Galois correspondence in Algebraic Topology

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Covering spaces

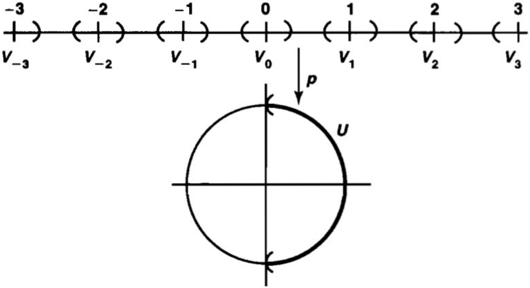
Definition 1 (Covering spaces)

$E \xrightarrow{p} X$ is said to be a *covering space* of X if every $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets S_i in E, each of which is mapped homeomorphically onto U by p.

Examples

Let us take $X = S^1 \subset \mathbb{C}$.

1. Take $E = \mathbb{R}$ and $p : \mathbb{R} \to S^1$ defined by $x \mapsto e^{2\pi\iota x}$.



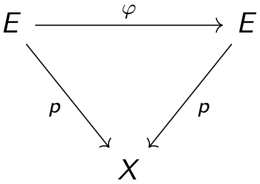
2. $E' = S^1$ and $p_n : S^1 \to S^1$ defined by $z \mapsto z^n$.

Group of covering transformations

Definition 2 (Group of covering transformations)

Given a covering space $E \xrightarrow{p} X$, the group G of *covering transformations* is the group of all homeomorphisms of E which preserves the fibers, that is, all those φ such that $p\varphi = p$.

Said differently, it is the group of all homeomorphic lifts of p.



Let us look at the group of covering transformations for the earlier examples.

1. $\mathbb{R} \xrightarrow{p} S^1$

   The desired homeomorphisms are precisely those of the form $x \mapsto x + n$ for $n \in \mathbb{Z}$. Thus, $G \cong \mathbb{Z}$ here.

2. $S^1 \xrightarrow{p_n} S^1$

   The desired homeomorphisms are precisely multiplication by n-th roots of unity. Thus, $G \cong \mathbb{Z}/n\mathbb{Z}$ here.

Disclaimer and Definitions

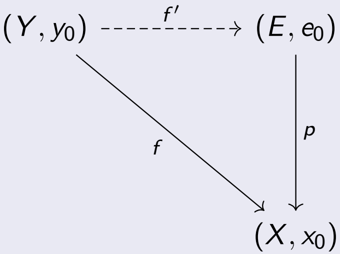
We shall now assume that all spaces are locally path-connected and path-connected.

Recall that given a map $E \xrightarrow{p} X$, we get a homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(X, p(e_0))$ as $[\tau] \mapsto [p \circ \tau]$.

A *map* between topological spaces is a continuous function between them.
A map of the form $f : (X, x_0) \to (Y, y_0)$ is a continuous function $f : X \to Y$ such that $f(x_0) = y_0$.

Theorem 1 (The lifting criterion)

Consider the situation



where p is a covering map and f an arbitrary map. The lift f' exists if and only if

$$f_* \pi_1(Y, y_0) \subset p_* \pi_1(E, e_0).$$

In such a case, the lift is unique.

Corollary 2

If Y is simply connected, the lift f' exists.

Proposition 3

If $Y \xrightarrow{f} X$ is also a covering space, and f' exists, then $Y \xrightarrow{f'} E$ is a covering space.

Thus, we get a diagram

$$\begin{array}{c}(Y, y_0)\\ \big\downarrow{\scriptstyle f'}\\ (E, e_0)\\ \big\downarrow{\scriptstyle p}\\ (X, x_0)\end{array}$$

of covering spaces.

Equivalent covering spaces

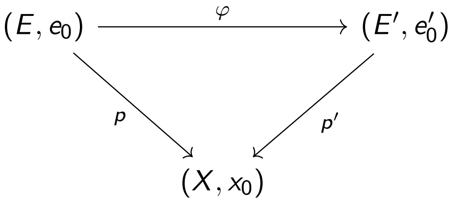
Corollary 4

If $(E, e_0) \xrightarrow{p} (X, x_0)$ and $(E', e_0') \xrightarrow{p'} (X, x_0)$ are covering spaces of X such that $p_*\pi_1(E, e_0) = p'_*\pi_1(E', e_0')$, then there is a unique homeomorphism

$$\varphi : (E, e_0) \to (E', e_0')$$

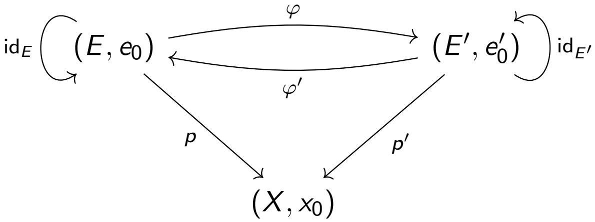
such that $p'\varphi = p$.

In such a case, we call the coverings *equivalent.*



Equivalent covering spaces: Proof

Proof. Since $p_*\pi_1(E, e_0) = p'_*\pi_1(E', e'_0)$, the lifts shown exist.



Uniqueness forces $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ to be the identity maps. Thus, φ is an homeomorphism proving the equivalence.

Note that we actually showed that any two φ and φ' as pictured must be inverses of each other. In particular, φ is unique. □

Universal covering space

If (X, x_0) has a covering space $(\tilde{X}, \widetilde{x_0}) \to (X, x_0)$ such that $\tilde{X}$ is simply connected, then $(\tilde{X}, \widetilde{x_0})$ is unique up to equivalence.

Definition 3 (Universal covering)

We call such a covering space "the" *universal covering space* of (X, x_0).

$\tilde{X}$ is *universal* in the sense that every other covering space is *below* it; it covers every other covering space.

Does every space have a universal covering space?

Well, note that a covering map is a local homeomorphism. Thus, small enough loops in X can be lifted to *loops* in $\tilde{X}$. Thus, these loops must be shrinkable to a point. This gives us a necessary condition. The space must be *semi-locally simply connected*. In fact, this is sufficient as well!

Theorem 5 (Existence of universal covering space)

If X is a semi-locally simply connected space, then X has a universal covering.

Lifting paths

The lifting criterion also shows that paths in X can always be lifted to paths in E. This is because $[0, 1]$ is simply connected. Moreover, we can choose any point in $p^{-1}(x_0)$ to be the starting point.

In fact, more is true. One can lift path homotopies as well.

Thus, if two loops are homotopic in X, then their lifts are homotopic as well. In particular, they have the same endpoint.

Computation of fundamental group

Theorem 6

Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering space with group of covering transformations G. If E is simply connected, then $\pi_1(X, x_0) \cong G$.

Sketch.

The isomorphism $\Phi : \pi_1(X, x_0) \to G$ is given as follows:

Given $[\sigma] \in \pi_1(X, x_0)$, pick a lift $\tilde{\sigma} : [0,1] \to E$ with $\tilde{\sigma}(0) = e_0$.

Set $e_1 := \tilde{\sigma}(1)$. Then, there exists a unique $g \in G$ with $g(e_0) = e_1$.

The map $[\sigma] \mapsto g$ is well-defined and an isomorphism. □

Corollary 7

$\pi_1(S^1, 1 + 0\iota) \cong \mathbb{Z}$.

Proof.

The covering map $\mathbb{R} \xrightarrow{p} S^1$ seen earlier had $G \cong \mathbb{Z}$ and $\mathbb{R}$ is simply connected. $\square$

Even actions

Now, we do something in the opposite direction. We start with a topological space E and a group G of homeomorphisms of E. From this, we construct a covering space.

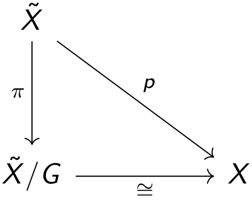
Definition 4 (Even action)

Let G be a group of homeomorphisms of E. G is said to act evenly on E if for every $e \in E$, there exists a neighbourhood U of E such that $U \cap gU = \varnothing$ for all $1 \neq g \in G$.

Even actions

If $(\tilde{X}, \widetilde{x_0}) \xrightarrow{p} (X, x_0)$ is a covering space with $\tilde{X}$ simply connected and G its group of covering transformations, then G acts evenly on E.

Moreover, $\tilde{X}/G$ is homeomorphic to X. This homeomorphism is the natural one as shown in the following diagram.



To summarise: we can recover X (up to homeomorphism) from $\tilde{X}$ and G.

Theorem 8 (Galois correspondence)

Let (X, x_0) be a (pointed) topological space with a universal covering space. Let H be a subgroup of $\pi_1(X, x_0)$. Then, there exists a covering space $(E, e_0) \xrightarrow{p} (X, x_0)$ unique up to equivalence such that

$$p_*\pi_1(E, e_0) = H.$$

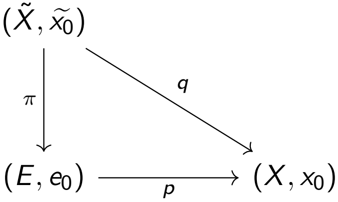
Thus, there is a one-one correspondence between the covering spaces of X and subgroups of $\pi_1(X, x_0)$.

In the example of $X = S^1$, $\mathbb{R}$ is the universal covering space corresponding to $\langle 0 \rangle$. For a nontrivial subgroup $n\mathbb{Z}$, we have $E = S^1$ and the map $z \mapsto z^n$.

Galois correspondence: Proof

Existence. Let $(\tilde{X}, \widetilde{x_0})$ be the universal covering space. Let G be the group of covering transformations.
Recall the isomorphism $\Phi : \pi_1(X, x_0) \to G$. Let $H' = \Phi(H)$. H' acts evenly on $\tilde{X}$.

Consider $(E, e_0) := (\tilde{X}/H', H'\widetilde{x_0})$. Now, we get an induced covering map p as follows.



Galois correspondence: Proof (contd.)

We now show that $p_*\pi_1(E, e_0) = H$.

Let $[\sigma]$ be an arbitrary element of $\pi_1(X, x_0)$.

Consider the lift $\tilde{\sigma}$ in $\tilde{X}$ with $\tilde{\sigma}(0) = \widetilde{x_0}$. Put $\widetilde{x_1} := \tilde{\sigma}(1)$.

Let $\tau = \pi \circ \tilde{\sigma}$. This is a path in E starting at e_0. Moreover, it is the unique lift of σ starting at e_0.
Thus, if τ is a loop, then $[\sigma] = p_*([\tau]) \in p_*\pi_1(E, e_0)$.
More importantly, if τ is *not* a loop, then $[\sigma] \notin p_*\pi_1(E, e_0)$.

Thus, $[\sigma] \in p_*\pi_1(E, e_0)$ iff τ is a loop.

Note that τ is a loop iff $\pi(\widetilde{x_0}) = \pi(\widetilde{x_1})$ iff there is a homeomorphism $h \in H'$ such that $h(\widetilde{x_0}) = \widetilde{x_1}$ iff $\Phi([\sigma]) \in H'$ iff $[\sigma] \in H$.

Thus, $[\sigma] \in H \iff [\sigma] \in p_*\pi_1(E, e_0)$. □