Morphisms of Schemes: Chevalley's Theorem

Aryaman Maithani
Mentor: Prof. Arvind Nair

June 14, 2021

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7. Given $f \in A$, A_f will denote the localisation of A at the multiplicative set $\{1, f, f^2, \ldots\}$.

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Moreover, the "obvious diagrams" must commute.

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The collection $\{D(f) : f \in A\}$ forms a basis for the above topology.

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To conclude, the only dense singleton subset of $\mathbb{A}^1_k$ is $\{\langle 0 \rangle\}$.

Structure sheaf

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In fact, (it follows that) the affine opens form a basis for X.

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The above is a morphism of affine schemes. That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

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Some definitions

Definition 14 (Compact morphism)

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A morphism $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of schemes is compact

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Let K be generated by $x_1, \ldots, x_n$, as a k-algebra.

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The End

Thank you for attending!

The reference for this talk has been Professor Ravi Vakil's (excellent) notes:

http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf