Lemma 3 are satisfied.

Let $W \subset X \times Y$ be open and $(x, y) \in W$.
Then, by def^n of prod. top., $\exists\, U \in \mathcal{J}_x,\ V \in \mathcal{J}_y$ s.t.

$$(x, y) \in U \times V \subset W.$$

Since $\mathcal{B}$ is a basis for $\mathcal{J}_x$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B \subset U$.

$\|^{ly}$ $\exists\, C \in \mathcal{C}$ s.t. $y \in C \subset V$.

$\Rightarrow (x, y) \in B \times C \subset U \times V \subset W$. $\boxtimes$

$\mathcal{D}$

Lecture 3 (14-01-2021)

14 January 2021 15:28

By last lecture's discussion, we know that

$$\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$$

is a basis for the product topology on $\mathbb{R}^2$.
This is called the standard topology on $\mathbb{R}^2$.

Def^n. Given any two sets X and Y, we have the two projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given as

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y \quad \forall (x, y) \in X \times Y.$$

(Projections)

Note that $\pi_1^{-1}(U) = U \times Y$ for any $U \subseteq X$ and similarly
$\pi_2^{-1}(V) = X \times V$ for any $V \subseteq Y$.

Thm 1. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \subseteq X \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \subseteq Y \text{ open}\}. \quad \subseteq \mathcal{P}(X \times Y)$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}_p$ denote the product topology on $X \times Y$.
Let $\mathcal{T}_S$ —"— topology generated by $\mathcal{S}$.

Note that any element of $\mathcal{S}$ is of the form $U \times Y$ or $X \times V$.
$U \subseteq X$ open $\quad V \subseteq Y$ open

Thus, $\mathcal{S} \subseteq \mathcal{T}_p$ since both the above are actually basis elements. Since $\mathcal{T}_p$ is a topology, it is closed under arbitrary unions of finite intersections. Thus, $\mathcal{T}_S \subseteq \mathcal{T}_p$.

On the other hand, consider any arbitrary basis elt. of $\mathcal{T}_p$.
It is of the form $U \times V$. $U \subseteq X$, $V \subseteq Y$ open.
Note now

$$U \times V = \underbrace{\pi_1^{-1}(U)}_{\in \mathcal{S}} \cap \underbrace{\pi_2^{-1}(V)}_{\mathcal{S}} \in \mathcal{T}_S$$

Thus, $U \times V \subset \mathcal{T}_S$. Since $\mathcal{T}_S$ is a topology, arbitrary union of basis elements of $\mathcal{T}_p$ is in $\mathcal{T}_S$.
Thus, $\mathcal{T}_p \subseteq \mathcal{T}_S$. ☒

Def^n. Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.
With this topology, $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$.

(Subspace topology)

(We will often just say "Y is a subspace of X" if it is clear.)

We now check that $\mathcal{T}_Y$ is actually a topology.

Check. (i) Since $\phi \in \mathcal{T}$, we get $\phi = \phi \cap Y \in \mathcal{T}_Y$.
$\quad Y = Y \cap X \in \mathcal{T}_Y$.

(ii, iii) Let $\{\mathcal{U}_i\}_{i \in I} \subseteq \mathcal{T}_Y$. Then, we have $\{\mathcal{U}_i'\}_{i \in I} \subseteq \mathcal{T}$ s.t.

$$\mathcal{U}_i' \cap Y = \mathcal{U}_i \qquad \forall i \in I.$$

Then,

$$\bigcup_{i \in I} \mathcal{U}_i = \bigcup_{i \in I} (\mathcal{U}_i' \cap Y) = \underbrace{\left(\bigcup_{i \in I} \mathcal{U}_i' \right)}_{\in \mathcal{T}} \cap Y$$

and similarly,

$$\bigcap_{i \in I} \mathcal{U}_i = \underbrace{\left(\bigcap_{i \in I} \mathcal{U}_i' \right)}_{\in \mathcal{T}} \cap Y.$$ ☒

and similarly, $\bigsqcup_{i \in I} \mathcal{U}_i = \left(\bigsqcup_{i \in I} \mathcal{U}_i \right) \cap Y$. ☒

$\uparrow$ $\mathcal{T}$ if $|I| < \infty$

Lemma 2. If $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, then the collection

$$\mathcal{B}_Y = \{ B \cap Y : B \in \mathcal{B} \}$$

is a basis for $(Y, \mathcal{T}_Y)$, i.e., the subspace topology on Y.

Proof. We use Lemma 3 from Lecture 2.
Using that, it suffices to show that for any $\mathcal{U} \in \mathcal{T}_Y$ and any $x \in \mathcal{U}$, $\exists B \in \mathcal{B}_Y$ s.t. $x \in B \subset \mathcal{U}$.

To this end, let x, $\mathcal{U}$ be as given. Then,
$\mathcal{U} = \mathcal{U}' \cap Y$ for some $\mathcal{U}' \in \mathcal{T}$.
Clearly, $x \in \mathcal{U}'$.
Then, $\exists B' \in \mathcal{B}$ s.t. $x \in B' \subset \mathcal{U}'$. ($\mathcal{B}$ is a basis for $\mathcal{T}$.)

Then, $B = B' \cap Y \in \mathcal{B}_Y$ and
$x \in B \subset \mathcal{U}$. ☒

Lemma 3. Let Y be a subspace of X. If $\mathcal{U}$ is open in Y and Y is open in X, then $\mathcal{U}$ is open in X.

Proof. $\mathcal{U} = \mathcal{U}' \cap Y$ for some $\mathcal{U}' \subset X$ open.
Since $\mathcal{U}'$ and Y are open in X, so is $\mathcal{U} = \mathcal{U}' \cap Y$. ☒
(finite intersection of open sets.)

EXAMPLES. (i) If $Y = [0, 1] \subset X = \mathbb{R}$ in subspace topology, it has a basis given as
$$\{ (a, b) \cap Y : a < b \in \mathbb{R} \}.$$

More explicitly, here we have

$$(a, b) \cap Y = \begin{cases} (a, b) & a \in Y \ni b \\ [0, b) & a \notin Y \ni b \\ (a, 1] & a \in Y \not\ni b \\ \phi \text{ or } Y & a \notin Y \not\ni b \end{cases}$$

(2) Consider $Y = [0, 1) \cup \{2\} \subseteq \mathbb{R}$.

Note that
$$\{2\} = (1.5, 2.5) \cap Y.$$
Thus, $\{2\}$ is open in Y. (Was not open in $\mathbb{R}$!)

Similarly, $[0, 1)$ is open in Y but not $\mathbb{R}$.

Thm 4. If A is a subspace of X and B of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Note that the above tells us that the following two ways of topolog-ising $A \times B$ are the same:

- consider A and B as spaces by themselves and give $A \times B$ the product topology
- consider the topological space $X \times Y$ in product topology. Note that $A \times B$ is a <u>subset</u> of $X \times Y$ and hence, can be given the subspace topology.

$$\xrightarrow{\text{subspace}}\ \xrightarrow{\text{product}} \ \overset{?}{=}\ \xrightarrow{\text{product}}\ \xrightarrow{\text{subspace}}$$

Proof. Note the following:

typical basis elt of $X \times Y$

$$\{ (U \times V) \cap (A \times B) : U \subseteq X,\ V \subseteq Y \text{ open} \}$$

basis for subspace topology on $A \times B$ by Lemma 2

$$= \{ (U \cap A) \times (V \cap B) : U \subseteq X,\ V \subseteq Y \text{ open} \}$$

$U \cap A$, $V \cap B$: a general open set in the subspace topology of $A \subseteq X$ or $B \subseteq Y$

basis for prod top. on $A \times B$

Thus, both the topologies have a common basis.

Def^n. A subset of a topological space is said to be closed if its complement is open.

(Closed set)

Examples. (1) $[a, b] \subseteq \mathbb{R}$ is closed because

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty) \text{ is open.}$$

(2) $[0, \infty) \times [0, \infty) \subseteq \mathbb{R}^2$ is closed because

$$\mathbb{R}^2 \setminus [0, \infty)^2 = \big((-\infty, 0) \times \mathbb{R} \big) \cup \big(\mathbb{R} \times (-\infty, 0) \big) \text{ is open.}$$

(3) In the discrete topology, every set is open and hence, every set is closed.

(4) Consider $Y = [-1, 0] \cup (2, 3) \subseteq \mathbb{R}$.

Both $\underset{(-2,\, 1) \cap Y}{\underset{\parallel}{[-1, 0]}}$ and $(2, 3)$ are open in Y.

Since they are complements of each other (in Y), we have that both the sets are closed as well, in Y.

Thm S. Let X be a topological space. Then,

(i) ϕ and X are closed,

(ii) arbitary intersection of closed sets is closed,

(iii) finite union of closed sets is closed.

Proof. $X \setminus \phi = X$, $\quad X \setminus X = \phi$.

Proof. $X \setminus \phi = X, \quad X \setminus X = \phi.$

$$X \setminus \left(\bigcap_{i \in I} C_i \right) = \bigcup_{i \in I} (X \setminus C_i).$$

$$X \setminus \left(\bigcup_{i=1}^{n} C_i \right) = \bigcap_{i \in I} (X \setminus C_i).$$

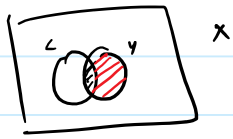
Conclude. ∎

Remark. The above is also a way to define a topology.

Thm 6. Let Y be a subspace of X and $A \subseteq Y$. Then, A is closed in Y iff A equals the intersection of a closed set (in X) with Y.

Proof. ($\Leftarrow$) Suppose $A = C \cap Y$ for some closed set $C \subseteq X$. Then,

$$Y \setminus A = \underbrace{\underbrace{(X \setminus C)}_{\text{open in } X} \cap Y}_{\text{open in } Y}$$



$\therefore$ A is closed in Y.

($\Rightarrow$) Suppose A is closed in Y. Then, $Y \setminus A$ is open in Y. Thus, $\exists\, \mathcal{U} \subseteq X$ open s.t.

$$Y \setminus A = \mathcal{U} \cap Y$$

Then,
$$\underset{\underset{A}{\parallel}}{Y \setminus (Y \setminus A)} = Y \setminus (\mathcal{U} \cap Y)$$
$$= Y \cap (\mathcal{U} \cap Y)^c$$
$$= Y \cap (\mathcal{U}^c \cup Y^c)$$
$$= (Y \cap \mathcal{U}^c) \cap (Y \cup Y^c)$$

The $(\,)^c$ is complement in X.

$\Rightarrow \quad A = Y \cap \mathcal{U}^c$

Since $\mathcal{U}^c \subseteq X$ is closed, we are done.

Remark. A set can be both open and closed. For example, $\emptyset$ and X.

A less trivial example: Take $X = [0,1] \cup [2,3]$.

Then, $A = [0,1] \subset X$ is both open & closed.

Lecture 4 (18-01-2021)

18 January 2021 15:24

Def^n. Given a topological space X and $A \subset X$, we define:

(Interior) The interior of A as the union of all open sets contained in A.

Notation: $\text{int } A$ or $\mathring{A}$. $(\phi \subset A)$

(Closure) The closure of A as the intersection of all closed sets containing A.

Notation: $cl(A)$ or $\bar{A}$. $(A \subset X)$

Remark. $\mathring{A}$ is an open set and $\bar{A}$ is a closed set. Further,

$$\mathring{A} \subset A \subset \bar{A}.$$

A is open iff $A = \mathring{A}$.

A is closed iff $A = \bar{A}$.

Def^n. Let $x \in X$. A neighbourhood of x is any set A such that there is an open set $U \subset X$ with

$$x \in U \subseteq A.$$

(Neighbourhood)

(That is, a neighbourhood is any set containing an open set containing the point. This is different from the def^n in Munkres!)

Thm 1. Let A be a subset of a topological space X.

Then, $x \in \bar{A}$ iff every neighbourhood U of x intersects A.

Proof. $(\Leftarrow)$ $x \notin \bar{A} \Rightarrow U = X \setminus \bar{A}$ is a nbd of x not intersecting A.

$(\Rightarrow)$ Suppose $\exists$ nbd C of x s.t. $C \cap A = \phi$.

Let U be open s.t. $x \in U \subseteq C$. (Def^n of nbd.)

Then, $X \setminus U$ is a closed set s.t. $A \subset X \setminus U$.

$\Rightarrow \bar{A} \subset X \setminus U$ (Why? $\because \bar{A}$ is the inter. of all closed sets cont. A.)

$\Rightarrow \bar{A} \cap U = \phi$

$\Rightarrow \quad \bar{A} \cap \mathcal{U} = \phi$

$\Rightarrow \quad x \notin \bar{A}.$ $\blacksquare$

Examples. ① $X = \mathbb{R}$ and $A = (0,1]$. Then, $\bar{A} = [0,1]$.
However, if $X = (0,1] = A$, then $\bar{A} = A$.

② $X = \mathbb{R}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, $\bar{B} = B \cup \{0\}$.

③ $C = \{0\} \cup (1,2)$. Then, $\bar{C} = \{0\} \cup [1,2]$

④ $\bar{\mathbb{Q}} = \mathbb{R}$

⑤ $\bar{\mathbb{N}} = \mathbb{N}$

⑥ $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{0\} = [0, \infty)$.

Def^n. Let X be a top. space and $A \subset X$. (Limit point)
A point $x \in X$ is said to be a limit point of A if every neighbourhood of x intersects A in some point other than x.
Notation : A'

Examples

	Subset of $\mathbb{R}$	Set of limit points
①	$(1,2]$	$[1,2]$
②	$\{\frac{1}{n} : n \in \mathbb{N}\}$	$\{0\}$
③	$\{0\} \cup (1,2)$	$[1,2]$
④	$\mathbb{Q}$	$\mathbb{R}$
⑤	$\mathbb{N}$	ϕ
⑥	$\mathbb{R}_+$	$\overline{\mathbb{R}_+}$

Thm 2. $\bar{A} = A \cup A'$. (Proof at the end)

Corollary 3. A is closed iff $A' \subset A$.
Proof. A is closed $\Leftrightarrow A = \bar{A} \underset{\text{Thm 2.}}{\Leftrightarrow} A' \subset A$.

Def^n. (Order relation or Simple order)

A relation C on set A is called an order relation (or a simple order) if it has the following properties:

(1) (Comparability) For every $x, y \in A$, $x \neq y \Rightarrow xCy$ or yCx.

(2) (Non reflexity) $\nexists x \in A$ s.t. xCx

(3) (Transitivity) xCy and $yCz \Rightarrow xCz$.

A set with a simple order is called an ordered set.

Example Usual '$<$' on $\mathbb{R}$ is a simple order.

Def^n. If X is a set and '$<$' a simple order relation. Then, we define "$x \leq y$" as "$x < y$ or $x = y$."

Let $A \subset X$. An element $a \in A$ is said to be the smallest element of A if

$$a \leq x \quad \forall x \in A.$$

Similarly, we define the largest element.

(We have used "the" since uniqueness is simple to check. Existence, however, is not guaranteed. $\mathbb{R}$ has no largest or smallest element. Neither does $(0, 1)$.)

Def^n. If $(X, <)$ is an ordered set, then for $a, b \in X$, we define the intervals

$$(a, b) := \{x \in X : a < x < b\},$$
$$(a, b] := \{x \in X : a < x \leq b\},$$
$$[a, b) := \{x \in X : a < x < b\},$$
$$[a, b] := \{x \in X : a < x < b\}.$$

(Intervals)

Def^n. (Order topology)

Let $(X, <)$ be an ordered set. Let $\mathcal{B}$ be the collection

Let $(X, <)$ be an ordered set. Let $\mathcal{B}$ be the collection of sets of the form:

(1) All (a, b) for $a, b \in X$,
(2) All $[a_0, b)$ for $b \in X$ where $a_0 \in X$ is the smallest element of X, if any.
(3) All $(a, b_0]$ for $a \in X$ where $b_0 \in X$ is the largest element of X, if any.

Then, $\mathcal{B}$ is a basis (check) and the topology generated is called the order topology on X.

Example. The standard topology on $\mathbb{R}$ is the order topology derived from the usual order on $\mathbb{R}$.

Def^n (Dictionary order)
Suppose that $(A, <_A)$ and $(B, <_B)$ are two ordered sets.
We can define $<$ on $A \times B$ by

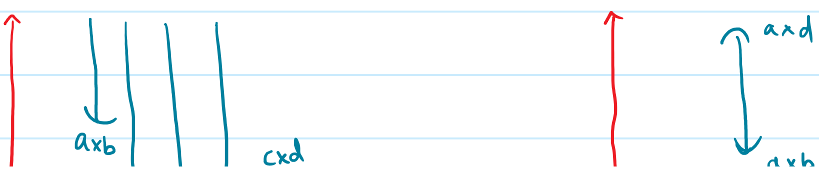
(we will denote elements of $A \times B$ by $a \times b$ instead of (a, b).)

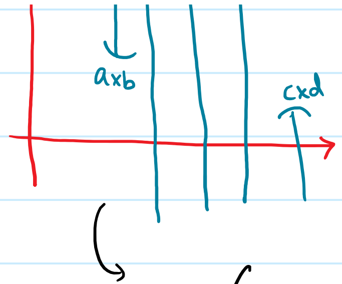
$$a_1 \times b_1 < a_2 \times b_2.$$

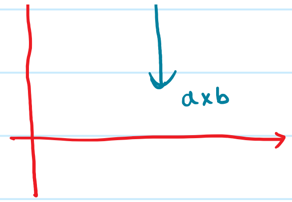
if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 < b_2$.

$<$ is a simple order on $A \times B$, called the dictionary order on $A \times B$.

Example $\mathbb{R} \times \mathbb{R}$ can be given an order topology in this dict. order.
A basis will be
$\{(a \times b, c \times d)\}$ where $a < c$ or $a = c$ & $b < d$.







$x \times y \in (a \times b,\ c \times d)$ iff
- $x = a$ and $b < y$ or
- $a < x < c$ and $y \in \mathbb{R}$
- $x = c$ and $y < d$

Remark If $Y = [0, 1) \cup \{2\}$, then $\{2\}$ is NOT open in the order topology.

Note that any basis element containing is of the form $B = (a, 2]$ with $a \in Y$.

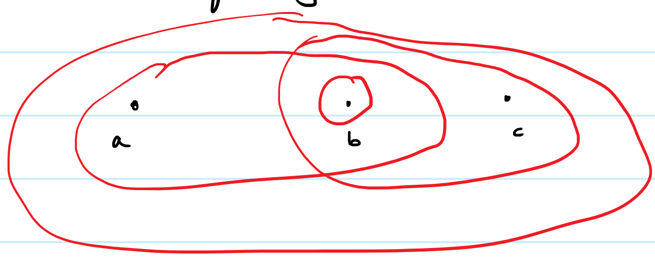
This means that $0 \le a < 1$ and hence, $\frac{a+1}{2} \in B$.

Thus, it always contains a point distinct from 2.

This shows that subspace and order topologies do not "commute"

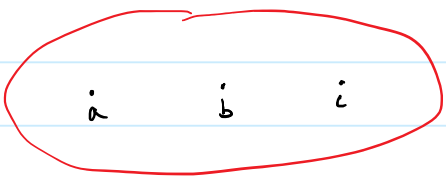
Remark. Singletons in $\mathbb{R}$ (or $\mathbb{R}^n$) are closed. This need not be true in general.

Consider the following topologies



$X = \{a, b, c\}$

$\mathcal{T} = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}$



$X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X\}$

$\{b\}$ is not closed in either of the above since $\{a, c\}$ is not open.

These spaces are not "nice". In fact, in the above spaces, a convergent sequence may have multiple limits. (Haven't defined this yet, though!)
We restrict ourselves to "nicer" spaces.

Def^n. A topological space X is called Hausdorff if for every distinct $x_1, x_2 \in X$, there exist neighbourhoods U_1, U_2 of x_1, x_2, respectively such that

$$U_1 \cap U_2 = \phi.$$

Thm 4. Every finite set in a Hausdorff space is closed.

Proof. It suffices to show the statement for singletons since finite unions of closed sets is closed.

Let $x_0 \in X$ be arbitrary. We show $\{x_0\}$ is closed.

Clearly, $\{x_0\} \subset \overline{\{x_0\}}$. Now, consider $y \in \{x_0\}^c$.

That is, $y \neq x_0$. By Hausdorffness, $\exists\, U_1, U_2$ nbds s.t.

$$x \in U_1, \quad y \in U_2 \quad \text{and} \quad U_1 \cap U_2 = \phi.$$

Thus, $U_2 \cap \{x_0\} = \phi$. Thus, $y \notin \overline{\{x_0\}}$. ∎

(Thm 1)

Proof of Thm 2. $\bar{A} = A \cup A'$.

($\subseteq$) Let $x \in \bar{A}$. Suppose $x \notin A$. We show $x \in A'$.

Let U be an arbit. nbd of x.

By Thm 1, $U \cap A \neq \phi$.

By assumption, $x \notin U \cap A$.

Thus, $x \in A'$, by def^n of A'.

($\supseteq$) $A \subset \bar{A}$ is clear. $A' \subset \bar{A}$ is also clear by def^n of A' and Thm 1. ∎

Lecture 5 (21-01-2021)

21 January 2021 15:36

Thm 1. Let X be a Hausdorff space, $A \subset X$, and $x \in X$.
Then, $x \in \bar{A} \Leftrightarrow$ every nbd of x contains infinitely many points of A.

Proof. ($\Leftarrow$) Trivial since infinitely many points imply one point apart from x.

($\Rightarrow$) Let x be a limit point.
For the sake of contradiction, let N be a nbd of x ($N \supseteq U \ni x$, U open) s.t. $A \cap (U \setminus \{x\}) = \{x_1, \ldots, x_n\}$ is finite.
Note $\{x_1, \ldots, x_n\}$ is closed since X is Hausdorff.
Thus, $V := U \cap (X \setminus \{x_1, \ldots, x_n\})$ is a nbd of x.
But $V \cap (A \setminus \{x\}) = \phi$. $\rightarrow\leftarrow$ $\boxtimes$

Note this makes sense even if $x \notin A$.

Recall from tutorial:

(1) Order top. is Hausdorff.
(2) Product of Hausdorff spaces is Hausdorff.
(3) Subspace of Hausdorff spaces is Hausdorff.

Def^n Continuous functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
A function $f: X \to Y$ is said to be **continuous** if $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_Y$.

In other words, inverse image of open sets (in Y) is open (in X).

Remark. By our earlier discussions, it is easily to see that it suffices to check that inverse images of basis (or subbasis) elements are open.

Recall $f^{-1}\left(\bigcup_{\alpha\in\Lambda} B_\alpha\right) = \bigcup_{\alpha\in\Lambda} f^{-1}(B_\alpha)$

$f^{-1}\left(\bigcap_{\alpha\in\Lambda} B_\alpha\right) = \bigcap_{\alpha\in\Lambda} f^{-1}(B_\alpha)$

Examples. (i) $f : \mathbb{R} \to \mathbb{R}_\ell$, $f(x) := x$ is not continuous since the topology of $\mathbb{R}_\ell$ is strictly finer.

(ii) $g : \mathbb{R}_\ell \to \mathbb{R}$, $g(x) := x$ is continuous.

Thm 2. Let X and Y be top. spaces and $f : X \to Y$.
TFAE
(i) f is continuous.
(ii) For every $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.
(iii) $f^{-1}(B)$ is closed for every closed $B \subset Y$.

Proof. (i) $\Rightarrow$ (ii)

Let $y \in f(\bar{A})$. Then, $y = f(x)$ for some $x \in \bar{A}$.
We show $x \in \overline{f(A)}$.
Let V be any open nbd. of y. (Want to show $V \cap f(A) \neq \phi$)
Then, $f^{-1}(V)$ is an open nbd of x.
Then, $A \cap f^{-1}(V) \neq \phi$. Let $x' \in A \cap f^{-1}(V)$.
Then, $f(x') \in f(A) \cap f(f^{-1}(V))$
$\Rightarrow f(x') \in f(A) \cap V$ (*)
Thus, $f(A) \cap V \neq \phi$, as desired.
Since any nbd contains an open nbd, we are done.

(ii) $\Rightarrow$ (iii) Let $B \subset Y$ be closed.
Put $A = f^{-1}(B)$. To show: A is closed.
A is closed $\Leftrightarrow A = \bar{A} \Leftrightarrow \bar{A} \subset A$.

$$x \in \bar{A} \Rightarrow f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B \quad (*)$$
$$\Rightarrow x \in f^{-1}(B)$$
$$\Rightarrow x \in A.$$

$(*)$ $f(f^{-1}(B)) \subset B$, in general. Equality if f onto.

(iii) $\Rightarrow$ (i) Obvious since $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$. $\boxtimes$

Def^n. Let X and Y be topological spaces and $f: X \to Y$ be a bijection. f is said to be a homeomorphism if f and f^{-1} are both continuous.

X and Y are said to be homeomorphic if there exists a homeomorphism $f: X \to Y$.

(Homeomorphism, homeomorphic)

A homeomorphism can also be defined as a bijection $f: X \to Y$ s.t. $f(U)$ is open in Y iff U is open in X.

Thus, f is not only a bijection of X and Y but also of $\mathcal{T}_X$ and $\mathcal{T}_Y$.

Def^n. Let $f: X \to Y$ be an injective continuous function. Let $Z = f(X)$ be the image of X in the supspace topology. Then, the restriction
$$f': X \to Z \quad \text{is a bijection.}$$
If f' is a homeomorphism, then we say that $f: X \to Y$ is a topological imbedding or an imbedding of X in Y.

(Imbedding)

Example (i) $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) := 2x + 4$ is a homeomorphism.

(ii) $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined $f(x) := \tan x$ is a homeomorphism.

(iii) $g : \mathbb{R}_\ell \to \mathbb{R}$ defined $g(x) := x$ is bijective and continuous but not a homeomorphism.

(iv) Let $S' := \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ be in subspace topology of $\mathbb{R}^2$.
Let $f : [0,1) \to S'$ be defined by

$$f(t) = (\cos 2\pi t, \ \sin 2\pi t).$$

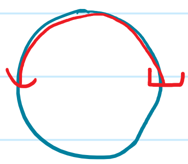
The f is bijective and continuous but f^{-1} is not continuous. To see the last part, consider

$$U = [0, 1/2) \in [0,1).$$

U is open but $f(U) \to$ top arc of S^1

not open in S'

note that $1 \times 0 \in$ top arc but no basis elt around that point.



Thm 3. Let X, Y, and Z be topological spaces.
If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof. Use $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. ∎

Def^n Box topology, Product Topology

Let J be an indexing set and $\{X_\alpha\}_{\alpha \in J}$ a collection of topological spaces.
Let us consider a basis for a topology on the Cartesian product

$$\prod_{\alpha \in J} X_\alpha ,$$

the collection of all set of the form

$$\prod_{\alpha \in J} U_\alpha ,$$

where each U_α is open in X_α. The topology induced is called the box topology.

Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \longrightarrow X_\beta$ be the projection map

$$\pi_\beta \left((x_\alpha)_{\alpha \in J} \right) = x_\beta .$$

Let $S_\beta = \{ \pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta \}$ and let

$$S = \bigcup_{\beta \in J} S_\beta .$$

Then S is a subbasis for a topology on $\prod_{\alpha \in J} X_\alpha$.
The topology generated is called the product topology.

Remark ① A typical basis elt. for prod. topology is

$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \qquad \left[\beta_1, \ldots, \beta_n \text{ p-wise distinct} \right]$$

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \begin{cases} U_{\beta_i} ; & \alpha = \beta_i \\ \ldots \end{cases}$$

$$= \prod_{\alpha \in J} U_\alpha \quad \text{where} \quad U_\alpha = \begin{cases} U_{\beta_i} & ; \ \alpha = \beta_i \\ X_\alpha & ; \ \text{else} \end{cases}$$

② If J is finite, both box and product coincide.

③ In general, box topology is finer than product.
If $|J| = \infty$, then it can be strictly finer.

(If each $X_\alpha = \mathbb{R}$, then strictly finer.
If each $X_\alpha = \{0\}$, then not.
If each X_α is in indsicrete topology, then not.)

Lecture 6 (25-01-2021)

25 January 2021 15:37

Thm 1. The box topology is finer than the product topology.

Proof. Every basis element of prod. topology is also one of box. ∎

Remarks (1) For finite products, the two are the same.

(2) If we simply refer to the product space, we shall mean the product topology, by default.

Thm 2. Let $f : A \longrightarrow \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \longrightarrow X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then, f is continuous iff each f_α is continuous

Proof. Note that $\pi_\beta : \prod X_\alpha \to X_\beta$ is continuous $\forall \beta$ since each $\pi_\beta^{-1}(U_\beta)$ is a subbasis element.

($\Rightarrow$) Now, suppose that $f : A \to \prod X_\alpha$ is continuous.
So, $f_\alpha = \pi_\alpha \circ f$ is continuous $\forall \alpha$.

($\Leftarrow$) Conversely, suppose each f_α is continuous.
It suffices to show that inverse images of subbasis elements are open.
A typical subbasis elt is $\pi_\alpha^{-1}(U_\alpha)$ for U_α open in X_α.
But $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha)$ ∎

open since f_α is continuous.

Remark. Above not true for box topology.

Take
$$f: \mathbb{R} \longrightarrow \prod_{i=1}^{\infty} \mathbb{R} \quad \text{given by}$$
$$t \longmapsto (t, t, \dots) \quad \text{is not continuous in box.}$$

Consider the open set $U = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$

Def^n. A metric d on a set X is a function
$$d: X \times X \longrightarrow \mathbb{R} \quad \text{satisfying}$$

(1) $d(x, y) \geq 0$ and $d(x,y) = 0 \iff x = y$.
(2) $d(x, y) = d(y, x)$
(3) $d(x, z) \leq d(x, y) + d(y, z)$

For a metric d on X, the number $d(x,y)$ is called the distance between x and y in metric d.
Given $\varepsilon > 0$, the set
$$B_d(x, \varepsilon) = \{ y \in X \mid d(x, y) < \varepsilon \}$$
is called the ε-ball centered at x.
We often write $B(x, \varepsilon)$ if d is understood.

The collection $\mathcal{B} = \{ B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0 \}$ is a basis and the topology induced is called the metric topology on X.

A topological space is called metrisable if there exists a metric on X which induces the given topology on X.

Examples. (1) Given a set X, define

$$d(x, y) = \begin{cases} 1 & ; x \neq y, \\ 0 & ; x = y. \end{cases}$$

This d is a metric and the topology induced is the discrete topology, since $B(x, 1) = \{x\}$.
(Thus, singletons are open and thus, every set is.)

(2) Standard topology on $\mathbb{R}$ is induced by
$$d(x, y) := |x - y|.$$

Note
$$(a, b) = B_d(x, \varepsilon) \quad \text{for} \quad x = \frac{a+b}{2} \text{ and}$$
$(a < b)$
$$\varepsilon = \frac{b-a}{2}.$$

(3) On $\mathbb{R}^n$, we have the Euclidean metric given by
$$d(x, y) = \|x - y\| = \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{1/2}.$$
Another example is the square metric
$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Both the metrics induce the same topology, which is the same as the usual product topology.

Thm 3. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function. Then,
f is continuous $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

Proof. Exercise. $\boxtimes$

Def^n. (Sequence and convergence)
Let X be a set. A sequence $(x_n)_{n=1}^{\infty}$ is a function $\mathbb{N} \to X$. $(n \mapsto x_n)$

It is said to converge to $x \in X$ if for every nbd U of x, $\exists n_0 \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq n_0$.
It is said to converge or be convergent if it converges to some $x \in X$.

Lemma. Let X be a topological space and $A \subset X$.
If $\exists$ a seq. $(x_n)_{n=1}^{\infty} \subset A$ which converges to $x \in X$, then $x \in \bar{A}$.
The converse is true if X is metrisable.

Proof. ($\Rightarrow$) Let $(x_n)_{n=1}^{\infty}$ and x be as in Lemma. Let U be an arbitrary nbd of x. We show $U \cap A \neq \phi$ to conclude.
By def^n of convergence, $\exists n_0 \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq n_0$.
Thus, $\phi \neq U \cap A$ since $x_{n_0+1} \in U \cap A$.

($\Leftarrow$) Assume d metrises X and $x \in \bar{A}$.
For each $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \cap A \neq \phi$.
For each $n \in \mathbb{N}$, pick $x_n \in B(x, \frac{1}{n})$. (Need some choice.)
Then, $d(x, x_n) < \frac{1}{n} \to 0$ and thus,
$$x_n \to x.$$
$\boxtimes$

(Note: An easy check that convergence of sequences in metric space coincides.)

Defn. A space X is said to have a countable basis at x if there is a countable collection $\mathcal{B}$ of open nbds of x s.t. each nbd of x contains an element $\mathcal{B}$. A space that has a countable basis at each $x \in X$ is said to be first countable.

Eg. $\mathbb{R}$, $\mathbb{R}^n$, take $\{ B(x, \frac{1}{n}) \mid n \in \mathbb{N} \}$ at each x.

Lemma 4½. The converse of Lemma 4 holds even if X is first countable. More generally, if $x \in \bar{A}$, then only countable basis at x is required.

Lecture 7 (28-01-2021)

28 January 2021 15:34

Thm 1. Let X, Y be topological spaces and $f: X \to Y$ be continuous.
(Suppose $x_n \to x$ in X. Then, $f(x_n) \to f(x)$ in Y.) $(*)$
If X is metrisable, then $(*)$ implies continuity.
(That is, if $f(x_n) \to f(x)$ for every convergent subsequence $x_n \to x$ for every $x \in X$, then f is continuous.)

Proof. Let $x_n \to x$ in X.
Let U be an arbitrary neighbourhood of $f(x)$.
Then, $f^{-1}(U)$ is a nbd of x.
Thus, $\exists N \in \mathbb{N}$ s.t. $x_n \in f^{-1}(U)$ $\forall n \geq N$.
Thus, $f(x_n) \in U$ $\forall n \geq N$ proving that $f(x_n) \to f(x)$.

Now, suppose that X is Hausdorff.
Assume that $(*)$ is satisfied.
It suffices to show $f(\bar{A}) \subset \overline{f(A)}$.
Let $A \subset X$ be arbitrary and let $y \in f(\bar{A})$.
Then, $y = f(x)$ for some $x \in \bar{A}$.
Thus, $\exists (x_n) \subseteq A$ s.t. $x_n \to x$. (Lemma 4 from Lec 6, X is metrisable.)
By our condition, $f(x_n) \to f(x)$ and $f(x_n) \in f(A)$.
Thus, $y = f(x) \in \overline{f(A)}$. (In general.) ∎

Remark. As in Lec 6, the "metrisable" can be relaxed to first countability.

Thm 2. If X is a topological space and $f, g : X \to \mathbb{R}$ are continuous, then $f \pm g$, $f \cdot g$ are continuous.
If $g(x) \neq 0$ $\forall x \in X$, then f/g is continuous.

Proof $+, -, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

$x \mapsto 1/x$ is continuous $\mathbb{R}\setminus\{0\} \to \mathbb{R}$.

Since $f \times g : X \to \mathbb{R} \times \mathbb{R}$ is continuous, we are done. ∎

Def^n. $\mathbb{R}^\omega := \prod_{n\in\mathbb{N}} X_n$ where $X_n = \mathbb{R}$ for all $n \in \mathbb{N}$.

Example (1) Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be defined by

$f(t) := (t, t, \ldots)$.

Then, (a) f is continuous if $\mathbb{R}^\omega$ is equipped with prod topology.

Thm 2 of Lec 6

(b) f is NOT continuous if $\mathbb{R}^\omega$ has box topology.

Note $B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$

$= \prod_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n})$ is a basis elt. of box topology.

Thus, Thm 2 of Lec 6 is not true

Then, $f^{-1}(B) = \{0\}$ is not open in $\mathbb{R}$.

(2) Again, take $\mathbb{R}^\omega$ in box topology.

Let $A = \{(x_1, x_2, \ldots) \mid x_i > 0 \ \forall i\}$.

Then, $\underline{0} \in \bar{A}$. (Here $\underline{0} = (0, 0, \ldots) \in \mathbb{R}^\omega$.)

std basis elt. around it: $B = (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) \times \cdots$

Then, $\underline{x} = (\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}, \ldots) \in B \cap \bar{A}$.

Claim. $\nexists\ (x_n) \subset A$ s.t. $\underline{x_n} \to \underline{0}$.

Proof. Assume not. Let $(\underline{x_n}) \subset A$ be s.t. $\underline{x_n} \to 0$.

Note that $\underline{x_n} = (x_{1n}, x_{2n}, \ldots)$ where $x_{in} > 0 \ \forall i$.

Define

$B = (-y_{11}, y_{11}) \times (-y_{22}, y_{22}) \times \cdots$.

Clearly, $\underline{0} \in B$ but $y_n \notin B \ \forall n$.

Thus, $y_n \not\to 0$.

Cor 3. $\mathbb{R}^\omega$ in box topology is not metrisable.

(Lemma 4 from Lec 6.)

Lemma 4 Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the metric

$$\rho(x,y) = \max\{|x_i - y_i| : 1 \le i \le n\}.$$

Then, ρ is a metric which induces the standard (product) topology on $\mathbb{R}^n$.

(The proof that ρ is indeed a metric is omitted.)

Proof. Let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a std. basis elt of prod. topology. Let $x = (x_1, \ldots, x_n) \in B$.

For each $i = 1, \ldots, n$, pick $\epsilon_i > 0$ s.t. $(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$.

Then, put $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$.

Then, $x \in B_d(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$

$$\subseteq (x_1 - \epsilon_1, x_1 + \epsilon_1) \times \cdots \times (x_n - \epsilon_n, x_n + \epsilon_n)$$

$$\subseteq B.$$

Conversely, each ϵ ball in the metric topology is a basis element of the product topology. ∎

Def^n. (Connected, separation)

Let X be a topological space.

A separation of X is a pair U, V of non-empty disjoint open subsets of X such that $U \cup V = X$.

X is said to be connected if no separation exists.

Lemma 5. A space X is connected iff the only clopen (closed as well as open) subsets of X are $\emptyset$ and X.

Proof. ($\Rightarrow$) Let U be a clopen set s.t. $\emptyset \ne U \ne X$.

Then, $V = U^c$ is also clopen and nonempty.

Then $X = U \cup V$. $\rightarrow\leftarrow$

($\Leftarrow$) Suppose X is not connected. Let U, V be a separation, then $\emptyset \neq U \neq X$ and $U = V^c$ is clopen. ☒

(Ex.) Let Y be a subspace of X and $A \subset Y$.
Then $\bar{A} \cap Y$ is the closure of A in Y.

Thm 6. A pair of disjoint non-empty sets A and B whose union is Y is a separation of Y iff neither contains a limit point of the other.

Proof. $(\Rightarrow)$ Then, $\underbrace{A = cl_Y(A)}_{\text{closure in } Y} = \underbrace{\bar{A}}_{\text{in } X} \cap Y$.

Claim. $\bar{A} \cap B = \emptyset$

Proof. $A = \bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B)$
$= A \cup (\bar{A} \cap B)$

Thus, $\bar{A} \cap B \subset A$.

Thus, $(\bar{A} \cap B) \cap A = \bar{A} \cap B$
$(\bar{A} \cap B) \cap A = \bar{A} \cap (B \cap A) = \emptyset$

Similarly, $A \cap \bar{B} = \emptyset$, as desired.

$(\Leftarrow)$ We only need to show that A and B are open in Y.
Equivalently, it suffices to show that A and B are closed in Y.
We know $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.
Thus, $\bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B)$
$= A \cup \emptyset = A$.

Thus, $cl_Y(A) = A$. Thus, A is closed in Y.
$\quad$ Similarly, B — " — . ☒

EXAMPLES (1) Any set in indiscrete topology is connected.

(2) $Y = (0,5) \cup (5,7) \subseteq \mathbb{R}$ is not connected.

$(0,5)$, $(5,7)$ form a separation.

(3) $Y = (0,5] \cup (5,7) = (0,7)$.

$(0,5]$, $(5,7)$ does NOT form a separation.

Note $(0,5]$ contains the limit point 5 of $(5,7)$.

Aliter: $(0,5]$ is not open in Y.

Later, we shall see that intervals in $\mathbb{R}$ are connected.

(4) $\mathbb{Q}$ is not connected. Let $I = (\sqrt{2}, \infty) \subset \mathbb{R}$.

$I \cap \mathbb{Q}$ is clearly open in $\mathbb{Q}$ since I is open in $\mathbb{R}$.

Now, $\mathbb{Q} \setminus (I \cap \mathbb{Q}) = (\mathbb{R} \setminus I) \cap \mathbb{Q}$

$= (-\infty, \sqrt{2}] \cap \mathbb{Q}$ $\quad \sqrt{2} \notin \mathbb{Q}$

$= (-\infty, \sqrt{2}) \cap \mathbb{Q}$

↳ also open. ∎

(5) Let $A = \mathbb{R} \times \{0\}$ and $B = \{(x,y) : x > 0,\ y = \frac{1}{x}\}$.

Put $Y = A \cup B \subseteq \mathbb{R}^2$ in subspace topology.

Then, A and B are closed in $\mathbb{R}^2$ and hence, in Y.

Since $A \cap B = \emptyset$, we are done. $(A \neq \emptyset \neq B)$

Lecture 8 (01-02-2021)

01 February 2021 20:58

Lemma 1. If the sets C and D form a separation of X and if Y is a connected subspace of X, then $Y \subseteq C$ or $Y \subseteq D$.

Proof. Note that $(Y\cap C)\cup(Y\cap D) = Y$ and $(Y\cap C)\cap(Y\cap D) = \emptyset$ with $Y\cap C$ and $Y\cap D$ open in Y. Thus, one must be empty. $Y \cap D = \emptyset \Rightarrow Y \subset C$ and $Y\cap C = \emptyset \Rightarrow Y \subseteq D$. ∎

Proof. Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of connected spaces.
Pick $p \in \bigcap A_\alpha$.
Put $Y = \bigcup A_\alpha$. Suppose, for the sake of contradiction, that $Y = C \cup D$ is a separation.
WLOG, $p \in C$. ($\because p \notin D$)
Now, given any $\alpha \in I$, we must have $A_\alpha \subset C$, by the previous theorem.
Thus, $A_\alpha \subset C$ $\forall \alpha$. Thus, $Y \subset C$ and hence, $D = \emptyset$. $\rightarrow\leftarrow$

Thm 3. If $A \subset X$ is connected and $B \subset X$ is such that $A \subset B \subset \bar{A}$, then B is connected.
In particular, $\bar{A}$ is connected.

Proof. Suppose $B = C \cup D$ is a separation.
Then, $A \subset C$ wlog. (A is connected.)
Thus, $\bar{A} \subset \bar{C}$. Moreover, $\bar{C} \cap D = \emptyset$, since (C, D) form a separation. Thus, $\bar{A} \cap D = \emptyset$.
(Thm 6, last lec)

form a separation. Thus, $\bar{A} \cap D = \phi$.
(Thm 6, last lec.)
$$B \cap D = D$$

Thus, $D = \phi$. ∎

Thm 4. Let $f: X \to Y$ be continuous. If X is connected, then $f(X)$ is connected.

Proof. Put $Z = f(X)$. Then, we get a function $f: X \to Z$.
Moreover, this new f is still continuous. (Z in subspace topology.)
(If $U \subseteq Z$ is open, then $U = V \cap Z$ for V open in Y.
Then, $f^{-1}(U) = f^{-1}(V \cap Z) = f^{-1}(V) \cap f^{-1}(Z) = f^{-1}(V) \cap X$
$= f^{-1}(V) \to$ open.)

We now look at the surjective map $f: X \to Z$.
Suppose $Z = A \cup B$ is a separation.
Then, $\underset{X}{f^{-1}(Z)} = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$.
non-empty and open
$\to \leftarrow$

Thm 5. Cartesian product of finitely many connected spaces is connected.

Proof. Since $X_1 \times \cdots \times X_n \simeq (X_1 \times \cdots \times X_{n-1}) \times X_n$ for $n \geq 3$,
it suffices to prove for $n = 2$.

Let X and Y be connected, we say $X \times Y$ is connected.
Fix a point $(a, b) \in X \times Y$. Let $x \in X$ be arbit.
Consider the connected sets $\{x\} \times Y$ $(\simeq Y)$ and $X \times \{b\}$ $(\simeq X)$.
Moreover, the slices have (x, b) in common. Thus,
$$T_x = (\{x\} \times Y) \cup (X \times \{b\})$$
is connected for each $x \in X$.
However, note that $(a, b) \in T_x \quad \forall x \in X$.

Thus, $\bigcup_{x \in X} T_x$ is connected. But $X \times Y = \bigcup_{x \in X} T_x$, as desired. ☒

Def^n. (Path, path-connected)

Let X be a topological space and $x, y \in X$.
A path from x to y in X is a function
$f: [0,1] \to X$ s.t. $f(0) = x$ and $f(1) = y$.
X is said to be path-connected if for any $x, y \in X$, there exists a path from x to y.

(Usually we may take $[a,b]$ instead of $[0,1]$.)

Fact: Intervals in $\mathbb{R}$ are connected. (Recall from $\mathbb{R}$ Analysis.)

Thm 6. Any path connected space is connected.

Proof. Suppose X is path-connected and $X = A \cup B$ is a sep.
Pick $x \in A$ and $y \in B$. By hypothesis, $\exists f: [0,1] \to X$ s.t. $f(0) = x$ & $f(1) = y$.
But $[0,1]$ is connected and thus, so is $f([0,1])$.
Thus, by Lemma 1, $f([0,1]) \subset A$ or $f([0,1]) \subset B$. $\rightarrow\leftarrow$

Examples. (1) The unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| < 1\} \subseteq \mathbb{R}^n$ is path-connected. (The straight line path works.)

(2) $\mathbb{R}^n \setminus \{\underline{0}\}$ is path-connected if $n > 1$.

Proof. Let $x, y \in \mathbb{R}^n \setminus \{\underline{0}\}$. If $\underline{0}$ does not lie on the line seg. joining x and y, take that line seg.
Else, pick z not on line and join x to z and z to y. ∎

If $n=1$, then $\mathbb{R}\setminus\{0\} = (-\infty, 0)\cup(0,\infty)$ is not even connected, let alone path-connected.

(3) For $n \geq 2$, define $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \subseteq \mathbb{R}^n$.
It is path-connected. To see this, define

$$g : \mathbb{R}^n \setminus\{0\} \to S^{n-1} \text{ by}$$
$$x \mapsto x/\|x\|.$$

Then, g is continuous and maps $\mathbb{R}^n\setminus\{0\}$ <u>onto</u> S^{n-1}.
The image of a path-connected space is path-connected and hence, S^{n-1} is path-connected.
(Ex.)

(Ex.) Continuous image of path connected space is path-connected.

<u>Proof.</u> Let $g : X \to Z$ be continuous and onto.
Pick $z_1, z_2 \in Z$. Then, $\exists x_1, x_2 \in X$ s.t. $x_1 \mapsto z_1$ & $x_2 \mapsto z_2$.
Now, $\exists \gamma : [0,1] \to X$ s.t. $x_1 \xrightarrow{\gamma} x_2$.

Then, $g\circ\gamma : [0,1] \to Z$ is continuous

and $(g\circ\gamma)(0) = z_1$ & $(g\circ\gamma)(1) = z_2$. ☒

Lecture 9 (08-02-2021)

08 February 2021 15:32

(4) Let $S = \{(x, \sin\frac{1}{x}) : 0 < x \le 1\}$.

The set $\bar{S}$ is called the topologist's sine curve.

(Topologist's sine curve)

$\bar{S} = S \cup \{0\} \times [-1, 1]$.

Note that S is connected, being the image of a connected set $(0, 1]$ under a continuous map $x \mapsto (x, \sin\frac{1}{x})$.

As seen, this implies $\bar{S}$ is connected.

Claim. However, $\bar{S}$ is not path-connected.

Proof. Suppose not. Let $f : [a, c] \to \bar{S}$ be a path from $(0, 0)$ to $(1, \sin 1)$.

Let $D = f^{-1}(\{0\} \times [-1, 1])$. $D \subset [a, c]$ is closed.

Thus, $b = \sup D \in D$.

$\therefore f : [b, c] \to \bar{S}$ has the property that $f(b) \in \{0\} \times [-1, 1]$ but $f(b) \in S$ for $x > b$.

WLOG, $[b, c] = [0, 1]$. Write $f(t) = (x(t), y(t))$.

Claim. $\exists (t_n) \subset (0, 1]$ s.t. $t_n \to 0$ and $y(t_n) = (-1)^n$.

Proof. For $n \in \mathbb{N}$, $x(1/n) > 0$. Thus, we can choose u_n s.t. $0 < u_n < x(1/n)$ and $\sin(1/u_n) = (-1)^n$.

By IVT, $\exists\, t_n$ s.t. $0 < t_n < 1/n$ and $x(t_n) = u_n$.

Thus, $y(t_n) = \sin(1/x(t_n)) = \sin(1/u_n) = (-1)^n$.

$0 < t_n < 1/n \Rightarrow t_n \to 0$. ☒

Thus, $t_n \to 0$ and $y(t_n)$ does not converge. Thus, y is not continuous. Therefore, f is not continuous. $\rightarrow\leftarrow$ ☒

Def^n. (Connected components)

Given X, define the equivalence relation $x \sim y$ if $\exists\, a$

connected subset of X containing x and y.
The equivalence classes are called the **components** or **connected components**.

Remark. Reflexive: $\{x\}$ is connected.
Sym.: Trivial.
Transitive: Let $x \sim y$ and $y \sim z$. $\exists A, B \subset X$ connected s.t. $x, y \in A$ and $y, z \in B$. Then, $A \cup B$ is connected since $y \in A \cap B$. But $x, z \in A \cup B$. $\therefore x \sim z$.

Thm 1. The components of X are connected disjoint subsets of X whose union is X, s.t. each connected subset of X intersects only one of them.

Proof. The part about being disjoint and union being X follows because $\sim$ was an equiv. relation.
Now, suppose A is a connected set s.t. A intersects the components C_1 and C_2. Let $x_1 \in A \cap C_1$ and $x_2 \in A \cap C_2$.
But then, $x_1, x_2 \in A$ and hence, $x_1 \sim x_2$. $\therefore C_1 = C_2$.
This proves the second part.
We just have to prove that each component C is connected.
Fix $x_0 \in C$. $\forall x \in C$, $x_0 \sim x$. $\therefore \exists A_x$ s.t. $x, x_0 \in A_x$ and A_x connected. By the earlier part, $A_x \subset C$.
$\therefore A_x \subset C \quad \forall x$

$$\Rightarrow C = \bigcup_{x \in C} A_x. \qquad \text{But } \bigcap_{x \in C} A_x \ni x_0.$$

$$\therefore C = \bigcup_{x \in C} A_x \text{ is connected.} \qquad \blacksquare$$

Def^n (Cover, open cover) A collection $\mathcal{U}$ of subsets of X is said to be a **cover** of X if $\bigcup_{U \in \mathcal{U}} U = X$.

If each $U \in \mathcal{U}$ is open, then $\mathcal{U}$ is said to be an open cover of X.

Def^n. (Compact) X is said to be compact if every open cover (of X) has a finite sub-cover.

Examples (1) $\mathbb{R} \to$ not compact

$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2)$$ but no finite subcover since $\mathbb{R}$ is not bounded.

(2) $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

Let $\mathcal{U}$ be an open cover of K.

$\exists\, U_0 \in \mathcal{U}$ s.t. $0 \in \mathcal{U}_0$.

Thus, $\exists N$ s.t. $K \cap [0, \frac{1}{N}) \subset U_0$.

Now, for $k = 1, \dots, N$, choose $U_k \in \mathcal{U}$ s.t. $\frac{1}{k} \in U_k$.

Then, $K \subset U_0 \cup U_1 \cup \dots \cup U_N$. ☒

(3) $(0, 1]$ not compact. $(0, 1] = \bigcup_{n \geq 2} (\frac{1}{n}, 1]$.

Def^n. If Y is a subspace of X, and $\mathcal{C}$ a collection of subsets of X, then $\mathcal{C}$ is said to cover Y if

$$Y \subseteq \bigcup_{C \in \mathcal{C}} C.$$

Lemma 2. Let Y be a subspace of X.
Then, Y is compact (in subspace topology) iff every covering of Y by sets open in X contains a finite subcollection covering Y. ☒

Lemma 3. Every closed subspace of a compact space is compact. ∎

Thm 4. Every compact subspace of a Hausdorff space is closed.

Proof. Let $Y \subseteq X$ be closed, where $X \leftarrow$ Hausdorff.

We prove that $X \setminus Y$ is open.

Let $x_0 \in X \setminus Y$. For each $y \in Y$, $\exists$ disjoint open nbds U_y and V_y of x_0 and y, resp. The collection

$$\{V_y : y \in Y\}$$

covers Y. Thus, $\exists\, y_1, \ldots, y_n \in Y$ s.t. $Y \subset V_{y_1} \cup \cdots \cup V_{y_n}$.

(Y is compact.)

Then, $U_{y_1} \cap \cdots \cap U_{y_n}$ is an open nbd of x contained in $(V_{y_1} \cup \ldots \cup V_{y_n})^c \supset Y^c$.

$\therefore$ $Y \setminus X$ is open.

Remark. The above proof shows the following:

If X is Hausdorff, $Y \subseteq X$ is compact, and $x_0 \notin Y$, then $\exists$ disjoint open sets U and V of X containing x_0 and Y, resp.

Lecture 10 (10-02-2021)

10 February 2021 16:05

Thm 1. The continuous image of a compact set is compact. ☒

Thm 2. Let $f: X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

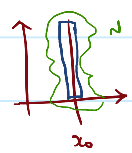
Proof. We need to show f^{-1} is continuous.
Let $K \subseteq X$ be closed. Then, K is closed, since X is compact.
Thus, $f(K) \subseteq Y$ is compact. Then, $f(K)$ is closed, since Y is Hausdorff. Thus, f is a closed map and hence, f^{-1} is continuous. ☒

Thm 3. The cartesian product of finitely many compact spaces is compact.

Proof. As for the case of connectedness, it suffices to prove for product of two spaces.

Proof. Let Y be a compact space.

Step 1. Suppose that $x_0 \in X$ and N is an open set in $X \times Y$ containing the "slice" $\{x_0\} \times Y$. We show that $\exists$ nbd W of x_0 in X s.t. $W \times Y \subseteq N \times Y$.
↳ called a "tube"



$N = \bigcup_{i \in J} U_i \times V_i$

First, we cover $\{x_0\} \times Y$ by $\{U_i \times V_i\}$ → open sets, basis elements.
By compactness, we can cover by finitely many, $i = 1, \dots, n$.
WLOG, assume that $(U_i \times V_i) \cap (\{x_0\} \times Y) \neq \emptyset$ for $i = 1, \dots, n$.
Then, $W = U_1 \cap \dots \cap U_n$ is a neighbourhood of x_0.
Then, $W \times Y \overset{\subseteq N}{}$ is the desired tube.

Now, assume X is also compact.

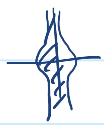
Step 2. Let $\mathcal{A}$ be an open covering of $X \times Y$.
Given $x_0 \in X$, $\{x_0\} \times Y$ is compact and hence covered by finitely many $A_1, \ldots, A_n \in \mathcal{A}$. Then,
$$N = A_1 \cup \cdots \cup A_n \text{ is an open set}$$
containing $\{x_0\} \times Y$.
By step 1, $\exists$ tube $W \times Y$ s.t. $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.

Thus, for each $x \in X$, $\exists W_x$ s.t. $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$. By compactness of X, X is covered by finitely many $W_{x_1}, \ldots, W_{x_n}$.
Each corresponding tube is covered by finitely many elements of $\mathcal{A}$. $\blacksquare$

Thm 4 (Tube Lemma) Let Y be compact and $x_0 \in X$. Let $N \subseteq X \times Y$ be open such that $\{x_0\} \times Y \subseteq N$. Then, $\exists$ open $W \subseteq X$ s.t. $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.

Proof. Step 1 of earlier. $\blacksquare$

Remark. Compactness of Y is needed.
Take $X = Y = \mathbb{R}$ and $0 \in X$ and
$$N = \left\{ (x, y) \in \mathbb{R}^2 : |x| < \frac{1}{1+y^2} \right\}.$$



No tube exists!

Lecture 11 (11-02-2021)

11 February 2021 15:36

Def^n. A collection $\mathcal{C}$ of subsets of X is said to satisfy the finite intersection condition if for every finite subcollection $\{C_1, \dots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \dots \cap C_n$ is non-empty.

Thm 1. Let X be a topological space. Then X is compact iff every collection $\mathcal{C}$ of closed sets in X satisfying the finite intersection condition satisfies $\bigcap_{C \in \mathcal{C}} C \neq \phi$.

Proof. For $\mathcal{C}$, define $\mathcal{U}_{\mathcal{C}} = \{X \setminus C : C \in \mathcal{C}\}$.

$\mathcal{C} \to$ closed sets, $\mathcal{U}_{\mathcal{C}} \to$ open sets

$\bigcap_{C \in \mathcal{C}} C = \phi \iff \mathcal{U}_{\mathcal{C}}$ is an open cover.

$\mathcal{C}$ has finite inter. property $\iff$ no finite subcollection of $\mathcal{U}_{\mathcal{C}}$ covers X.

Conclude! ☒

Cor 2. If X is compact and $X \supset C_1 \supset C_2 \supset \cdots$ with $C_n \neq \phi$ for all $n \in \mathbb{N}$, then $\bigcap_{n \geq 1} C_n \neq \phi$.

Proof. $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ satisfies finite intersection property. ☒

Def^n. (Limit point compact) A space X is said to be limit point compact if every infinite subset of X has a limit point.

Thm 3. Compactness $\Rightarrow$ Limit point compactness.

Proof. Let X be compact and $A \subset X$ be s.t. A has no limit point. We show A is finite.
Since $A' = \phi$, $\bar{A} = A$; i.e., A is closed.
For each $a \in A$, we can choose an open nbd U_a of a that does not intersect $A \setminus \{a\}$.
(Since a is not a lt. point of A.)
Note that $\{U_a : a \in A\}$ cover A. Since A is closed in X, A is compact. Thus, it has a finite subcover.
But $|U_a \cap A| = 1$ $\forall a \in A$, we see that A is finite. ☒

Remark. The converse of the above is not true.
Consider any set Y with two points $\{\bar{0}, \bar{1}\}$ and give it the indiscrete topology.
Consider $X = \mathbb{N} \times Y$ in product topology.
Then, any non-empty subset of X has a lt. point.
(A basis of X is $\{\{n\} \times Y : n \in \mathbb{N}\}$. Thus, given any $\emptyset \neq A \subseteq X$, pick $(n, x) \in A$. Then, $(n, 1-x)$ is in any nbd of (n, x).)
However, X is NOT compact. We have
$$X = \bigcup_{n \in \mathbb{N}} \{n\} \times Y.$$ ☒

Def^n. (Sequentially compact) A space X is said to be **sequentially compact** if every sequence has a convergent subsequence.

Thm 4. Let X be a metrisable space. TFAE:
(i) X is compact.
(ii) X is limit point compact.
(iii) X is sequentially compact.

Proof. (i) $\Rightarrow$ (ii) Previous theorem.

(ii) $\Rightarrow$ (iii)

Let (a_n) be a sequence in X. If (a_n) has a constant subsequence, we are done. Thus, assume not. Then, $A = \{a_n : n \in \mathbb{N}\}$ is infinite and hence, has a limit point x.

Pick an element in $A \cap B(x,1)$. It is of the form a_{k_1} for some $k_1 \in \mathbb{N}$.

Assume we have chosen $k_1 < k_2 < \dots < k_n$ s.t.

$$a_{k_i} \in B(x, 1/i) \cap A \quad \forall\, i = 1, \dots, n.$$

Now, $B(x, 1/n+1) \cap A$ is infinite.

Thus, can choose $k_{n+1} > k_n$ s.t. $a_{k_{n+1}} \in B(x, \frac{1}{n+1}) \cap A$.

Then, $a_{n_k} \longrightarrow x$ as $k \longrightarrow \infty$.

(iii) $\Rightarrow$ (i) ① We show the following statement:

Let $\mathcal{A}$ be an open cover of X. Then $\exists \delta > 0$ s.t. for each subset of X having diameter less than δ, there is an element of $\mathcal{A}$ containing it.

$\to$ Let $\mathcal{A}$ be an open cover for which no such δ exists. Taking $\delta = 1/n$, we get sets B_n such that $\operatorname{diam}(B_n) < \delta$ and $B_n \not\subset A \quad \forall\, A \in \mathcal{A}$.

Choose an $x_n \in B_n \quad \forall n$. Then (x_n) has a convergent subsequence (x_{n_k}). Let $x \in X$ be the limit. Choose $\epsilon > 0$ s.t. $B(x, \epsilon) \subset A$.

Eventually, $x_{n_k} \in B(x, \epsilon/2)$.

By choose k large enough, $B_{n_k} \subset B(x, \epsilon) \subset A$. $\rightarrow\leftarrow$

② We show the following:

For every $\epsilon > 0$, $\exists$ a finite covering of X by ϵ-balls.

→ Assume $\exists \epsilon > 0$ for which the above does not hold.

Choose $x_1 \in X$ arbitrarily. Then, $B(x_1, \epsilon) \neq X$.

Choose $x_2 \in X - B(x_1, \epsilon)$.

Inductively, we can choose

$$x_{n+1} \in X - \bigcup_{i=1}^{n} B(x_i, \epsilon).$$

Then, $(x_n)_{n=1}^{\infty}$ has a convergent subsequence (x_{n_k}).

But $d(x_i, x_j) \geq \epsilon \quad \forall i \neq j$. $\therefore$ No subseq can conv. $\Rightarrow\Leftarrow$

③ Now, we show X is compact.

Let $\mathcal{A}$ be an open cover of X. Choose $\delta > 0$ such that ① holds.

By ②, X can be covered by finitely many $\delta/3$-balls.

Since each have diameter $\leq 2\delta/3$, each of them lie in an element of $\mathcal{A}$. Choose one such element for each ball (of which there are finitely many). These elements form a finite subcover. $\blacksquare$

Remark. δ is called the Lebesgue number of $\mathcal{A}$.

Lecture 12 (18-02-2021)

18 February 2021 15:36

Defn: Let $p: X \to Y$ be a surjective function. The map is said to be a quotient map if any $U \subseteq Y$ is open iff $p^{-1}(U) \subseteq X$ is open.

Can replace "open" with "closed" since $p^{-1}(U^c) = (p^{-1}(U))^c$.

(quotient map)

Remarks:

(1) A quotient map is continuous.

(2) It need not be bijective.

(3) A homeomorphism is a quotient map.

(4) Quotient + Injective $\Rightarrow$ Homeomorphism

(5) Surj. + open map $\Rightarrow$ Quotient ($\Leftarrow$) not true!

(6) Surj. + closed map $\Rightarrow$ Quotient ($\Leftarrow$) not true!

Defn. A subset $C \subseteq X$ is said to be saturated w.r.t. p if

$$p^{-1}(\{y\}) \cap C \neq \emptyset \Rightarrow p^{-1}(\{y\}) \subseteq C.$$

(Saturated)

(That is, if C contains one pre-image, it contains all.)

Remark. Thus, p is a quotient map iff p is a continuous surjection that maps open saturated sets to open sets.
(Or "closed" instead of "open"!)

Example ① Let $X = [0,1] \cup [2,3] \subseteq \mathbb{R}$ and $Y = [0,2] \subseteq \mathbb{R}$.

Define $p: X \to Y$ by

$$p(x) = \begin{cases} x & ; x \in [0,1] \\ x-1 & ; x \in [2,3] \end{cases}$$

p is continuous and surjective. Moreover, it is closed because X is compact and Y Hausdorff.
Thus, p is a quotient map.
(Not homeomorphism since $p(1) = p(2)$.)
However, p is NOT open. $[0,1] \subset X$ is open but

$p([0,1]) = [0,1] \subset Y$ is NOT open.

(Note that $[0,1]$ is NOT saturated since $p^{-1}(\{1\}) \cap [0,1] \neq \emptyset$ but $p^{-1}(\{1\}) \not\subset [0,1]$.)

② Let $A = [0,1) \cup [2,3] \subseteq X$.

Define $q: A \to Y$ by $q = p|_A$.

Then, q is a bijection and thus, every subset is saturated. However, $[2,3]$ is open in X but $q([2,3])$ is not open in Y. (Note q is continuous!)

Def^n. Let X be a topological space A a set. Let $p: X \to A$ be a surjective function. Then, there exists a unique topology on A which q a quotient map. This is called the quotient topology on A.

(Quotient topology)

Proof.

Let $\mathcal{T} = \{U \subset A : p^{-1}(U) \text{ is open in } X\}$.

T.S.T. $\mathcal{T}$ is a topology.

① $\emptyset = p^{-1}(\emptyset)$ and $A = p^{-1}(X)$. $\therefore \emptyset, A \in \mathcal{T}$.

② $p^{-1}\left(\bigcup_{\alpha \in \Lambda} U_\alpha\right) = \bigcup_{\alpha \in \Lambda} p^{-1}(U_\alpha)$ and

③ $p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$ Show

closure under finite intersec. and arbitrary union.

Uniqueness is clear. That $p: X \to A$ is a quotient map is also clear.

Def^n. Let X be a topological space and X^* a partition of X. Let $p: X \to X^*$ be the natural projection map. (This is surjective) The space X^*

with the quotient topology induced by p is called a quotient space of X.

Recall that partitions of a set are equivalent to an equivalence relation X.
(no pun intended)

A subset $\mathcal{U} \subseteq X^*$ is a collection of equivalence classes and $p^{-1}(\mathcal{U}) \subset X$ is simply the union of those equivalence classes.

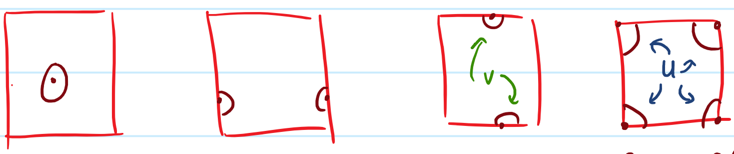
Example. ③ Let $X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$ be the closed unit disc in $\mathbb{R}^2$.

Let $X^* = \{\{(x,y)\} : x^2+y^2 < 1\} \cup \{S^1\}$.

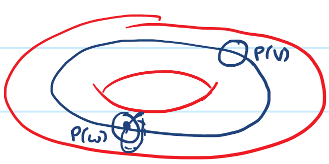
(All $\{(x,y)\}$ for $(x,y) \in X^\circ$ and S^1.)

④ $X = [0,1] \times [0,1]$.

$$X^* = \{\{(x,y)\} : (x,y) \in \mathring{X}\} \cup \{\{(0,y),(1,y)\} : 0<y<1\} \cup \{\{(x,0),(x,1)\} : 0<x<1\} \cup \{\{(0,0),(1,0),(1,1),(0,1)\}\}.$$



some saturated sets



Thm. Let $p : X \to Y$ be a quotient map.
Let A be a subspace which is saturated w.r.t. p.
Let $q : A \to p(A)$ be the restriction $p|_A$.
Then,

(1) If A is either open or closed, then q is

a quotient map.

(2) If p is either open or closed, then q is a quotient map.

Proof. Step 1.

Claim. (a) $V \subset p(A) \Rightarrow q^{-1}(V) = p^{-1}(V)$ and

(b) $p(U \cap A) = p(U) \cap p(A)$ if $U \in X$.

Proof. (a)($\subseteq$) Let $x \in q^{-1}(V)$.

Then $q(x) \in V \subset p(A)$.

$\therefore$ $q(x) \in p(A)$. Thus, $q(x) = p(a)$ for some $a \in A$.
$= q(a)$

Since a is saturated, $x \in p^{-1}(\{p(a)\}) \subset A$.

($\supseteq$) Let $x \in p^{-1}(V)$. Then, $p(x) \in V \subset p(A)$. Same argument again.

(b) $p(U \cap A) \subset p(U) \cap p(A)$ is true in general.

($\supseteq$) Let $y \in p(U) \cap p(A)$.

Then, $y = p(u) = p(a)$ for some $a \in A$ and $u \in U$.

$a \in p^{-1}\{y\} \cap A$.

$\therefore$ $p^{-1}\{y\} \subset A$ and hence, $u \in A$.

$\therefore u \in U \cap A$.

Step 2. ① Suppose A is open in X.

Claim. q is a quotient map.

Proof. Let $V \subseteq p(A)$.

Suppose $q^{-1}(V)$ is open in A. A is open in X.

Then, $q^{-1}(V)$ is open in X. (a) of Step 1.
$\parallel$

Then, $p^{-1}(V)$ is open in X.

Then, V is open in Y since p is quotient.

In particular, V is open in $p(A)$. ∎

② Suppose p is open.

Claim. q is a quotient map.

Proof. Let $V \subseteq p(A)$ be s.t. $q^{-1}(V)$ is open (in A).

Then, $p^{-1}(V) = q^{-1}(V)$ is open.

$p^{-1}(V)$ is open in A and hence,

$$p^{-1}(V) = U \cap A \quad \text{for } U \subseteq X \text{ open.}$$

$$\Rightarrow p(p^{-1}(V)) = p(U \cap A) \quad \text{(b) from Step 1.}$$

onto

$$\Rightarrow V = p(U) \cap p(A)$$

But U is open and p is an open map.

Thus, V is open in $p(A)$.

Thus, q is a quotient map.

Step 3. Do the same as prev. step by replacing "open" with "closed". ∎

Lecture 13

21 February 2021 11:19

Prop^n 1 Let $p: X \to Y$, $q: Y \to Z$ be quotient maps.
Then $q \circ p : X \to Z$ is a quotient map.

Proof. Clearly, $q \circ p$ is surjective and continuous.
Let $U \subseteq Z$ be s.t. $(q \circ p)^{-1}(U)$ is open.
That is, $p^{-1}(q^{-1}(U))$ is open.
Since p is quotient, $q^{-1}(U)$ is open. Since q is quotient, U is open. ∎

Thm 2. Let $p: X \to Y$ be a quotient map. Let Z be a topological space. Let $g: X \to Z$ be a map such that g is constant on each $p^{-1}(\{y\})$ for $y \in Y$.

In other words, $p(x) = p(x') \Rightarrow g(x) = g(x')$.
Let us refer to this as "g respects p."

Then, g induces a map $f: Y \to Z$ s.t.
$f \circ p = g$ and

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

(i) f is continuous iff g is continuous.
(ii) f is a quotient map iff g is a quotient map.

Proof Since g respects p and p is onto, we get a unique well-defined map $f: Y \to Z$ defined by
$$f(p(x)) = g(x).$$
(Since each $y \in Y$ is of the form $p(x)$ and if $p(x) = p(x')$, then $g(x) = g(x')$.)

(i) If f is continuous, then $g = f \circ p$ is continuous, being the composition of continuous maps.

Conversely, suppose g is continuous.

Let $U \subseteq Z$ be open.

Then, $p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U) = g^{-1}(U)$ is open since g is cont. But p is quotient. Thus, $f^{-1}(U)$ is open. Hence, f is continuous.

(ii) If f is quotient, then $g = f \circ p$ is, by Prop^n 1

Conversely, let g be a quotient map.

$\therefore$ g is onto and hence, so is f.

Also, g is continuous and hence, so is f, by (i).

Now, let $U \subseteq Z$ be s.t. $f^{-1}(U)$ is open.

TS: U is open.

Note $f^{-1}(U)$ open $\xRightarrow{p \text{ cont.}}$ $p^{-1}(f^{-1}(U))$ is open $\Big\rangle$ $g = f \circ p$

$\Rightarrow$ $g^{-1}(U)$ is open $\Big\rangle$ g quotient

$\Rightarrow$ U is open ∎

Cor 3. Let $g : X \to Z$ be a surjective continuous map

Let X^* be the partition induced by the equivalence relation $\sim$ on X given by

$$x \sim x' \iff g(x) = g(x').$$

$$\left(X^* = \{ g^{-1}(\{z\}) : z \in Z \}. \right)$$

Consider X^* with the quotient topology induced by the natural $p : X \twoheadrightarrow X^*$.

(a) g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism iff g is a quotient map.

(b) If Z is Hausdorff, then so is X^*.

Proof. Here, $Y = X^*$ and p is the canonical map.

By construction, f is bijective. (It was already onto, it is 1-1, since X^* is precisely the partition based on fibers of g.)

By the previous theorem, $f : X^* \longrightarrow Z$ is continuous and bijective.

(a) Now, f is a homeo $\Leftrightarrow$ f is quotient
$$\Updownarrow \quad \text{Thm 2 (ii)}$$
g is quotient

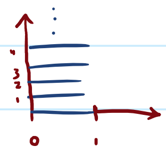
(b) Let Z be Hausdorff.

Let $x, y \in X^*$ be s.t. $x \neq y$. Since f is 1-1, $f(x) \neq f(y)$ in Z.

$\therefore \exists\, U \ni f(x),\ V \ni f(y)$ open s.t. $U \cap V = \phi$.

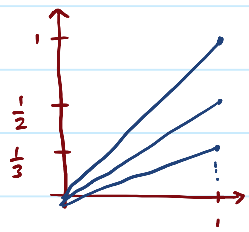
Then, $f^{-1}(U) \cap f^{-1}(V) = \phi$ and are nbds of x and y, resp.

Example Let $X = \bigcup_{n \in \mathbb{N}} \left([0,1] \times \{n\} \right) \subseteq \mathbb{R}^2$



be in subspace topology and let

$$Z = \left\{ \left(x, \frac{x}{n} \right) : x \in [0,1],\ n \in \mathbb{N} \right\} \subseteq \mathbb{R}^2$$



also be in subspace top.

Define $g : X \longrightarrow Z$ by

$$g(x, n) = \left(x, \frac{x}{n} \right).$$

Now, if $z \in Z \setminus \{(0,0)\}$, then $g^{-1}(\{z\})$ is a singleton. But if $z = (0, 0) \in Z$, then

$$g^{-1}(\{z\}) = \{(0, n) : n \in \mathbb{N}\}.$$

Now, take $X^* = \{g^{-1}(\{z\}) : z \in Z\}$. ← give quotient topology

By the earlier, we have a bijective continuous map

$$f : X^* \longrightarrow Z.$$

Q. Is f a homeomorphism?

A. No.

Consider the set $A = \{(\frac{1}{n}, n) \in X : n \in \mathbb{Z}\}$.

A is closed since $A' = \phi$. Moreover A is saturated w.r.t. g. However,

$$g(A) = \{(\frac{1}{n}, \frac{1}{n^2}) : n \in \mathbb{N}\}$$

does have a limit point outside $g(A)$.

Thus, $g(A)$ is not closed. Thus, g is not a quotient map and hence, f is not a homeo. ∎

Lecture 14 (03-03-2021, 04-03-2021)

03 March 2021 16:10

Def^n A space is said to be second countable if it has a countable basis. (second countable)

Remark Metric spaces need not be second countable. (They are first countable, though.) Take X uncountable with discrete.

Ex. $\mathbb{R}$ with standard topology. $\{(a,b) : a, b \in \mathbb{Q}\}$ is a countable basis.
Even $\mathbb{R}^n$. $\{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{Q}\}$ is —"—.
The space $\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} \mathbb{R}$ (in prod. top.) is also second countable.
↳ A countable basis : $\left\{ \prod_{n \in \mathbb{N}} U_n : \begin{array}{l} U_n = (a_n, b_n) \text{ for } a_n, b_n \in \mathbb{Q} \text{ for fin many } n. \\ U_n = \mathbb{R} \text{ else.} \end{array} \right\}$

Def^n. $A \subset X$ is said to be dense in X if $\bar{A} = X$. (dense)

Def^n. (a) A space for which every open covering contains a countable subcovering is called a Lindelöf space. (Lindelof, Lindelöf)

(b) A space having a countable dense subset is said to be separable. (separable)

Thm 1. Let X be second countable.
Then,
(a) X is Lindelöf.
(b) X is separable.

Proof. let $\{B_n : n \in \mathbb{N}\}$ be a countable basis for X.
(a) Let $\mathcal{C}$ be an open covering of X.

Let $I = \{ n \in \mathbb{N} : B_n \subset C$ for some $C \in \mathcal{C} \}$.

For each $n \in I$, pick some $C \in \mathcal{C}$ s.t. $B_n \subset C$ and call it C_n. (Such a C exists by choice of I.)

Let $\mathcal{C}' = \{ C_n : n \in I \}$. Clearly $\mathcal{C}'$ is countable.

<u>Claim</u> $\mathcal{C}'$ covers X.

<u>Proof.</u> Let $x \in X$. Let $C \in \mathcal{C}$ be an element s.t. $x \in C$. $\because$ C is open and $\{B_n\}_{n \in \mathbb{N}}$ a basis, $\exists n \in \mathbb{N}$ s.t. $x \in B_n \subseteq C$. $\therefore$ $n \in I$.

Thus, $x \in C_n \in \mathcal{C}'$. ∎

(b) WLOG, assume $B_n \neq \emptyset$ $\forall n$.

For each B_n, pick some $x_n \in B_n$.

Let $D = \{ x_n : n \in \mathbb{N} \}$.

<u>Claim</u> D is dense in X.

<u>Proof.</u> Let $x \in X$. So, $\exists$ a basis elt. B_n containing x. But $B_n \cap D \neq \emptyset$. ∎ ∎

<u>Example</u> $\mathbb{R}_\ell$ is first countable because for each $x \in \mathbb{R}_\ell$, there is a countable basis $\{ [x, x + 1/n) : n \in \mathbb{N} \}$.

$\mathbb{R}_\ell = \overline{\mathbb{Q}}$ and thus, separable.

We now see that $\mathbb{R}_\ell$ is not second countable.

Let $\mathcal{B}$ be a basis of $\mathbb{R}_\ell$. We show $\mathcal{B}$ is uncountable.

Choose for each $x \in \mathbb{R}_\ell$, an element $B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq [x, x+1)$. Because $x = \inf B_x$, we see that $B_x \neq B_y$ for $x \neq y$. Thus, $\mathcal{B}$ is uncountable.

<u>Remark</u> $\mathbb{R}_\ell$ is Lindelöf.

Second countable $\Rightarrow$ First countable, in general.

Prop^n 2. Every separable metrisable space is second countable.

Proof. Let $D \subseteq X$ be a countable dense subset.
Consider the countable set
$$\mathcal{B} = \{ B(d, r) : d \in D, \ r \in \mathbb{Q}^+ \}.$$
Claim. $\mathcal{B}$ is a basis.

Proof. Let $x \in X$ and let $U \ni x$ be open.
let $\epsilon > 0$ be s.t. $B(x, \epsilon) \subseteq U$.

To show: $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

Now, pick $a \in D \cap B(x, \epsilon/3)$.

Pick $r \in \mathbb{Q}^+$ s.t. $d(a, x) < r < \epsilon/3$.

(Note $d(a,x) < \epsilon/3$ since $a \in B(x, \epsilon/3)$.)

Then, $x \in B(a, r) \subseteq B(x, 2\epsilon/3) \subset U$. ☒

Cor 3. $\mathbb{R}_\ell$ is not metrisable.

Proof. $\mathbb{R}_\ell$ is separable but not second countable. ☒

Def^n. Let X be a space s.t. $\{x\}$ is closed for all $x \in X$.

① X is said to be *regular* if for every pair (x, B) with $x \in X \setminus B$ and $B \subseteq X$ closed, there exist disjoint open sets containing x and B.

② The space X is said to be *normal* if for each pair (A, B) of closed subsets $A, B \subset X$, $\exists$ disjoint open sets containing A and B.

Remark. Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff.

Lemma 4. Let X be a topological space s.t. singletons are closed.

(a) X is regular iff given any point x and open $U \ni x$, then $\exists V^{\ni x}$ open s.t. $\bar{V} \subset U$.

(b) X is normal iff given any closed set C and open $U \supseteq C$, then $\exists\, V$ open s.t. $\bar{V} \subset U$.

Proof. (a) ($\Rightarrow$) Assume X regular.

Let $x \in X$ and $U \ni x$ be open.

Consider $B = X \setminus U$. Then, B is closed.

$\therefore\ \exists\, V^{\ni x}, V'^{\supseteq B}$ open s.t. $V \cap V' = \phi$.

Thus, $V \subseteq V'^{c}$ ← closed and thus, $\bar{V} \subseteq V'^{c} \subseteq B^{c} = U$.

($\Leftarrow$) Now, let $x^{\in X}$ and $B \not\ni x$ be closed.

Let $U = X \setminus B$. Then, $x \in U$.

$\therefore\ \exists\, V \ni x$ open s.t. $\bar{V} \subset U$.

Now, $X \setminus \bar{V} \supset X \setminus U = B$.

Then, V and $X \setminus \bar{V}$ are sets which prove regularity.

(b) The same type of arguments work. ∎

Thm 5. Let $\{X_\alpha : \alpha \in I\}$ be an indexed family of spaces.

Let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in I$.

Let $\prod X_\alpha$ be equipped be either of product or box topology. Then,

$$\prod_{\alpha \in I} \overline{A_\alpha} = \overline{\prod_{\alpha \in I} A_\alpha}.$$

Proof ($\subseteq$) Let $x = (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \overline{A_\alpha}$. $(x_\alpha \in \overline{A_\alpha}\ \forall \alpha)$

Let $U = \prod U_\alpha^{\ni x}$ be a basis for box / prod top.

Then, each $U_\alpha \subseteq X_\alpha$ is open. Then, we get

$A_\alpha \cap U_\alpha \neq \phi \quad \forall \alpha$ since $x_\alpha \in U_\alpha$ and $x_\alpha \in \overline{A_\alpha}$.

$\Rightarrow\ U \cap \left(\prod A_\alpha\right) \neq \phi$

$\therefore\ x \in \overline{\prod A_\alpha}$.

($\supseteq$) Let $x \in \overline{\prod A_\alpha}$.

Fix $\alpha_0 \in I$. We show $x_{\alpha_0} \in \overline{A_{\alpha_0}}$:

Let $U_{\alpha_0} \subseteq X_{\alpha_0}$ be open. Consider $U = \prod_{\substack{\alpha \in I \\ \alpha \neq \alpha_0}} X_\alpha \times U_{\alpha_0}$.

Then, U is open in both topologies and contains x

Thus, $U \cap \prod A_\alpha \neq \phi$.

$\Rightarrow \quad U_\alpha \cap A_\alpha \neq \phi \quad \forall \alpha$

$\Rightarrow \quad U_{\alpha_0} \cap A_{\alpha_0} \neq \phi$. ∎

Lecture 15 (05-03-2021)

05 March 2021 21:17

Thm 1. A subspace Y of a regular space X is regular.
A product of regular spaces is regular.

Proof. Clearly, one point sets are closed in Y.
Let $x \in Y$ and $B \subseteq Y$ be closed with $x \notin B$.
Then, $\bar{B} \cap Y = B$.
closure in X — closure in Y

Since $x \notin B$ and $x \in Y$, we see that $x \notin \bar{B}$.
Use regularity of X, $\exists\, U, V$ open disjoint nbds of x and $\bar{B}$.
Then, $U \cap Y$ and $V \cap Y$ are the required open sets in Y.

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of regular spaces.
Note that each X_α is then Hausdorff.
Thus, $\prod X_\alpha$ is Hausdorff. Thus, singletons are closed.
Let $x = (x_\alpha)_{\alpha \in J} \in X$ and let U be a basis elt of x in X.
We will use lemmas 4 and 5 from last lecture to show that
$\exists V$ open s.t. $x \in V \subset \bar{V} \subset U$ and then use $\overline{\prod V_\alpha} = \prod \overline{V_\alpha}$.

Write $U = \prod U_\alpha$. Use regularity for each α to get
$V_\alpha \ni x_\alpha$ s.t. $x_\alpha \in V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$.
If $U_\alpha = X_\alpha$, then take $V_\alpha = X_\alpha$ instead.

Then, $\prod V_\alpha =: V$ is an open nbd of x. Moreover,

$$\bar{V} = \overline{\prod V_\alpha} = \prod \overline{V_\alpha} \subseteq \prod U_\alpha = U. \quad \blacksquare$$

Examples. (1) The space $\mathbb{R}_K$ is Hausdorff but not regular.
($\mathbb{R}$ Hausdorff $\Rightarrow$ $\mathbb{R}_K$ Hausdorff)
But consider $x = 0$ and $K = \{1, 1/2, \dots\}$.

Note K is closed in $\mathbb{R}_K$.

Let us assume $\mathbb{R}_K$ is regular.

Let $U \ni x$ be a basis elt and $V \supseteq K$ be open disjoint from V.

Note U must be of the form $(a, b) \setminus K$.

Choose $n \in \mathbb{N}$ so that $\frac{1}{n} \in (a, b)$.

Then, choose a basis element $(c, d) \subset V$ containing $1/n$.

Pick $\epsilon < 1/n$ s.t. $\epsilon > \max\{c, 1/n+1\}$. Then, $\epsilon \subset U \cap V$. $\rightarrow\leftarrow$

(2) We show $\mathbb{R}_\ell$ is normal (and hence, regular).

Note singletons are closed since they are closed in $\mathbb{R}$.

Now, let A and B disjoint in $\mathbb{R}_\ell$.

For each $a \in A$, choose $U_a = [a, x_a)$ s.t. $U_a \cap B = \phi$.

$\parallel^{\text{ly}}$ take $V_b = [b, x_b)$ for each $b \in B$ s.t. $V_b \cap A = \phi$.

Put $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$.

<u>Claim</u>: $[a, x_a) \cap [b, x_b) = \phi \quad \forall a \in A \ \forall b \in B$

<u>Proof</u>. WLOG $a \leq b$. Then $b < x_a$.

But then $b \in [a, x_a) \cap B$.

Thus, $U \cap V$ is empty. ☒

Lecture 16 (08-03-2021)

08 March 2021 15:38

- Recall : Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff

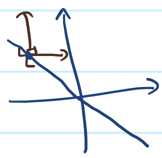
  Normal $\not\Leftarrow$ Regular (will see now!), Regular $\not\Leftarrow$ Hausdorff ($\mathbb{R}_K$)

Defn. $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is called the Sorgenfrey plane. (Sorgenfrey plane)

- Note $\mathbb{R}_\ell^2$ is regular since product of reg. spaces is regular. We will show it is not normal. Recall $\mathbb{R}_\ell$ was normal. Thus, product of normal spaces needn't be normal. Moreover, regular $\not\Rightarrow$ normal

Thm. $\mathbb{R}_\ell^2$ is not normal.

Proof. Assume $\mathbb{R}_\ell^2$ is normal. Consider $L = \{(x, -x) : x \in \mathbb{R}\}$.
L is closed in $\mathbb{R}^2$ and hence, in $\mathbb{R}_\ell^2$.
The set $[x, y) \times [-x, z)$ is open in $\mathbb{R}_\ell^2$ and its intersection with L is $\{(x, -x)\}$.
Thus, L has discrete topology.



Thus, given any $A \subset L$, A and $L - A$ are closed in L and hence, in $\mathbb{R}_\ell^2$. (Since L is closed in $\mathbb{R}_\ell$.)

By (assumption of) normality of $\mathbb{R}_\ell^2$, $\exists\, U_A$ and V_A open in $\mathbb{R}_\ell$ disjoint s.t. $U_A \supseteq A$ and $V_A \supseteq L \setminus A$.
(Fix such U_A and V_A $\forall A$.)

Let $D = \{(x, y) \in \mathbb{R}_\ell^2 : x, y \in \mathbb{Q}\}$.
On considering the basis $\{[a, b) \times [c, d) : a < b,\ c < d \in \mathbb{R}\}$ of $\mathbb{R}_\ell^2$, it is clear that D is dense in $\mathbb{R}_\ell^2$.

Define $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ by $\quad$ ($\mathcal{P}(X) \to$ pow. set of X)

Define $\Theta : \wp(L) \to \wp(D)$ by

$$\Theta(A) = D \cap U_A \quad \text{for } \phi \neq A \neq L,$$
$$\Theta(\phi) = \phi,$$
$$\Theta(L) = D.$$

($\wp(X) \to$ pow. set of X)

Claim 1. Θ is injective.

Proof. If $\phi \neq A \subsetneq L$, then $U_A \neq \phi$ and denseness of D gives $\Theta(A) \neq \phi$. Moreover, $\Theta(A) \neq D$ because $D \cap V_A \neq \phi$ and thus, $D \setminus \Theta(A) \neq \phi$.

Now, suppose $\phi \neq B \subsetneq L$ with $A \neq B$.
wlog, $A \setminus B \neq \phi$. Pick $x \in A \setminus B$. Then, $x \in L \setminus B$
and thus, $x \in V_B \cap U_A$. $\leftarrow$ thus, non-empty (and open)
Thus, $D \cap (V_B \cap U_A) \neq \phi$.
$\Rightarrow \Theta(A) \cap V_B \neq \phi$.
OTOH, $\Theta(B) \cap V_B = \phi$. Thus, $\Theta(A) \neq \Theta(B)$. ∎

However, $|L| = |\mathbb{R}|$ and D is countably infinite.
Thus, $|\wp(D)| = |L| < |\wp(L)|$. Claim 1 is a contradiction.
Then, $\mathbb{R}_\ell^2$ is not normal. ∎

Thm 1. Every second countable regular space is normal.

Proof. Let X be a regular space with a countable basis $\mathcal{B}$.
Let A and B be disjoint closed subsets of X.
Since B is closed, $\exists$ an open nbd. U_x of each point $x \in A$ s.t. $U_x \cap B \neq \phi$. Since X is regular, $\exists$ nbd $V_x \ni x$ s.t. $\bar{V} \subseteq U$.

So, $\exists$ a basis element C_x of $\mathcal{B}$ s.t. $x \in C \subseteq V_x$.
Now, $A \subseteq \bigcup_{x \in A} C_x$. Note $\overline{C_x} \cap B = \phi$.

Thus, $\exists$ a countable covering $\{C_n : n \in \mathbb{N}\}$ (of A) s.t. $\overline{C_n} \cap B = \phi \ \forall n$.

$\|^{ly}$ $\{W_n : n \in \mathbb{N}\} \subseteq \mathcal{B}$ s.t. $B \subset \bigcup W_n$ & $A \cap \overline{W_n} = \emptyset \ \forall n$.

(However, $\bigcup W_n$ and $\bigcup C_n$ need not be disjoint.)

Define $C_n' = C_n - \bigcup_{i=1}^{n} \overline{W_i}$ and $W_n' = W_n - \bigcup_{i=1}^{n} \overline{C_i}$.

Clearly, C_n' and W_n' are open in X $\forall n \in \mathbb{N}$.

Moreover, $A \subset \bigcup_{n \in \mathbb{N}} C_n'$ since $A \cap \bigcup_{n \geq 1} \overline{W_i} = \emptyset$.

Similarly, $B \subset \bigcup_{n \in \mathbb{N}} W_n'$. However, now $\left(\bigcup C_n'\right) \cap \left(\bigcup W_n'\right) = \emptyset$.

Suppose $x \in C_n' \cap W_m'$ for $n, m \in \mathbb{N}$.
WLOG $n \leq m$. Then, $W_m' = W_m - \bigcup_{i=1}^{m} \overline{C_i}$.

C_n appears as a subset

Thus, $\bigcup_{n \in \mathbb{N}} C_n'$ and $\bigcup_{n \in \mathbb{N}} W_n'$ have the desired properties. ☒

Thm 2. Every metrisable space is normal.

Proof. Let (X, d) be a metric space. (Singletons are closed.)
Let A and B disjoint closed subsets of X.
For each $a \in A$, choose $\varepsilon_a > 0$ s.t. $B(a, \varepsilon_a) \cap B = \emptyset$.
Similarly, $\forall b \in B$, choose $\varepsilon_b > 0$ s.t. $B(b, \varepsilon_b) \cap A = \emptyset$.

Define $\mathcal{U} = \bigcup_{a \in A} B(a, \frac{\varepsilon_a}{2})$ and $V = \bigcup_{b \in B} B\left(b, \frac{\varepsilon_b}{2}\right)$.

Then, $\mathcal{U}$ and V are open sets containing A and B, resp.

Claim. $\mathcal{U} \cap V = \emptyset$.

Proof. Suppose $z \in \mathcal{U} \cap V$.

Then, $z \in B(a, \frac{\varepsilon_a}{2}) \cap B\left(b, \frac{\varepsilon_b}{2}\right)$ for some $a \in A$, $b \in B$.

By triangle inequality

$$d(a, b) < \frac{\varepsilon_a + \varepsilon_b}{2} \leq \varepsilon_a. \quad (\text{wlog})$$

$$\therefore \quad b \in B(a, \varepsilon_a) \cap B \quad \rightarrow\leftarrow.$$

$\blacksquare$

Thm 3 Every compact Hausdorff space is normal.

Proof. Seen in tuts and midsem. $\blacksquare$

Lecture 17 (11-03-2021)

11 March 2021 15:35

Thm 1 (Urysohn Lemma) Let X be a normal and $A, B \subset X$ be closed and disjoint. Let $a, b \in \mathbb{R}$ with $a < b$. Then, there exists a continuous map

$$f : X \to [a, b]$$

such that $f(x) = a \;\forall x \in A$ and $f(y) = b \;\forall y \in B$.

Proof. Wlog, $[a, b] = [0, 1]$.
Eneumerate $P = \mathbb{Q} \cap [0, 1]$ as $\{1, 0, x_3, x_4, \ldots\}$.

<u>Step 1.</u> We define open $U_p \subset X$ for $p \in P$ s.t.

$$\overline{U_p} \subset U_q \quad \text{whenever} \quad p < q. \qquad (*)$$

Let $U_1 := X \setminus B$. By normality, $\exists\, U_0$ s.t.

$$A \subset U_0 \subset \overline{U_0} \subset U_1.$$

Let $P_n = \{x_1, \ldots, x_n\}$.

Suppose $U_{x_1}, \ldots, U_{x_n}$ have been defined. $(n \geq 2)$
We now define $U_{x_{n+1}}$.

$P_{n+1} = P_n \cup \{x_{n+1}\}$.
Since $x_{n+1} \neq 0, 1$ it has an immediate successor and predecessor. Let $s, p \in P_{n+1}$ be these, resp.

Then, $\overline{U_p} \subset U_s$. By normality, $\exists$ open V s.t.

$$\overline{U_p} \subset V \subset \overline{V} \subset U_s.$$

Call this $U_{x_{n+1}}$. Then, $(*)$ is still maintained.

Thus, we have constructed the family $\{U_p\}_{p \in P}$ as desired.

Step 2. We now define U_p for all $p \in \mathbb{Q}$.
Put
$$U_p = \begin{cases} \emptyset & \text{for } p < 0, \\ X & \text{for } p > 1. \end{cases}$$

Note that (*) still holds.

Step 3. For $x \in X$, define
$$Q(x) := \{ p \in \mathbb{Q} : x \in U_p \}.$$

Note that $Q(x)$ is bounded below by 0 and every rational > 1 is in $Q(x)$.
Thus, for each $x \in X$, $\inf Q(x)$ exists and is in $[0, 1]$.

Thus, defining $f(x) := \inf Q(x)$ gives a map
$$f : X \to [0, 1].$$

We now show that f has the desired property.

- If $x \in A$, then $0 \in Q(x)$. $\therefore f(x) = 0$.
- If $x \in B$, then $Q(x) = (1, \infty) \cap \mathbb{Q}$.
  $\therefore f(x) = 1$.

Now, we prove continuity.
By (*), ① if $x \in \overline{U_r}$, then $x \in U_s$ $\forall s > r$.
Thus, $f(x) \leq r$.

② $x \notin U_r \Rightarrow f(x) \geq r$
$\Downarrow$ $\Uparrow$
$x \notin U_s$ $\forall s \leq r$

Now, let $x_0 \in X$ and $(c, d) \ni f(x_0)$. We show that $\exists\, U \ni x_0$ s.t. $f(U) \subset (c, d)$. (d > 1 or c < 0 is allowed.)
open

Choose rationals p, q s.t. $c < p < f(x_0) < q < d$.

<u>Claim</u>. $U = U_q \setminus \overline{U_p}$ has the property.

<u>Proof</u>. Clearly, U is open in X.

Since $p < f(x_0)$, we have $x_0 \notin \overline{U_p}$.

Since $f(x_0) < q$, we have $x_0 \in U_q$.

Thus, $x_0 \in U_q \setminus \overline{U_p}$.

Now, let $x \in U$.

Then, $f(x) \geq p$ since $x \notin \overline{U_p}$. Similarly $f(x) \leq q$.

Thus, $f(x) \in [p, q] \subset (c, d)$. ∎

The above shows that f is continuous. ∎

<u>Defn</u>. If $A, B \subset X$ are such that $\exists$ a continuous function $f : X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$, then we say that A and B can be separated by a continuous function.

Remark. UL says that disjoint closed sets in a normal space can be separated by a continuous function.

The converse is true too, as can be seen by considering $f^{-1}[0, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, 1]$.

Lecture 18 (15-03-2021)

15 March 2021 15:38

Thm 1 (Tietze extension theorem)

Let X be a normal space and $A \subset X$ be closed.

(a) Any continuous function $f: A \to [a,b]$ can be extended to $X \to [a,b]$.

(b) Any continuous function $f: A \to \mathbb{R}$ can be extended to $X \to \mathbb{R}$.

Proof. Step 1. If $f: A \to [-r, r]$ is continuous, then $\exists$ a continuous function $g: X \to [-r, r]$ s.t.

$$|g(x)| \leq \frac{1}{3} r \quad \forall x \in X \text{ and}$$

$$|g(a) - f(a)| \leq \frac{2}{3} r \quad \forall a \in A.$$

Proof. Divide $[-r, r]$ into the following intervals of length $\frac{2r}{3}$:

$$I_1 = [-r, -r/3], \quad I_2 = [-r/3, r/3], \quad I_3 = [r/3, r].$$

Put $B = f^{-1}(I_1)$, $C = f^{-1}(I_3)$. Then, B and C are disjoint closed subsets of A and hence, of X.

By Urysohn's lemma, $\exists g: X \to [-\frac{r}{3}, \frac{r}{3}]$ continuous

s.t. $g(B) = \{-\frac{r}{3}\}$ and $g(C) = \{r/3\}$.

By construction, $|g(x)| \leq r/3 \quad \forall x \in X$.

Now, let $a \in A$.

- $a \in A \setminus (B \cup C) \Rightarrow f(a), g(a) \in I_2 \Rightarrow |f(a) - g(a)| \leq \frac{2r}{3}$.
- $a \in B \Rightarrow g(a) = -r/3$, $f(a) \in I_1 \Rightarrow |f(a) - g(a)| \leq \frac{2r}{3}$.
- $a \in C \Rightarrow |f(a) - g(a)| \leq 2r/3$.

This finishes step 1. $\blacksquare$

Step 2. We prove (a) now.
Assume $[a, b] = [-1, 1]$, wlog.
By step 1, $\exists\, g_1 : X \to [-1/3, 1/3]$ s.t.
$$|g_1(a) - f(a)| \leq 2/3 \quad \forall a \in A.$$

Thus, $f - g_1$ maps A into $[-2/3, 2/3]$.
Use Step 1 again to get $g_2 : X \to [-2/9, 2/9]$
and so on.
We get $g_1, \dots, g_n : X \to \mathbb{R}$ s.t.
$$|f(a) - g_1(a) - \cdots - g_n(a)| \leq \left(\tfrac{2}{3}\right)^n \quad \forall a \in A.$$

Apply step 1 to get $g_{n+1} : X \to \mathbb{R}$ s.t.
$$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n \quad \forall x \in X \text{ and}$$

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \quad \forall a \in A.$$

By induction, we get $\{g_n\}_{n \in \mathbb{N}}$.
Since $\sum \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n < \infty$, Weierstrass M-test gives

$$S_n = \sum_{i=1}^{n} g_i \quad \text{converges uniformly.}$$

Let $g = \lim_n S_n$. Then, g is continuous, since g_i were. We see that $g(a) = f(a)$ $\forall a \in A$.
Moreover,
$$|g(x)| \leq \sum_{i=1}^{\infty} |g_i(x)| \leq \sum \left(\frac{2}{3}\right)^n \cdot \frac{1}{3} = 1.$$

$\therefore$ g maps into $[-1, 1]$.

Step 3. We now prove (b).

Wlog, replace $\mathbb{R}$ with $(-1, 1)$. (Both are homeomorphic)

Thus, we have $f : A \to (-1,1)$.

By Step 2, we may extend it to $g : A \to [-1, 1]$.

Put $D = g^{-1}(\{1\}) \cup g^{-1}(\{1\})$.

D is closed in A and hence, in X.

Since $g(A) = f(A)$, it follows that $D \cap A = \phi$.

By Urysohn's lemma, $\exists\ \phi : X \to [0, 1]$ s.t. $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$.

Let $h(x) = \phi(x)\, g(a)$.

h is continuous and

$$h(a) = \phi(a)\, g(a) = g(a) = f(a) \quad \forall a \in A$$

and $$h(d) = \phi(d)\, g(d) = 0 \quad \forall d \in D.$$

Thus, $h : X \to \mathbb{R}$ maps into $(-1, 1)$ (why?)

and agrees with f on A. ∎

Lecture 19 (18-03-2021)

18 March 2021 15:49

<u>Recall</u>: X is compact iff every collection $\mathcal{C}$ of closed subsets having the finite intersection property (FIP) satisfies
$$\bigcap_{C \in \mathcal{C}} C \neq \phi.$$

Let $\{X_\alpha : \alpha \in J\}$ be an arbitrary family of compact sets and $X = \prod_{\alpha \in J} X_\alpha$ in product topology.

Let $\mathcal{A}$ be a collection of closed subsets of X having FIP. For each $\beta \in J$, let $\pi_\beta : X \to X_\beta$ denote the projection. Then $\{\overline{\pi_\beta(A)} : A \in \mathcal{A}\}$ also has the FIP for each $\beta \in J$.
Since X_β is compact, for each $\beta \in J$, the set $\bigcap_{A \in \mathcal{A}} \overline{\pi_\beta(A)}$ is non-empty.
But if we choose $x_\beta \in \bigcap_{A \in \mathcal{A}} \overline{\pi_\beta(A)}$, it need not be the case that
$$x = (x_\beta)_{\beta \in J} \in \bigcap_{A \in \mathcal{A}} A.$$

To deal with this, we expand $\mathcal{A}$. Since π_β are not closed maps, the set $\pi_\beta(A)$ need not be closed even if A is. So we need not assume $\mathcal{A}$ is a collection of closed sets.

<u>Lemma 1</u>. Let X be a set $\mathcal{A}$ be a collection of subsets of X having FIP. Then, $\exists$ a collection $\mathcal{D}$ of subsets of X s.t.
① $\mathcal{A} \subset \mathcal{D}$,
② $\mathcal{D}$ has FIP,
③ $\mathcal{D}$ is maximal (w.r.t. $\subseteq$) with the above properties.

Proof. We use Zorn's Lemma.

Let $\mathbb{A} = \{ \mathcal{B} \subseteq P(X) : \mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ has FIP$\}$.

$\mathbb{A} \neq \phi$ since $\mathcal{A} \in \mathbb{A}$.

Now, given a chain $\mathbb{B} \subseteq \mathbb{A}$, put

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}.$$

Then, $\mathcal{A} \subseteq \mathcal{C}$, clearly. Moreover $\mathcal{C}$ has FIP since given $C_1, \dots, C_n \in \mathcal{C}$, $\exists\, \mathcal{B}_i \in \mathbb{B}$ containing C_i for $i=1,\dots,n$.

Since $\mathbb{B}$ is a chain, $\exists\, k$ s.t. $C_1, \dots, C_n \in \mathcal{B}_k$.

$\because \mathcal{B}_k$ has FIP, $C_1 \cap \dots \cap C_n \neq \phi$.

$\therefore \mathcal{C} \in \mathbb{A}$. Clearly $\mathcal{C}$ is an upper bound of $\mathbb{B}$.

Thus, every chain has an upper bound and thus, $\mathbb{A}$ has a maximal element, as desired. ∎

Lemma 2. Let X be a set and $\mathcal{D}$ be a collection of subsets of X that is maximal w.r.t. FIP.

Then,

(1) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(2) If $A \subset X$ is s.t. $A \cap D \neq \phi$ $\forall\, D \in \mathcal{D}$, then $A \in \mathcal{D}$.

Proof. (1) Let $D_1, D_2 \in \mathcal{D}$. Put $D = D_1 \cap D_2$.

Claim. $\mathcal{D} \cup \{D\}$ has FIP.

Proof. Let $E_1, \dots, E_n \in \mathcal{D} \cup \{D\}$.

If $E_i \neq D$ $\forall i$, then $E_i \in \mathcal{D}$ $\forall i$ & $\bigcap E_i \neq \phi$.

$\therefore$ assume $E_1 = D$. Then, $\bigcap E_i = D_1 \cap D_2 \cap E_2 \cap \dots \cap E_n \neq \phi$. ∎

By maximality, $D \in \mathcal{D}$. By induction, all finite intersections are in $\mathcal{D}$.

(2) Claim. $\mathcal{D} \cup \{A\}$ has FIP.

Proof. Let $E_1, \dots, E_n \in \mathcal{D}$. Then, $E_1 \cap \dots \cap E_n \in \mathcal{D}$ by earlier.

$\therefore A \cap (E_1 \cap \dots \cap E_n) \neq \phi$, by assumption. ∎

$\therefore A \in \mathcal{D}$, by maximality. $\boxtimes$

Lecture 20 (22-03-2021)

22 March 2021 21:07

Thm (Tychonoff Theorem) An arbitrary product of compact spaces is compact.

Pf. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of compact spaces and put

$$X = \prod_{\alpha \in J} X_\alpha, \quad \text{in product topology.}$$

Let $\mathcal{A}$ be any collection of X having FIP.
To show X is compact, it suffices to show that $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \phi$.

By lemma 1 (Lec 19), $\exists\ \mathcal{D} \supseteq \mathcal{A}$ maximal with FIP.
Enough to show $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \phi$.

Fix $\beta \in J$. Consider $\pi_\beta : X \to X_\beta$. The collection $\{\pi_\beta(D) : D \in \mathcal{D}\}$ has FIP (had seen earlier).
Thus, so does $\{\overline{\pi_\beta(D)} : D \in \mathcal{D}\}$. Now, since X_β is compact,
$\bigcap_{D \in \mathcal{D}} \overline{\pi_\beta(D)} \neq \phi$.

Now, for each $\beta \in J$, we can choose $x_\beta \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\beta(D)}$.

Put $x = (x_\beta)_{\beta \in J} \in X$.

Claim. $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$.

Proof. Let the subbasis element $\pi_\beta^{-1}(U_\beta)$ contain x.
Thus, $x_\beta \in U_\beta \leftarrow$ open. Then, $U_\beta \cap \pi_\beta(D) \neq \phi$ for any $D \in \mathcal{D}$. Thus, $\exists\ y \in D$ s.t. $\pi_\beta(y) \in U_\beta$ or $y \in \pi_\beta^{-1}(U_\beta) \cap D$.

By lemma 2 (lec 19), we get $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$.
Again, using above lemma, we get that every basis element

belong to $\mathcal{D}$.

Now, given any $D \in \mathcal{D}$ and any basis ell. $U \ni x$, we have $U \in \mathcal{D}$ and hence, $D \cap U \neq \emptyset$, by FIP.

Thus, $x \in \overline{D}$ $\forall$ $D \in \mathcal{D}$. ⊠

The claim proves the result. ⊠

Def^n. (Locally compact) X is said to be locally compact at $x \in X$ if $\exists$ a compact neighbourhood of x. (Recall our nbds only contain an open set containing x. Not necessarily open itself.)

X is said to be locally compact if X is locally compact at x $\forall x \in X$.

Examples. (1) $\mathbb{R}$ is locally compact.

(2) $\mathbb{Q}$ is not locally compact.

(3) $\mathbb{R}^n$ is locally compact.

(4) The countable product $\mathbb{R}^\omega$ is not locally compact.

Let $\bar{0} \in \mathbb{R}^\omega$. Assume K is a compact nbd of $\bar{0}$.

Then, $\exists$ $\epsilon_1, \dots, \epsilon_n > 0$ s.t. $U = (-\epsilon_1, \epsilon_1) \times \cdots \times (-\epsilon_n, \epsilon_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$

Then, $\overline{U} \subset \overline{K} = K$ since compact is closed in Haus.

But $\overline{U}$ is not compact. ⊠

Thm 2. Let X be a space. Then X is a locally compact Hausdorff space iff $\exists$ space Y s.t.

(i) X is a subspace of Y,

(ii) $Y - X$ is a singleton,

(iii) Y is a compact Hausdorff space.

Moreover, if there is another such Y', then $\exists f : Y \to Y'$ homeo s.t. $f|_X = id_X$.

Proof. Step 1. We show uniqueness first.

Proof. $Y = X \sqcup \{p\}$, $Y' = X \cup \{p'\}$.

Define $h : Y \to Y'$ by

$$x \mapsto \begin{cases} x & ; \; x \in X, \\ p' & ; \; x = p. \end{cases}$$

Clearly, f is a bijection. Suffices to show h is an open map.

If $U \subseteq Y$ is open and $U \subseteq X$, then we are done.
Suppose $p \in U$. Now, $C = Y \setminus U \subseteq X$ is closed in Y and hence, compact. Thus, $h(C) = C$ is compact and hence, closed.
Thus, $h(U) = Y' \setminus C$ is open. Hausdorff

This proves Step 1. $\blacksquare$

Step 2. Suppose X is LCH.
Put $Y = X \cup \{\infty\}$ and define a topology $\mathcal{T}$ on Y as:

$$\left.\begin{array}{ll} \text{(i)} & U \text{ open in } X \quad \text{or} \\ \text{(ii)} & \{ Y - C : C \subseteq X \text{ compact} \} \end{array}\right\} \Leftrightarrow \mathcal{T}$$

Claim. $\mathcal{T}$ is indeed a topology.

(a) $\phi \to$ type (i) open set.
$Y = Y - \phi$ and ϕ compact $\to$ type (ii)

(b) $U_1 \cap U_2 \to$ open ✓

$(Y - C_1) \cap (Y - C_2) = Y - \underbrace{(C_1 \cup C_2)}_{\text{compact}} \longrightarrow$ open ✓

$U \cap (Y - C) = \underbrace{U \cap (X - C)}_{\text{open}} \to$ type (i) ✓

(c) $\underbrace{\bigcup U_\alpha}_{U} \cup \bigcup (Y - C_\beta) = U \cup (Y - \underbrace{\cap C_\beta}_{C})$

$= U \cup (Y - C) = U \cup \underbrace{(X - C)}_{\text{open}}$ ✓

Thus, $\mathcal{T}$ is indeed a topology on Y.
X does have the subspace topology, clearly.
Moreover, $Y - X$ is indeed a singleton.

Claim Y is Haus.
Proof. If $y_1 \neq y_2 \in X$, done.
Suppose $x \in X$. To show: x and ∞ have disjoint nbd.

Since X is locally compact, $\exists\ U, C$ s.t. $x \in U \subseteq C$.
Thus, $Y - C$ is a nbd of ∞ disjoint from $U \ni x$. ∎

Claim. Y is compact.
Proof. Take a cover $\{U_\alpha\}$. Then, $\exists\ \alpha_0$ s.t. $\infty \in U_{\alpha_0}$.
Thus, it contains a nbd of the form $Y - C$.
Cover C by finitely many. ∎

We are done now.

Step 3. Given such a Y, X is LCH. (H is clear.)
Proof. $Y = X \cup \{\infty\}$. Let $x \in X$. Let $U \ni x$, $V \ni \infty$ be disjoint. Then, $V = Y - C$ for some compact C.
Then, $x \in U \subset C$. ∎

Lecture 21 (25-03-2021)

25 March 2021 15:30

Def^n. If Y is a topological space and $X \subsetneq Y$ a proper subspace, and $\overline{X} = Y$, then Y is called a compactification of X.
If $Y - X$ is a singleton, then Y is called a one point compactification of X.

(compactification, one point compactification)

Remark. If X itself was compact + Hausdorff, then the point ∞ in our construction of Y was isolated. Indeed $Y - X = \{\infty\}$ would then be open. In particular $\overline{X} = X \neq Y$ and thus, Y is NOT a one pt. compactification of X.

On the other hand, if X is not compact, then $Y - X$ is not open and thus $\overline{X} \supsetneq X$. $\therefore$ $Y = \overline{X}$ and Y is the one point compactification.
(Y here is as in the theorem earlier.)

Recall from Real Analysis.

- Cauchy sequences
- Complete metric spaces : Every Cauchy sequence converges.

Examples.
① $\mathbb{R}^n$ is complete $\forall n \in \mathbb{N}$.
② $\mathbb{Q}$ is not complete. Take $x_n = \dfrac{\lfloor \sqrt{2}\, 10^n \rfloor}{10^n} \in \mathbb{Q}$.

Then, $x_n \longrightarrow \sqrt{2}$ in $\mathbb{R}$.

Thus, $(x_n)_n$ is Cauchy but does not converge in $\mathbb{Q}$.
(Limit in $\mathbb{R}$ is unique.)

③ $(-1, 1)$ is not complete. Take $x_n = 1 - \frac{1}{n}$. Then,

$$|x_n - x_m| \leq \left|\frac{1}{n} - \frac{1}{m}\right| \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

Interesting! ④ Consider $X = \{\frac{1}{n} : n \in \mathbb{N}\}$.

X is not complete w.r.t. the metric $d(x,y) = |x-y|$.

But it is complete w.r.t the discrete metric d'.

However, d and d' induce the same (discrete) topology!

Thus, completeness is not preserved by homeomorphism.

Also recall:

Lemma 1. A metric space X is complete if every Cauchy sequence in X has a convergent subsequence.

(Show that the limit of the subsequence is of the sequence.)

Corollary 2. A compact metric space is complete.

Def^n. A metric space (X, d) is called totally bounded if for every $\varepsilon > 0$, $\exists$ a finite covering of X by ε balls.

Remark. Totally bounded $\Rightarrow$ Bounded.

($\not\Leftarrow$) $\bar{d}(a, b) = \min\{1, |a-b|\}$ on $\mathbb{R}$ defines a bounded metric on $\mathbb{R}$. But taking $\varepsilon = 1/2$ shows it's not totally bounded.

Example.

① $\mathbb{R}$ is std. Topology is complete but not totally bounded.

② $(-1, 1)$ is totally bounded but not complete.

③ $[-1, 1]$ is both.

Thm 3 A metric space (X, d) is compact iff it is complete and totally bounded.

Proof. ($\Rightarrow$) Already saw that compact $\Rightarrow$ complete.
To show: totally bounded.
Let $\varepsilon > 0$ be given. $\{B(x, \varepsilon)\}_{x \in X}$ is an open cover.
Conclude.

($\Leftarrow$) Let X be complete and totally bounded.
We shall prove that X is sequentially compact.
(This is sufficient since X is a metric space.)

Let $(x_n)_{n=1}^{\infty}$ be an arbitrary seq. We show it has a Cauchy subsequence. Completeness ensures convergence.

- Cover X by finitely many balls $\{B_i\}_{i=1}^{N_1}$ of radius 1.
  $\exists B_1$ s.t. $B_1 \ni x_n$ for infinitely many n.
  $J_1 = \{n \in \mathbb{N} : x_n \in B_1\}$.

- Cover by balls of radius $1/2$.
  $\exists B_2 \to$ contains infinitely many x_n for $n \in J_1$.
  Create J_2, J_3, so on...

Then, pick $n_1 \in J_1$, $n_1 < n_2 \in J_2$, $n_2 < n_3 \in J_3, \ldots$
(x_{n_k}) is Cauchy. ∎

Lecture 22 (31-03-2021)

31 March 2021 16:14

Thm 1 Let $C_1 \supset C_2 \supset C_3 \supset \cdots$ be closed sets in a complete metric space X. If $\operatorname{diam}(C_n) \to 0$, then $\bigcap_{n=0}^{\infty} C_n \neq \phi$.

Proof. Choose $x_n \in C_n$ for each n.
Let $\varepsilon > 0$ be given. Then, $\exists N \in \mathbb{N}$ s.t. $\operatorname{diam}(C_n) < \varepsilon$ $\forall n \geq N$.
Thus, if $m, n \geq N$, then
$$d(x_m, x_n) \leq \operatorname{diam}(C_N) < \varepsilon.$$
Thus, $x_n \to x \in X$.
Now, given any $N \in \mathbb{N}$, $\{x_n\}_{n \geq N} \subseteq C_N$.
Since C_N is closed, $x \in C_N$. This is true for all N.
Thus, $x \in C$.

Def^n. A space X is said to be a Baire space if the following holds:
Given any countable collection $\{C_n\}_{n \in \mathbb{N}}$ of closed sets with $C_n^\circ = \phi$ $\forall n \in \mathbb{N}$, it is the case that $\left(\bigcup_{n \in \mathbb{N}} C_n\right)^\circ = \phi$.

(Recall $A^\circ = \operatorname{int}(A)$.) (Baire space)

Example. (1) $\mathbb{Q}$ is NOT a Baire space.
All singletons in $\mathbb{Q}$ are closed with empty interior.
However, the (countable!) union of all is $\mathbb{Q}$ but interior of $\mathbb{Q}$ (in $\mathbb{Q}$) is not empty.

(2) $\mathbb{N}$ is vacuously a Baire space since $A^\circ = \phi \Leftrightarrow A = \phi$.

Def^n. A subset A of a space X is said to be of first category in X if it is contained in a countable union of closed sets with empty interior.

Otherwise it is said to be of second category.

(first category, second category)

Remark. A space X is a Baire Space iff every non-empty open subset of X is of second category.

Prop^n 2. TFAE:

(1) X is a Baire space.

(2) If $\{U_n\}_{n\in\mathbb{N}}$ is a collection of open, dense sets, then $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

Proof. Note that given $A \subseteq X$, TFAE:

(i) A is closed and has empty interior.

(ii) A^c is open and dense in X.

Conclude now using De Morgan's laws. ∎

Lecture 23 (05-04-2021)

05 April 2021 15:38

Thm 1 (Baire category theorem) If X is a complete metric space, then X is a Baire space

Proof Let $\{D_n\}_{n=1}^{\infty}$ be a countable collection of dence open subsets of X

For arbitrary, $x_0 \in X$ and $r_0 > 0$, consider $U_0 = B(x_0, r_0)$ Suffices to show

$$U_0 \cap \left(\bigcap_{n \geq 1} D_n \right) \neq \phi$$

Since D_1 is open and dense, $U_0 \cap D_1$ is open and non-empty Pick $x_1 \in D_1 \cap U_0$ and $r_1 \in (0,1)$ s.t $\overline{B(x_1, r_1)} \subseteq D_1 \cap U_0$

Let $U_1 = B(x_1, r_1)$

Proceed inductively to get $r_n \in (0, 1/n)$, $x_n \in D_n \cap U_{n-1}$ s.t. $\overline{U_n} = \overline{B(x_n, r_n)} \subseteq D_n \cap U_{n-1}$.

Note that $\{\overline{U_n}\}_{n \in \mathbb{N}}$ is a nested sequence of closed sets in the complete metric space X with diam $\to 0$

Thus, $\bigcap_{n \geq 1} \overline{U_n} \neq \phi$ Moreover, $\overline{U_n} \subset \left(\bigcap_{m=1}^{n} D_n \right) \cap U_0$

Thus, $\left(\bigcap_{n \geq 1} D_n \right) \cap U \neq \emptyset$. $\square$

Cor 2 $\mathbb{R}^n$ is a Baire space $\square$

Thm 3 $\mathbb{R}^{\omega}$ is metrisable.

Proof Let $\bar{d}$ $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $d(a, b) := \min\{|a-b|, 1\}$.

Define D $\mathbb{R}^{\omega} \times \mathbb{R}^{\omega} \longrightarrow \mathbb{R}$ by

$$D(x, y) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}_{i \geq 1}$$

for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in $\mathbb{R}^\omega$

It is easy to see that $\bar{d}$ and D are metrics

Claim D gives the product topology on $\mathbb{R}^\omega$

Proof Let $x \in \mathbb{R}^\omega$ and $\varepsilon > 0$ be arbitrary

Choose N s.t $1/N < \varepsilon$

Put

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

Then, if $y \in V$, then $D(x, y) < \varepsilon$

Thus, $x \in V \subseteq B_D(x, y)$

This shows that $(\mathbb{R}^\omega, \Pi)$ is finer than $(\mathbb{R}^\omega, D)$

Conversely, let V be a basis elt. of $(\mathbb{R}^\omega, \Pi)$ of the form

$$(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \times \cdots \times (x_N - \varepsilon_N, x_N + \varepsilon_N) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

(x fixed as before) Assume $\varepsilon_i < 1$ $\forall i$

Choose $\varepsilon = \min\{\varepsilon_1, \frac{\varepsilon_2}{2}, \ldots, \frac{\varepsilon_N}{N}\} > 0$ Now, if $y \in \mathbb{R}^\omega$ is s.t.

$D(x, y) < \varepsilon$, then

$$\frac{|y_i - x_i|}{i} < \varepsilon \quad \forall i = 1, \ldots, N$$

or $|y_i - x_i| < i\varepsilon \leq \varepsilon_i \quad \forall i = 1, \ldots, N$.

Thus, $y \in V$ ☒

(If V is a general basis elt, pick $\varepsilon_i = 1/2$ for the initial few $\mathbb{R}$s.)

This shows that $(\mathbb{R}^\omega, D)$ is finer than $(\mathbb{R}^\omega, \Pi)$. ☒

Thm 4. (Urysohn's metrisation theorem)

Every regular space with a countable basis is metrisable

Proof Let X be regular, second countable. We show $X \hookrightarrow \mathbb{R}^\omega$ and hence, is metrisable

Step 1. We know X is normal. Thus, given $x_0 \in X$ and $U \ni x_0$ open, $\exists f : X \to [0,1]$ cts. s.t. $f(x_0) = 1$ and $f(X - U) = \{0\}$ (Use Urysohn's lemma)

Let $\{B_n : n \in \mathbb{N}\}$ be a countable basis. For each $n, m \in \mathbb{N}$ for which $\overline{B_n} \subset B_m$, apply Urysohn's lemma to choose a continuous $g_{n,m} : X \to [0,1]$ s.t. $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$.

Now, given any $x_0 \overset{X}{\in} \overset{\text{open}}{U}$, choose B_m s.t. $x_0 \in B_m \subseteq U$.
By regularity, $\exists n$ s.t. $x_0 \in B_n \subseteq \overline{B_n} \subseteq B_m$.
Then, $g_{n,m}(x_0) = 1$ and $g_{n,m}(X - U) = \{0\}$

$\{g_{n,m}\}$ is countable. Relabel as $\{f_n\}_{n \in \mathbb{N}}$.

Step 2 Let $\mathbb{R}^\omega$ be in product topology. Define
$$F : X \to \mathbb{R}^\omega \text{ as}$$
$$F(x) = (f_1(x), f_2(x), \quad)$$

F is continuous since each f_i is

- F is 1-1. If $x \neq y \in X$, choose $U \subseteq X$ open s.t. $x \in U \not\ni y$. Let N be s.t. $f_N(x) = 1$ and $f_N(X - U) = \{0\}$. Then, $f_N(y) = 0$

Thus, $F : X \to F(X) \subset \mathbb{R}^\omega$ is a bijection
To show F is an imbedding, it suffices to show that for each open U, $F(U)$ is open in $F(X) =: Z$

Claim. Let U be open and $z_0 \in F(U)$
$\exists$ W open in Z s.t. $z_0 \in W \subseteq F(U)$.

Proof. Let $x_0 \in U$ be s.t. $F(x_0) = z_0 \in Z$.
Choose an N for which $f_N(x_0) > 0$ and $f_N(X - U) = \{0\}$
Let $V = \pi_N^{-1}((0, \infty)) \subset \mathbb{R}^\omega$. $V \subseteq \mathbb{R}^\omega$ is open since π_N is cts

Let $W = V \cap Z$ This is open in Z
Note $z_0 \in V$ since $\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$
($z_0 \in Z$ to begin. Thus, $z \in W$)
Moreover, if $z \in W$, then $\pi_N(z) > 0$ and $z = F(x)$
for some $x \in X$. Since f_N vanishes outside U and
$f_N(x) = \pi_N(z) > 0$, we get $x \in U$. Thus, $z = F(x)$
for $x \in U$ $\therefore$ $z \in F(U)$ ∎

This proves the theorem ∎