

Morphisms of Schemes: Chevalley's Theorem

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- 1 X and Y will denote topological spaces.
- 2 U, V, W will denote open subsets of the ambient topological space.
- 3 By a cover $\{U_i\}$ of U , we mean that $U = \bigcup_i U_i$. In particular, $U_i \subset U$ for all i .
- 4 A and B will denote a commutative ring with 1. (All our rings will be of this form!)
- 5 $\text{Spec } A$ will denote the set of prime ideals of A .
- 6 Given $S \subset A$, $\langle S \rangle$ will denote the ideal generated by S .
- 7 Given $f \in A$, A_f will denote the localisation of A at the multiplicative set $\{1, f, f^2, \dots\}$.

Definition 1 (Presheaf)

Let X be a topological space. A **presheaf (of rings)** \mathcal{F} on X is the following collection of data:

- 1 For each open set $U \subset X$, we are given a ring $\mathcal{F}(U)$.
- 2 For open sets $U \subset V \subset X$, we have a ring map $\text{res}_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, called the **restriction map**.

The above data is required to satisfy the following conditions:

- 1 $\text{res}_{U,U} = \text{id}_{\mathcal{F}(U)}$ for all open $U \subset X$.
- 2 If $U \subset V \subset W$ are open sets, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\text{res}_{W,V}} & \mathcal{F}(V) \\ & \searrow \text{res}_{W,U} & \downarrow \text{res}_{V,U} \\ & & \mathcal{F}(U) \end{array}$$

Definition 2 (Sheaf)

Let X be a topological space. A **sheaf (of rings)** \mathcal{F} on X is a presheaf \mathcal{F} on X satisfying the following:

Given an open set $U \subset X$, an open cover $\{U_i\}$ of U , and elements $f_i \in \mathcal{F}(U_i)$ such that $\text{res}_{U_i, U_i \cap U_j}(f_i) = \text{res}_{U_j, U_i \cap U_j}(f_j)$ for all i, j , there **exists** a **unique** $f \in \mathcal{F}(U)$ such that

$$\text{res}_{U, U_i}(f) = f_i$$

for all i .

Slogan 3

Given elements on patches which are compatible, we **can** glue them **uniquely**.

Definition 4 (Ringed space)

A **ringed space** is a tuple (X, \mathcal{O}_X) , where X is a topological space and \mathcal{O}_X is a sheaf on X .

Definition 5 (Morphism of ringed spaces)

Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be ringed spaces. A **morphism** $\pi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is the following data:

- 1 A continuous map $\pi : X \rightarrow Y$.
- 2 For every open $V \subset Y$, we have a ring map

$$\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(\pi^{-1}(V)).$$

Moreover, the “obvious diagrams” must commute.

Goal: Turn $\text{Spec } A$ into a ringed space. First, we need a topology.

Definition 6 (Distinguished and Vanishing sets)

Let A be a ring, and $f \in A$. Define

$$D(f) := \{\mathfrak{p} \in \text{Spec } A : f \notin \mathfrak{p}\}.$$

Given a subset $S \subset A$, define

$$V(S) := \{\mathfrak{p} \in \text{Spec } A : S \subset \mathfrak{p}\}.$$

(Check: $D(f) = \text{Spec } A \setminus V(f)$.)

Simple check 1: Given $S \subset A$, we have $V(S) = V(\langle S \rangle)$.

Simple check 2: If $D(g) \subset D(f)$, then f is invertible in A_g . Thus, there is a natural map $A_f \rightarrow A_g$.

Definition 7 (Zariski topology)

Let A be a ring. Then, the collection

$$\{V(I) : I \subset A \text{ is an ideal}\}$$

describes a topology on $\text{Spec } A$ by denoting the collection of *closed* subsets. This is called the **Zariski topology** on $\text{Spec } A$.

Proposition 8 (A basis for the Zariski topology)

The collection $\{D(f) : f \in A\}$ forms a basis for the above topology.

A Helper Example

Let k be a field. We denote $\text{Spec } k[x]$ by \mathbb{A}_k^1 .

Since $k[x]$ is a PID, the prime ideals are $\langle 0 \rangle$ and the maximal ideals.

The set $\{\langle 0 \rangle\}$ is dense in \mathbb{A}_k^1 .

The closed sets are given precisely as:

- 1 The empty set.
- 2 The whole space.
- 3 Sets containing finitely many maximal ideals.

In particular, maximal ideals are *closed points*, i.e., $\{\mathfrak{m}\}$ is closed. Consequently, $\{\mathfrak{m}\}$ is not dense in \mathbb{A}_k^1 .

To conclude, the only dense singleton subset of \mathbb{A}_k^1 is $\{\langle 0 \rangle\}$.

We now describe a sheaf $\mathcal{O}_{\text{Spec } A}$. However, we shall cheat a bit. We only define the objects and arrows on the level of basis elements. One must check that this does indeed a sheaf on the whole space.

Definition 9 (Structure sheaf)

Let A be a ring. Given $f \in A$, we define

$$\mathcal{O}_{\text{Spec } A}(D(f)) := A_f.$$

Given $D(g) \subset D(f)$, the restriction map is the natural map $A_f \rightarrow A_g$.

This is called the **structure sheaf** on $\text{Spec } A$.

Definition 10 (Affine scheme)

An **affine scheme** is a ringed space which is isomorphic to some $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

Definition 11 (Scheme)

A **scheme** is a ringed space (X, \mathcal{O}_X) such that every $p \in X$ has an open neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Slogan 12

A scheme can be covered by **affine opens**.

In fact, (it follows that) the affine opens form a basis for X .

Morphisms of affine schemes

Let $\pi^\# : A \rightarrow B$ a map of rings. This induces a map $\pi : \text{Spec } B \rightarrow \text{Spec } A$ given by $\mathfrak{p} \mapsto (\pi^\#)^{-1}(\mathfrak{p})$. This is continuous.

Moreover, this also induces a morphism of ringed spaces. More explicitly, given $f \in A$, we have the map

$$\begin{array}{ccc} \mathcal{O}_{\text{Spec } A}(D(f)) & \longrightarrow & \mathcal{O}_{\text{Spec } B}(\pi^{-1}(D(f))) = \mathcal{O}_{\text{Spec } B}(D(\pi^\# f)) \\ \parallel & & \parallel \\ A_f & \longrightarrow & B_{\pi^\# f} \end{array} .$$

The above is a [morphism of affine schemes](#). That is, a morphism of affine schemes is a morphism of ringed spaces that is induced by some ring map as above.

Definition 13 (Morphism of schemes)

A **morphism of schemes** $\pi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces that “locally looks like” a morphism of affine schemes.

More precisely, for each choice of affine open sets $\text{Spec } A \subset X$, $\text{Spec } B \subset Y$, such that $\pi(\text{Spec } A) \subset \text{Spec } B$, the restricted morphism is one of affine schemes.

Definition 14 (Compact morphism)

A morphism $\pi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of schemes is **compact** if the preimage of any compact open subset is compact.

Definition 15 (Finite type morphism)

A *compact* morphism $\pi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of schemes is **of finite type** if for every affine open $\text{Spec } B \subset Y$, $\pi^{-1}(\text{Spec } B)$ can be covered by affine open subsets $\text{Spec } A_i$, so that each A_i is a finitely generated B -algebra.

Definition 16 (Noetherian schemes)

A scheme (X, \mathcal{O}_X) is said to be **Noetherian** if X can be covered by finitely many affine opens $\text{Spec } A_i$ such that each A_i is a Noetherian ring.

Some topology

Definition 17 (Locally closed set)

A subset of a topological space X is said to be **locally closed** if it is the intersection of an open subset and a closed subset.

Definition 18 (Constructible set)

A subset of a topological space X is said to be **constructible** if it can be written as a finite disjoint union of locally closed sets.

Example 19 (Simple example)

$X \subset X$ is a constructible subset. $\{\langle 0 \rangle\} \subset \mathbb{A}_k^1$ is not.

Caution 20

What we call “compact” is usually called *quasicompact*.

The definition of “constructible set” above is not the standard one. However, for Noetherian topological spaces (whatever those are), the two are equivalent.

Theorem 21 (Chevalley)

If $\pi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a finite type morphism of Noetherian schemes, then the image of any constructible set is constructible. In particular, the image of π is constructible.

Corollary 22 (Nullstellensatz)

Let $k \subset K$ be a field extension. Suppose K is a finitely generated k -algebra. Then, K is a finite extension of k .

Proof.

Let K be generated by x_1, \dots, x_n , as a k -algebra. It suffices to show that each x_i is algebraic over k . Suppose some x_i is not. Then, we have an inclusion of rings $k[x_i] \hookrightarrow K$, and $k[x_i]$ is isomorphic to the polynomial ring over k .

This corresponds to a dominant morphism $\pi : \text{Spec } K \rightarrow \mathbb{A}_k^1$. Since $\text{Spec } K$ is a singleton, so is the image of π . By dominance of π (and the **Helper** example), the image is $\{\langle 0 \rangle\}$. But this is not constructible (**Simple** example). This contradicts Chevalley's Theorem. □

Thank you for attending!

The reference for this talk has been Professor Ravi Vakil's (excellent) notes:

<http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf>