MA 214: Tutorial 3

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1. Give an example of a polynomial $P_{2n+2}(x)$ of degree $2n+2$ such that Gaussian Quadratures (with $n+1$ nodes) is not exact for $P_{2n+2}(x)$.

Solution.
Simply consider $P_{2n+2}(x) = (Q_{n+1}(x))^2$.
It is clear that $P_{2n+2}(x)$ is indeed of degree $2n+2$ as Q_{n+1} has degree $n+1$.
Now, note that $I = \displaystyle\int_{-1}^{1} P_{2n+2}(x)\mathrm{d}x > 0$. This is because the integrand is a nonnegative continuous function that is not identically zero. (It being continuous is required.)

On the other hand, if we calculate the the approximate sum, we get

$$S = \sum_{i=0}^{n} c_i P_{2n+2}(x_i),$$

where x_i are the roots of $Q_{n+1}(x)$. However, this means that they are also roots of P_{2n+2}. This gives us that $S = 0$.
Thus, I is clearly not equal to S.

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4. Consider $I = \displaystyle\int_0^1 \sin\left(x^3\right)\mathrm{d}x$.
a) How many subdivisions of the interval $[0,1]$ are needed so that the trapezoid rule gives an error of 10^{-4} (or less)?
Solution. Let N be the number of divisions used in approximating the integral via the composite trapezoidal rule. Recall that the error given by the composite trapezoid rule will be:

$$E_C^T = -f''(\xi)h^2\frac{1}{12} \qquad \text{for some } \xi \in [0,1],$$

where $h = \frac{1-0}{N}$.

In this case, we have $f''(\xi) = 6\xi\cos(\xi^3) - 9\xi^4\sin(\xi^3)$. As $|\xi| \le 1$, we have it that $|f''(\xi)| \le 15$.

Thus, we see that $|E_C^T| \le 15 \cdot \dfrac{1}{N^2} \cdot \dfrac{1}{12}$.

Now, one way to ensure that $|E_C^T|$ is $\le 10^{-4}$, we may simply "equate" the RHS to be $\le 10^{-4}$.
This gives us $15 \cdot \frac{1}{N^2} \cdot \frac{1}{12} \le 10^{-4}$ or $N^2 \ge 12500$ which implies $N \ge 111.8$. Now, we can simply choose $N = 112$.

b) (Same question as a) but for Simpson's rule)
Solution.
With similar notations, we now have the error as

$$E_C^S = -\frac{1}{180}f^{(4)}(\xi)\left(\frac{1}{2N}\right)^4.$$

Note that the fourth derivative of $\sin(x^3)$ is given as

$$9(x^2(9x^6 - 20)\sin(x^3) - 36x^5\cos(x^3)).$$

Thus, we have $|f^{(4)}(\xi)| \le 585$.

Doing the same thing as earlier sets up the inequality:

$$\frac{585}{180}\frac{1}{16N^4} \leq 10^{-4}.$$

The smallest natural number satisfying the above is $N = 7$. That is our answer.

Remark. Note that these are quite loose bounds. That is, it is quite possible that even a smaller value of N works. However, our method guarantees that the N that we do get will indeed work.