Galois correspondence in Algebraic Topology

Aryaman Maithani
(Supervisor: Prof. Rekha Santhanam)

Department of Mathematics
IIT Bombay

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Covering spaces

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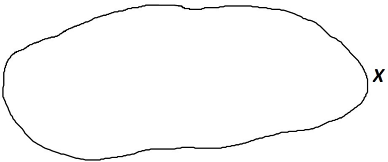

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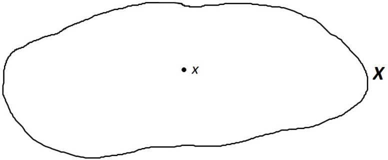

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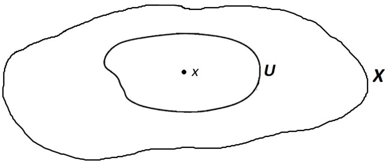

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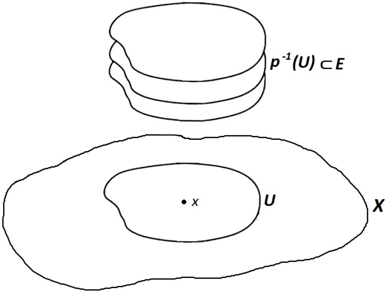

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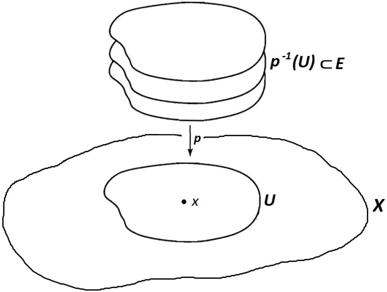

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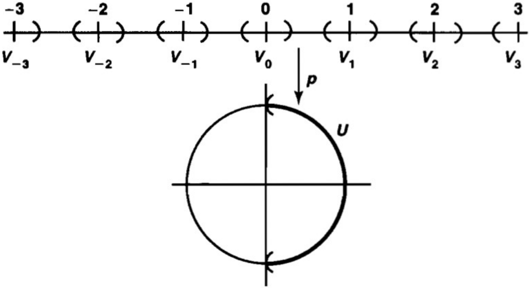

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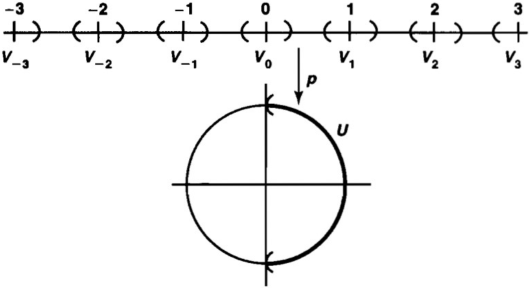

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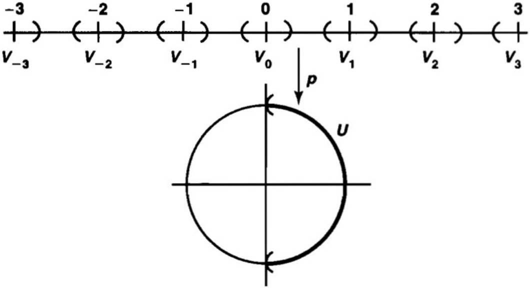

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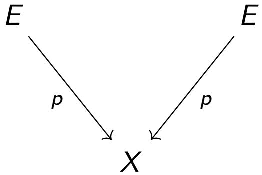

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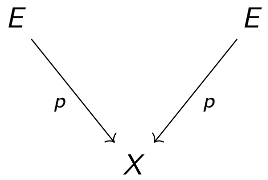

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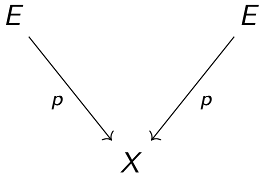

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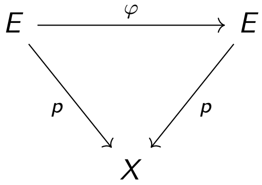

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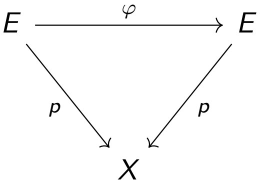
Said differently, it is the group of all homeomorphic lifts of p.



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A map of the form $f : (X, x_0) \to (Y, y_0)$ is a continuous function $f : X \to Y$ such that $f(x_0) = y_0$.

Lifting criterion

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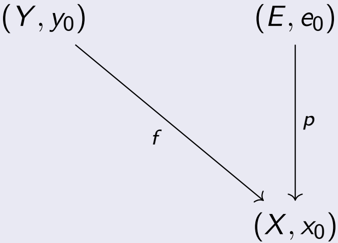

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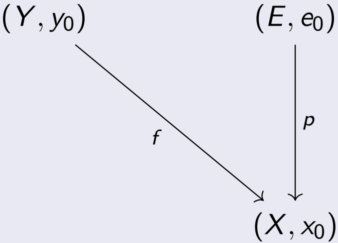

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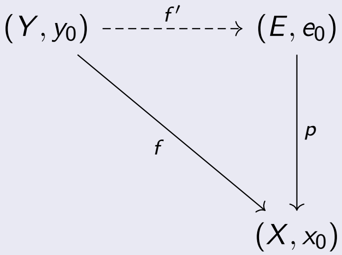

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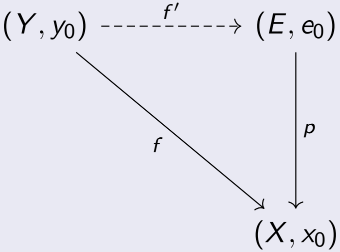

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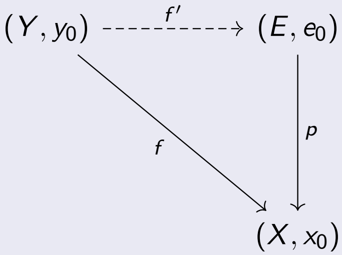

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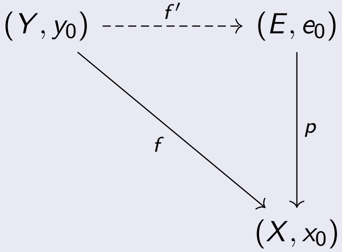

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In such a case, the lift is unique.

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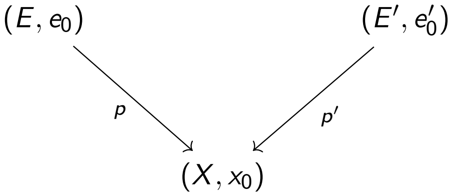

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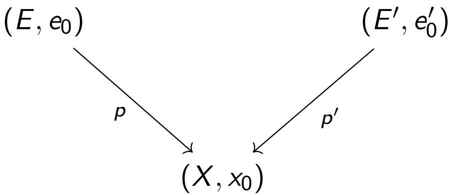

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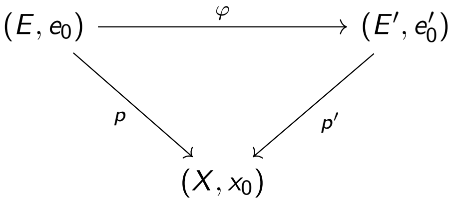

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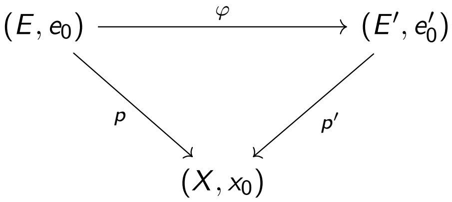
In such a case, we call the coverings *equivalent.*



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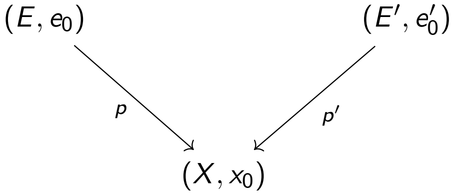

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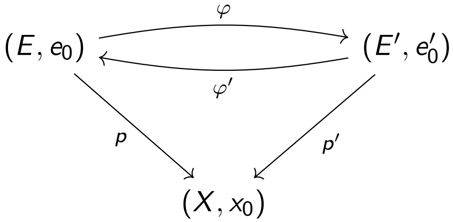

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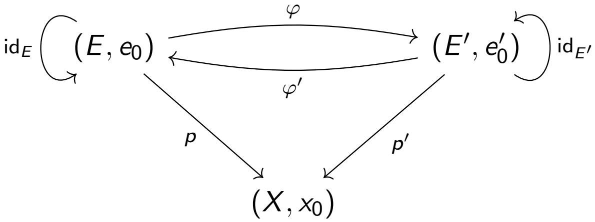

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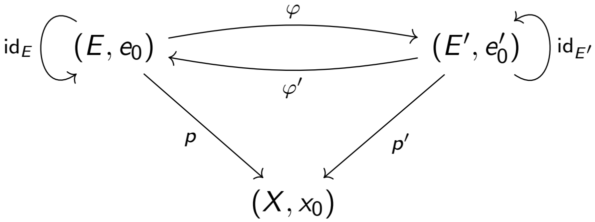

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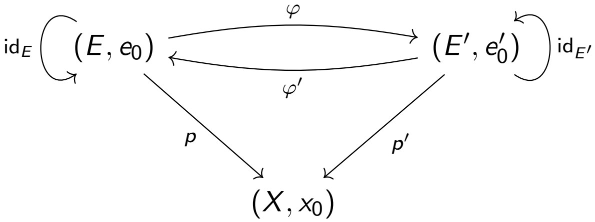

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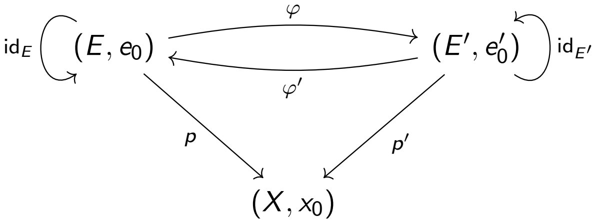

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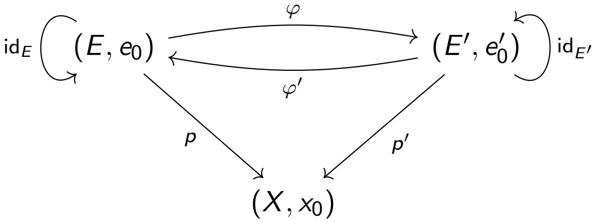

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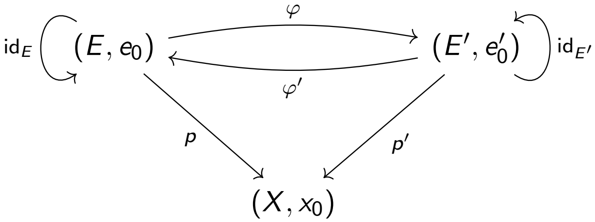

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Does every space have a universal covering space?

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Theorem 5 (Existence of universal covering space)

If X is a semi-locally simply connected space, then X has a universal covering.

Lifting paths

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Computation of fundamental group

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The map $[\sigma] \mapsto g$ is well-defined and an isomorphism. $\square$

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$\pi_1(S^1, 1 + 0\iota) \cong \mathbb{Z}$.

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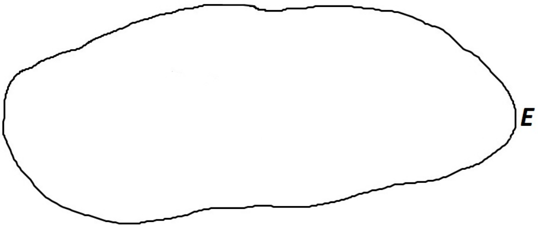

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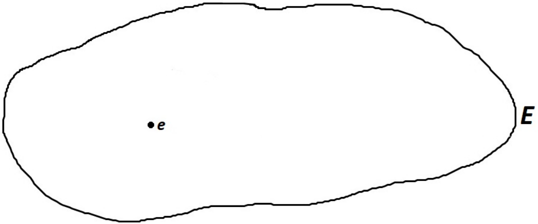

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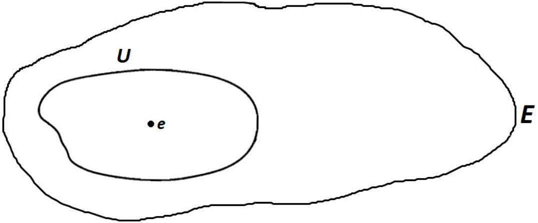

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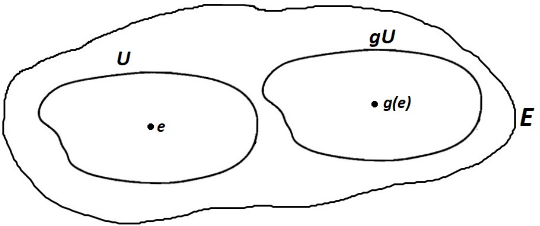

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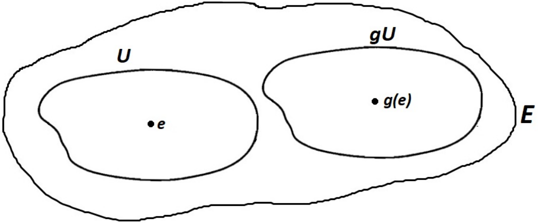

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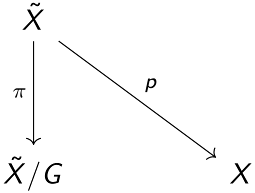

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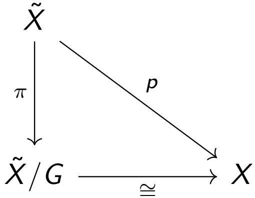

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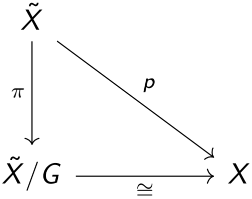

To summarise: we can recover X (up to homeomorphism) from $\tilde{X}$ and G.

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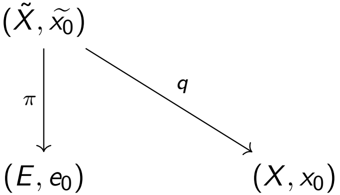

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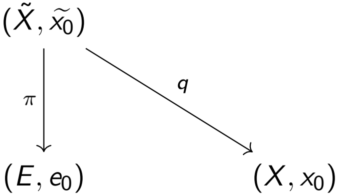

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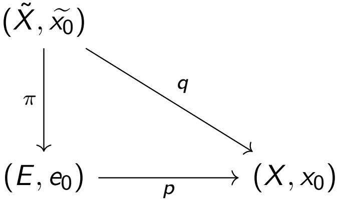

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We now show that $p_*\pi_1(E, e_0) = H$.

Let $[\sigma]$ be an arbitrary element of $\pi_1(X, x_0)$.

Consider the lift $\tilde{\sigma}$ in $\tilde{X}$ with $\tilde{\sigma}(0) = \widetilde{x_0}$. Put $\widetilde{x_1} := \tilde{\sigma}(1)$.

Let $\tau = \pi \circ \tilde{\sigma}$. This is a path in E starting at e_0. Moreover, it is the unique lift of σ starting at e_0.
Thus, if τ is a loop, then $[\sigma] = p_*([\tau]) \in p_*\pi_1(E, e_0)$.
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