MA 214: Tutorial 5

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4. Show that $g(x) = \pi + 0.5\sin\left(\frac{x}{2}\right)$ has a unique fixed point in $[0, 2\pi]$.

Solution.
First method:
In this method, we don't use anything that's taught in MA 214 but just stuff we know from before.
We want to show that $g(x) = x$ has a unique solution in $[0, 2\pi]$. Define $f(x) := g(x) - x$ for $x \in [0, 2\pi]$. Thus, the given problem is equivalent to showing that f has a unique root in $[0, 2\pi]$.

Existence. Note that $f(0) = g(0) - 0 = \pi > 0$ and $f(2\pi) = g(2\pi) - 2\pi = -\pi < 0$.
As f is continuous, there is some $\xi \in (0, 2\pi)$ such that $f(\xi) = 0$.

Uniqueness. Suppose that there exists two distinct roots $a, b \in [0, 2\pi]$ of f. Then, by Rolle's theorem, there exists some c between a and b such that $f'(c) = 0$.
However, note that $f'(x) = g'(x) - 1 = \frac{1}{4}\cos\left(\frac{x}{2}\right) - 1 \leq -\frac{3}{4} < 0$ for all $x \in (0, 2\pi)$. A contradiction.

Second method:
In this method, we use the following theorem done in Lecture 9:
Let $I = [a, b]$ be an interval and $g : I \to I$ be a continuous function. Then, g has a fixed point. Further, if g is differentiable on I and if there exists some $K < 1$ such that $|g'(x)| \leq K$, then the fixed point is unique.

In our case, we have $I = [0, 2\pi]$. It can be easily checked that for $x \in I$, we have $g(x) \in I$. Moreover, $|g'(x)| \leq \frac{1}{4}$.
Thus, we are done.

Note: To find the fixed point, start with any $x_0 \in I$ and keep computing $g(x_0), g(g(x_0)), g(g(g(x_0))), \ldots$.