MA 214: Tutorial 6

Aryaman Maithani

04-03-2020

1. Let $I = [a, b] \subset \mathbb{R}$ for some $a < b$ and $g : I \to I$ be a twice differentiable function such that there exists some $k \in \mathbb{R}$ such that $|g'(x)| \leq k < 1$ for all $x \in I$.
   Let ξ denote the unique fixed point of g. Suppose that $g'(\xi) = 0$ and $g''(\xi) \neq 0$. Show that the fixed point iteration has quadratic rate of convergence.

   Solution.
   Note that g is twice continuously differentiable and thus, by Taylor, we have that for any $h \in \mathbb{R}$:

   $$g(\xi + h) = g(\xi) + g'(\xi)h + \frac{1}{2}g''(c)h^2,$$

   for some c between ξ and $\xi + h$.
   As $g(\xi) = \xi$ and $g'(\xi) = 0$, we get that

   $$g(\xi + h) - \xi = \frac{1}{2}g''(c)h^2,$$

   for some c between ξ and $\xi + h$.
   Now, set $h = x_n - \xi = e_n$ to get:

   $$g(x_n) - \xi = \frac{1}{2}g''(\eta_n)(x_n - \xi)^2,$$

   for some η_n between x_n and ξ.
   Note that $g(x_n) = x_{n+1}$ and thus, $g(x_n) - \xi = -e_{n+1}$. Also, $x_n - \xi = -e_n$. Thus, we have

   $$\frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2}g''(\eta_n).$$

   Now, we note two things:

   (a) As η_n lies between x_n and ξ and $x_n \to \xi$, we get that $\eta_n \to \xi$. (Sandwich theorem.)
   (b) g'' is given to be twice continuously differentiable. Thus, $g''(\eta_n) \to g''(\xi)$.

   Thus, $\displaystyle\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2}g''(\xi) \neq 0$. Thus, it converges quadratically. (Since $g''(\xi) \neq 0$.)

2. If f has a double root at ξ, then show that the iteration

   $$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

   converges quadratically to ξ if x_0 is sufficiently close to ξ.

   Solution.
   Let $g(x) := x - \dfrac{2f(x)}{f'(x)}$ when $f'(x) \neq 0$. At ξ, we define it to be the limit.
   I will also be assuming that g is nice enough, that is, differentiable twice continuously. (Also assuming that f is continuously differentiable thrice.)
   Note that

   $$\lim_{x\to\xi} g(x) = \lim_{x\to\xi}\left(x - 2\frac{f(x)}{f'(x)}\right) = \xi - \lim_{x\to\xi} 2\frac{f'(x)}{f''(x)} = \xi.$$

   Thus, $g(\xi) = \xi$.
   Now, differentiating gives us $g'(x) = 1 - 2\dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = -1 + 2\dfrac{f(x)f''(x)}{(f'(x))^2}$.

Computing $\lim_{x\to\xi} g'(x)$ is easy using L'Hospital and you get $g'(\xi) = 0$.
Now, we calculate $g''(x)$ for $x \neq \xi$. We get:

$$\begin{aligned} g'' &= \frac{(f')^2[2ff''' + 2f'f''] - 4ff'(f'')^2}{(f')^4} \\ &= \frac{f'[2ff''' + 2f'f''] - 4f(f'')^2}{(f')^3} \\ &= \frac{2(f')^2f'' - 4f(f'')^2}{(f')^3} + \frac{2ff'''}{(f')^2} \end{aligned}$$

We now calculate the limit $x \to \xi$ for both the terms using L'Hospital appropriately. Let us do the second term first as that's easier.

$$\begin{aligned} \lim_{x\to\xi} \frac{2f(x)f'''(x)}{(f'(x))^2} &= f'''(\xi) \lim_{x\to\xi} \frac{2f(x)}{(f'(x))^2} \\ &= f'''(\xi) \lim_{x\to\xi} \frac{2f'(x)}{2f'(x)f''(x)} \\ &= \frac{f'''(\xi)}{f''(\xi)} && (\because f''(\xi) \neq 0) \end{aligned}$$

The first term is:

$$\begin{aligned} \lim_{x\to\xi} \frac{2(f'(x))^2f''(x) - 4f(x)(f''(x))^2}{(f'(x))^3} &= 2f''(\xi) \lim_{x\to\xi} \frac{(f'(x))^2 - 2f(x)f''(x)}{(f'(x))^3} \\ &= 2f''(\xi) \lim_{x\to\xi} \frac{\cancel{2f'(x)f''(x)} - 2f(x)f'''(x) - \cancel{2f'(x)f''(x)}}{3(f'(x))^2f''(x)} \\ &= -\frac{4}{3}\frac{\cancel{f''(\xi)}}{\cancel{f''(\xi)}} \lim_{x\to\xi} \frac{f(x)f'''(x)}{(f'(x))^2} \\ &= -\frac{4}{3}f'''(\xi) \lim_{x\to\xi} \frac{f'(x)}{2f'(x)f''(x)} \\ &= -\frac{2}{3}\frac{f'''(\xi)}{f''(\xi)} \end{aligned}$$

Note that we have kept using $f''(\xi) \neq 0$ in the above calculations.
Thus, we finally get:

$$\lim_{x\to\xi} g''(x) = \frac{1}{3}\frac{f'''(\xi)}{f''(\xi)}.$$

Assuming g'' to be continuous gives us that $g''(\xi) = \dfrac{1}{3}\dfrac{f'''(\xi)}{f''(\xi)}$. With the further assumption that $f'''(\xi) \neq 0$, we are almost done, by the previous case.

We still need to get the 'k' and I as in the previous question.
To do this, we note that g' is continuous and $g'(\xi) = 0$. Thus, there is some $\delta > 0$ such that $|g'(\xi) - g'(x)| < 1/2$ for all $|x - \xi| < \delta$. (Note that $1/2$ is arbitrary, we could take any $\epsilon > 0$. But for the purpose of this question, we shall also take $\epsilon < 1$.)
Let $k := 1/2$. Clearly, $k < 1$.
Thus, for $x \in (\xi - \delta, \xi + \delta)$, we have that $|g'(x)| < k$. Let $I = \left[\xi - \frac{\delta}{2}, \xi + \frac{\delta}{2}\right]$. Note that I is a closed interval. We continue to have the property that $|g'(x)| < k$ for $x \in I$.

Now we need to show that: given any $x \in I$, we have that $g(x) \in I$. This is clearly true if $x = \xi$. Assume $x \neq \xi$.
Then, we have $g(x) - g(\xi) = g'(\eta)(x - \xi)$ for some η between x and ξ. (LMVT)
Thus, $|g(x) - g(\xi)| \leq |x - \xi| \leq \dfrac{\delta}{2}$. But $g(\xi) = \xi$. Thus, $|g(x) - \xi| \leq \dfrac{\delta}{2}$ giving us $g(x) \in I$.

Now, we are in the same set up as 1.

3. Let A be a given positive constant. Set $g(x) := 2x - Ax^2$.
   (a) Show that if the fixed point iteration converges to a nonzero limit, then the limit is $P = 1/A$.
   Solution.

We are given that the sequence satisfying

$$x_{n+1} = 2x_n - Ax_n^2,\ n \geq 0$$

converges to some nonzero limit P.
Noting that $\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} x_n$, we get that $P = 2P - AP^2$ or $AP^2 = P$.
As $P \neq 0$, we see that $P = A^{-1}$, as desired.

(b) Find an interval about $1/A$ for which the fixed point iteration converges.
Solution.
The idea is the same as the last question. First we choose some arbitrary $k \in (0, 1)$. I like $1/2$, so I choose $k = 1/2$.
Now, let us try to find a closed interval containing A^{-1} such that $|g'(x)| \leq k$ on that interval.
Note that $|g'(x)| = 2|1 - Ax| = 2A|A^{-1} - x|$.
As we want $|g'(x)| \leq k$, we see that $|A^{-1} - x|$ must be $\leq (4A)^{-1}$. Thus, let $I = \left[\dfrac{1}{A} - \dfrac{1}{4A}, \dfrac{1}{A} + \dfrac{1}{4A}\right]$.
Once again, like before, we can show that $g(x) \in I$ for all $x \in I$. As we have $|g'(x)| \leq k < 1$ for $x \in I$, we are done.
That is, I is the desired interval.

4. Use fixed point iteration to find a root of $2\sin(\pi x) + x = 0$ in $[1, 2]$.

Solution.
Consider $g(x) = \dfrac{1}{\pi}\sin^{-1}\left(-\dfrac{x}{2}\right) + 2$ for $x \in [1, 2]$.
Check that $g(x) \in [1, 2]$ for all $x \in [1, 2]$.
Also, check that $|g'(x)| = \dfrac{1}{\pi}\dfrac{1}{\sqrt{4 - x^2}}$.
Note that g' shoots to infinity near 2. We want a closed interval on which $|g'(x)| \leq k$ for some $k < 1$.

Let $x_0 = \sqrt{4 - \dfrac{1}{\pi^2}}$. Note that $1 < x_0 < 2$ and $g'(x_0) = 1$. Choose $x_1 = \frac{1}{2}(1 + x_0)$.
Then, we have $1 < x_1 < x_0 < 2$. As g' is clearly increasing on $[1, 2]$, we have that $|g'(x)| \leq g'(x_1) < 1$ for all $x \in [1, x_1]$. Letting $I = [1, x_1]$ and $k = g'(x_1)$ does the job as earlier. That is, we know that we may pick any $x_0 \in I$ and we'll get that the sequence defined by $x_{n+1} = g(x_n)$ will converge to the fixed point.

5. Show that if A is any positive real number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}} \qquad \text{for } n \geq 1$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

Solution.

Claim 1. $x_n > 0$ for all $n \geq 0$.

Proof. It would be an insult to my time and yours if I write a proof of this evidently trivial fact. □

Claim 2. $x_n \geq \sqrt{A}$ for $n \geq 1$.

Proof.

$$\begin{aligned} x_n &= \frac{1}{2}\left(x_{n-1} + \frac{A}{x_{n-1}}\right) \\ &\geq \sqrt{A} && (\text{AM} \geq \text{GM and } x_{n-1} > 0) \end{aligned}$$

□

Claim 3. $x_{n+1} \leq x_n$ for all $n \geq 1$.

Proof.

$$\begin{aligned}
x_{n+1} - x_n &= x_n - \frac{1}{2}\left(x_{n-1} + \frac{A}{x_{n-1}}\right) \\
&= \frac{1}{2}\left(-x_{n-1} + \frac{A}{x_{n-1}}\right) \\
&= \frac{1}{2}\left(\frac{A - x_n^2}{x_{n-1}}\right) \\
&\leq 0 && \text{(By previous claim.)}
\end{aligned}$$

□

Thus, (x_n) is an eventually decreasing sequence which is bounded below. Thus, it converges. (Had done this in MA 105.)
(Note that the "eventually" is necessary because x_0 might be $< \sqrt{A}$.) If you have forgotten MA 105, then you may look at the aliter.

Aliter.
If $x_0 = \sqrt{A}$, then it's clear that $x_n = \sqrt{A}$ for all $n \geq 0$ and thus, $x_n \to \sqrt{A}$.
Suppose $x_0 \neq \sqrt{A}$. Then, by the claims given earlier, we have that $\sqrt{A} \leq x_n \leq x_1$ for all $n \geq 1$.
Consider the function $g(x) := \frac{1}{2}\left(x - \frac{A}{x}\right)$ for $x \in I = [\sqrt{A}, x_1]$.
Note that $g'(x) = \dfrac{1}{2}\left(1 - \dfrac{A}{x^2}\right)$. Clearly $g'(x) \leq \dfrac{1}{2} < 1$. Also, $x^2 > A$ gives us that $g'(x) > 0$. Thus, $|g'(x)| \leq \frac{1}{2} < 1$ for all $x \in I$.

Also, note that if $x \in I$, we have $g(x) \in I$. (Why? It is clear that $g(x) \geq A$ by AM-GM again. To see that $g(x) \leq x_1$, do the same sort of argument like Claim 3 to show that $g(x) - x \leq 0$.)
Thus, we are again done by our theorem about fixed point iterations. (What theorem?)