MA 214: Tutorial 4

Aryaman Maithani

12-02-2020

4. Suppose $p(x)$ is a polynomial of degree ≤ 3.
Show that if we compute T'_{2N} in Romberg integration, then we get the exact value of the integral.

Solution.
Note that Romberg integration uses composite trapezoidal rule for the approximations.
The idea is to show that the T'_{2N} computed is actually the value that is approximated by the composite Simpson's rule. However, we know that the error for Simpson's rule is 0 when integrating a polynomial of degree ≤ 3. (There's the $f^{(4)}(\xi)$ term.)

The rest is just simple calculation:
First, we have that

$$T_N = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(a+ih) + f(x_N)\right], \text{ where } h = \frac{b-a}{N}, \text{ and}$$

$$T_{2N} = \frac{h}{4}\left[f(x_0) + 2\sum_{i=1}^{2N-1} f\left(a+i\frac{h}{2}\right) + f(x_{2N})\right],$$

where the h is as before.
T_{2N} can be rearranged to be better written as:

$$T_{2N} = \frac{1}{2}T_N + \frac{h}{2}\sum_{i=1}^{N} f\left(a+(2i-1)\frac{h}{2}\right),$$

for the same $h = \frac{b-a}{2N}$.

Now, using the formula

$$T'_{2N} = T_{2N} - \frac{T_N - T_{2N}}{4-1} = \frac{4T_{2N} - T_N}{3},$$

we get:

$$T'_{2N} = \frac{1}{3}\left(2T_N + 2h\sum_{i=1}^{N} f\left(a+(2i-1)\frac{h}{2}\right) - T_N\right)$$

$$= \frac{1}{3}\left(\frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(a+ih) + f(x_N)\right] + 2h\sum_{i=1}^{N} f\left(a+(2i-1)\frac{h}{2}\right)\right)$$

$$= \frac{h}{6}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(a+ih) + f(x_N) + 4\sum_{i=1}^{N} f\left(a+(2i-1)\frac{h}{2}\right)\right]$$

$$= \frac{h}{6}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(a+ih) + 4\sum_{i=1}^{N} f\left(a+(2i-1)\frac{h}{2}\right) + f(x_N)\right] \qquad \text{where} h = \frac{b-a}{N}$$

The above is precisely composite Simpson's rule with N divisions. Thus, we are done.