Group Theory

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IIT Bombay

23rd July 2020

Greetings

Hi,

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group

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Credits: Aneesh Bapat

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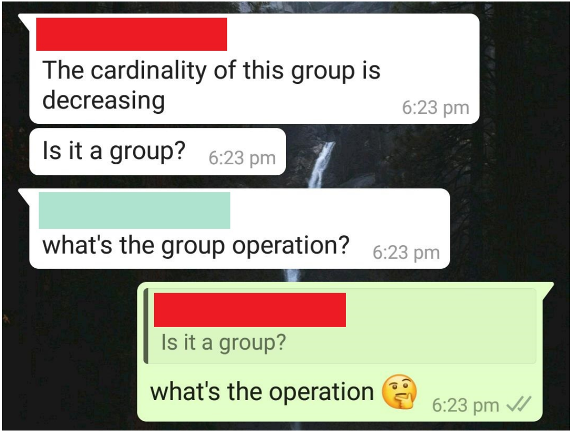
Note the "along with." We don't talk about a group by just talking about a set. It is necessary to have an operation on it as well.

Joke


[redacted]
The cardinality of this group is decreasing 6:23 pm
Is it a group? 6:23 pm
[redacted]
what's the group operation? 6:23 pm
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First, a definition.

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Now, we define what a group is.

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Review

The discussion we had is what led to us agreeing upon the above axioms. So, let us only discuss what went wrong with the non-examples.

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Recall vector spaces?

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Recall vector spaces? Verify that any vector space along with its $+$ forms a group.

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- "G is a group" when $\cdot$ is clear from context.

Notations

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For $n > 0$, we define

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Definition 5 (Order (element))

The order of an element $x \in G$ the smallest positiver integer n such that

$$x^n = e.$$

(Where e is the identity of G.)
If no such n exists, then we say the the element has infinite order.
It is denoted by $|x|$.

Finite groups

Proposition 1

Every element of a finite group has finite order.

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Proof.

Let G be a finite group and let $x \in G$.
It suffices to show that $x^n = e$ for *some* $n \in \mathbb{N}$.
Note that $x^0, x^1, \ldots, x^{|G|}$ are $|G| + 1$ elements of G. By PHP, two of them must be equal. Thus,

$$x^n = x^m$$

for some $0 \leq n < m \leq |G|$.
The above equation gives us

$$e = x^{m-n}.$$

Since $m - n \in \mathbb{N}$, we are done. □

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One may note that the identity element of $(G, \cdot)$ is always present in H and moreover, it is also the identity of $(H, \cdot|_H)$.

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- $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$.
- The set of $n \times n$ invertible upper triangular (real) matrices is a subgroup of the group of all invertible $n \times n$ (real) matrices.

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Note that different elements could correspond to the same coset. That is, a coset may have different representatives. In fact, we now see precisely when that is possible.

Properties of cosets

Proposition 2 (Equality of cosets)

Let $a, b \in G$. Then,

$$aH = bH \quad \text{iff} \quad b^{-1}aH = H \quad \text{iff} \quad b^{-1}a \in H.$$

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Note that H itself is a coset since it equals $e \cdot H$. (Or $h \cdot H$ for any $h \in H$.)

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Remark. This shows that any two cosets have the same cardinality.

Lagrange's Theorem

With the concept of cosets, we can prove a (quite fundamental) result of group theory.

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Theorem 1 (Lagrange's Theorem)

Let G be a finite group and $H \leq G$.
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That completes our proof. □

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The case of groups is particularly simple since there's pretty one much thing that gives the group its structure, the group operation. This leads to the following definition.

Homomorphisms

Definition 8 (Homomorphism)

Let $(G, \cdot)$ and $(H, \star)$ be groups. A function

$$\varphi : G \to H$$

is said to be a *group homomorphism* if

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Examples

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- The map $\exp : \mathbb{R} \to \mathbb{R}^\times$ defined by

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- Given any $n \in \mathbb{Z}$, the map $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined as

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  is a homomorphism.
- In general, if G is an abelian group and $n \in \mathbb{Z}$, the map $x \mapsto x^n$ is a homomorphism.

Isomorphisms

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While the groups are not equal (since they don't have the same element), they pretty much are same in terms of the group properties.
This idea can formalised as follows.

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Definition 9 (Isomorphism)

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One can note that $\cong$ is an "equivalence relation".

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- The map $\exp : \mathbb{R} \to \mathbb{R}^+$ is an isomorphism. (Note that $\mathbb{R}$ is a group under $+$ whereas $\mathbb{R}^+$ is a group under $\cdot$.)

Kernels

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Proposition 5

With the same notations as above, we have

$$\ker \varphi \leq G.$$

A curious property about kernels

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The above says that $aKa^{-1} \subset K$

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This can be written in yet another way as $aK = Ka$.

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Let us keep this in mind for now. We shall come back to it later. Note that the only property we used was that $gK = Kg$ and not really that K was a kernel.

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Said even more differently, given any $n \in N$, and $g \in G$, we must have $gng^{-1} \in N$.

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Another thing to note is that any subgroup of an abelian group is normal.

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The above group is what is called $\mathbb{Z}/5\mathbb{Z}$. Of course, this works for all values of 5.
In fact, this is the group (up to isomorphism) which we saw earlier as $\{1, \dots, n-1\}$ with addition modulo n.

Exercises

1. Let $(G, \cdot)$ be a finite group and let $x \in G$.
Show that $H = \{1, x, x^2, \ldots\}$ is a (finite) subgroup of G.

Show that H has order $|x|$.

Conclude that $|x|$ divides $|G|$.
In particular, we have $x^{|G|} = e$.

2. Let $n > 1$ be a natural number. Define

$$(\mathbb{Z}/n\mathbb{Z})^* = \{x : 1 \leq x \leq n, \gcd(x, n) = 1\}.$$

Show that $|(\mathbb{Z}/n\mathbb{Z})^*| = \varphi(n)$, where φ is the Euler totient function.

Show that $(\mathbb{Z}/n\mathbb{Z})^*$ is a group under the operation "multiplication mod n" with identity being 1.

Conclude that $a^{\varphi(n)} = 1$ for all $a \in (\mathbb{Z}/n\mathbb{Z})^*$. (Note that this equality is in the group $(\mathbb{Z}/n\mathbb{Z})^*$.)

Conclude that $a^{\varphi(n)} \equiv 1 \bmod n$ for all a with $\gcd(a, n) = 1$.

This is Euler's theorem (in number theory) and Fermat's little theorem is a special case of it.

3. Let G be a finite group and let p be a prime dividing $|G|$. Let $\mathcal{S}$ denote the set of p-tuples of elements of G the product of whose coordinates is 1 :

$$\mathcal{S} = \{(x_1, \ldots, x_p) : x_i \in G \text{ and } x_1 x_2 \cdots x_p = 1\}.$$

(a) Show that $\mathcal{S}$ has $|G|^{p-1}$ elements, hence has order divisible by p.

Define the relation $\sim$ on $\mathcal{S}$ be letting $\alpha \sim \beta$ if β is a cyclic permutation of α.

(b) Show that a cyclic permutation of an element of $\mathcal{S}$ is again an element of $\mathcal{S}$.
(c) Prove that $\sim$ is an equivalence relation on $\mathcal{S}$.
(d) Prove that an equivalence class contains a single element if and only if it is of the form $(x, \ldots, x)$ with $x^p = 1$.

(e) Prove that every equivalence class has order 1 or p (this uses the fact that p is a *prime*). Deduce that $|G|^{p-1} = k + pd$ where k is the number of classes of size 1 and d is the number of classes of size p.
(f) Since $\{(1, \ldots, 1)\}$ is an equivalence class of size 1, conclude from (e) that there must be a nonidentity element x in G with $x^p = 1$, i.e., G contains an element of order P. (Show $p \mid k$ and so $k > 1$.)

The previous exercise proves the following theorem.

Theorem 2 (Cauchy's Theorem)

If G is a finite group and $p \mid |G|$, then there exists $x \in G$ such that $p = |x|$.

With $H = \{1, x, \ldots\}$ as in Exercise 1, this shows that there exists a subgroup of order p.
This is a partial converse to Lagrange's theorem (and the statement shown in Exercise 1).
(And that's the best we can get.)

Credits: The above style of proof was published in Amer. Math. Monthly, 66 (1959), p. 199 by James McKay.
The above exercise has been taken from Abstract Algebra by Dummit and Foote.

4. Let $n \geq 1$ be a natural number and define

$$[n] = \{1, \ldots, n\}.$$

Let S_n denote the set of all *bijections* from $[n]$ to $[n]$.
Let $\circ$ denote the usual composition operation of functions.

(a) Show that $\circ$ is a binary operation on S_n.
(b) Show that S_n is a group under $\circ$.
This is known as the *symmetric group* on n elements.
This is a very common example of a (family of) group.
(c) Show that S_n is abelian if and only if $n \leq 2$.

A remarkable theorem called *Cayley's theorem* says that any (finite) group is isomorphic to a subgroup of S_n for some n.