MA 214: Tutorial 1

Aryaman Maithani

22-01-2020

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $g \ge 0$ be an integrable function on $[a, b]$. Show that

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x \qquad \text{for some } \xi \in [a, b].$$

Solution.
As f is a continuous function defined on a closed and bounded interval, it is bounded and moreover, it attains these bounds. (Extreme value theorem, done in 105.)
In other words, there exist $m, M \in [a, b]$ such that $f(m) \le f(x) \le f(M)$ for all $x \in [a, b]$.
(Note that it's not necessary that $m \le M$ but that's not required anyway.)
As $g \ge 0$, the above gives us that

$$f(m)g(x) \le f(x)g(x) \le f(M)g(x) \quad \forall x \in [a, b].$$

Integrating on all three sides gives that $\int_a^b f(x)g(x)\mathrm{d}x$ lies between $If(m)$ and $If(M)$ where $I = \int_a^b g(x)\mathrm{d}x$.
Now, note that $h(x) := If(x)$ defined for $x \in [a, b]$ is a continuous function and thus, by intermediate value property, there exists ξ between m and M such that $h(\xi) = \int_a^b f(x)g(x)\mathrm{d}x$. As $m, M \in [a, b]$, we also get that $\xi \in [a, b]$.

Thus, we get that $f(\xi)\int_a^b g(x)\mathrm{d}x = \int_a^b f(x)g(x)\mathrm{d}x$, as desired.

---

5. Prove that the k^{th} divided difference $p[x_0, \ldots, x_k]$ of a polynomial p of degree $\le k$ is independent of the interpolation points $x_0, x_1, \ldots, x_k$.

Solution.
Let $p(x)$ be a polynomial of degree $\le k$.
Then, $p(x) = a_0 + a_1x + \cdots a_kx^k$ for some $a_0, \ldots, a_k \in \mathbb{R}$. ($a_k$ may be zero.)
Now, we show that the k^{th} divided difference equals a_k independent of the choice of $x_0, \ldots, x_k$. This would clearly prove the desired result.

To see this, simply observe that $P_k(x)$ is a polynomial of degree $\le k$ such that $P_k(x_i) = p(x_i)$ for all $i \in \{0, \ldots, k\}$. In other words, p and q agree at $k + 1$ points. By the uniqueness theorem seen earlier, this forces $p(x) = P_k(x)$. In turn, this forces that $p(x)$ and $P_k(x)$ have the same leading coefficient. We already know the leading coefficient of $p(x)$ is a_k, by definition. On the other hand, recalling the definition of $P_k(x)$ gives us:

$$\begin{aligned} P_k(x) := p[x_0] &+ p[x_0, x_1](x - x_0) \\ &+ p[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \ldots \\ &\ \vdots \\ &+ p[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1)\cdots(x - x_{k-1}) \end{aligned}$$

Thus, the leading coefficient of $P_k(x)$ is $p[x_0, \ldots, x_k]$.
This completes the proof as we conclude that $a_k = p[x_0, \ldots, x_k]$, as desired.