MA 205

Complex Analysis Tutorials

(TUT - 2)

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General Info

① Keep yourself muted unless you wish to say something.

② If you wish to say something, directly unmute yourself and speak. No point in using "Raise hand" function.

③ If you feel like I've made a mistake, just point it out there.

Case 1

I made a mistake.

I correct the mistake.

Case 2

I did not make a mistake.

I point out the confusion.

I clarify the point better.

Everyone benefits.
(Especially you.)

④ Alternately, you may put it in the chat and someone else may point out your/my mistake. I would view the chat later and adress it if the need may be.

⑤ When unmuting and saying something, I won't be seeing your name since I'll be on another screen.
It is encouraged (but not forced) that you just mention your name at the beginning.
(Since it's being recorded, I understand if you don't want to.)

§ TUTORIAL - 1

25·08·2020

Recap. A set $U \subset \mathbb{C}$ is open if for every $z_0 \in U$, $\exists\, \delta > 0$ s.t.

$$B_\delta(z_0) \subset U.$$

$$\{ z \in \mathbb{C} : |z - z_0| < \delta \}$$

open ball of radius δ at z_0

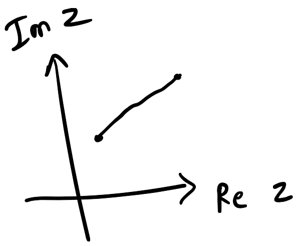
δ - neighbourhood of z_0

- A subset $K \subset \mathbb{C}$ is said to be closed if $\mathbb{C} \setminus K$ is open.

- A subset $P \subset \mathbb{C}$ is said to be path-conn. if for every $z_0, z_1 \in P$, there exists a continuous function

$$\boxed{\sigma : [0, 1] \to P} \text{ s.t.}$$

$$\sigma(0) = z_0 \text{ and } \sigma(1) = z_1.$$



Differentiation

A function $f : \Omega \to \mathbb{C}$ is said to be (complex) differentiable at $z_0 \in \Omega$ if

open subset of $\mathbb{C}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

$f'(z_0)$, if it exists

CR → Cauchy - Riemann Equations

$$u_x = v_y \qquad u_y = -v_x$$

Any (complex) function $f : \Omega \to \mathbb{C}$ can be viewed as a function

$$f : \Omega \to \mathbb{R}^2.$$

also regard as a subset of $\mathbb{R}^2$

$$f(x, y) = \big(u(x, y),\ v(x, y)\big)$$

where $v, u : \Omega \to \mathbb{R}$. (Real-valued)

Thus, u_x, u_y, v_x, v_y make sense.

What CR eq^n's tell us is:

If f is (complex) differentiable at $z_0 = x_0 + iy_0$, then

$$u_x(x_0, y_0) = v_y(x_0, y_0),$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

(existence of partial deriv. is part of the thm)

One more thing that makes sense is the total derivative. (Since f is being viewed as a function from $\Omega_{\subseteq \mathbb{R}^2}$ to $\mathbb{R}^2$.)

Thm. If f is complex diff at $z_0 \in \Omega$, the the total derivative of f at z_0 exists.

Remark. Total deriv. $\not\Rightarrow$ Complex der.

CR eq^n's $\not\Rightarrow$ Complex der.

Q4. $\left\{ \begin{array}{l} \text{Total Deriv.} \\ \& \\ \text{CR eq}^{n}\text{'s} \end{array} \right.$ at (x_0, y_0) $\Longleftrightarrow$ Complex diff. at $z_0 = x_0 + iy_0$.

MA105: Let $f : \Omega \overset{\subseteq \mathbb{R}^2}{} \to \mathbb{R}^2$ with $f = (u, v)$.
If u_x, u_y, v_x, v_y are continuous at some $(x_0, y_0) \in \Omega$, then f has a total derivative at (x_0, y_0).

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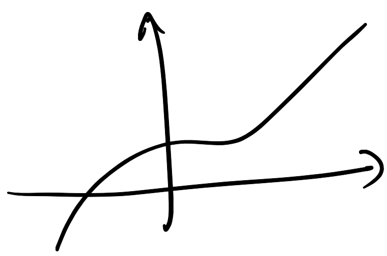
f is said to have total derivative [$A \to$ linear trans. from $\mathbb{R}^2$ to $\mathbb{R}^2$] at $\bar{x}_0 = (x_0, y_0)$ if

$$\lim_{\substack{h \to 0 \\ \in \mathbb{R}^2}} \frac{\| f(\bar{x}_0 + h) - f(\bar{x}_0) - Ah \|}{\|h\|} = 0.$$

$f : \mathbb{R} \to \mathbb{R}$



$f : \mathbb{C} \to \mathbb{C}$ graph? $\to$ 4D

4. Check for real differentiability and holomorphicity:

1. $f(z) = c$ $\longrightarrow$
2. $f(z) = z$
3. $f(z) = z^n, n \in \mathbb{Z}$
4. $f(z) = \mathrm{Re}(z)$
5. $f(z) = |z|$
6. $f(z) = |z|^2$ $\longrightarrow$ Real diff everywhere & complex diff precisely at 0
7. $f(z) = \bar{z}$ $\longrightarrow$ same as (4)
8. $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

bit.ly/ca-205

1. $f(z) = c$. Take $z_0 \in \mathbb{C}$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{0}{z - z_0} = 0.$$

Thus, f is (complex) diff everywhere.
↳ holo
↳ real diff.

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3. $f(z) = z^n$.

① $n \geq 0$ $\quad$ f is defined on $\mathbb{C}$. (Conv: $0^0 = 1$)

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \frac{\cancel{(z - z_0)}\left(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1}\right)}{\cancel{(z - z_0)}}$$

$$= \lim_{z \to z_0} \left(\sum_{i=0}^{n-1} z^i z_0^{n-1-i} \right)$$

$$= \sum_{j=0}^{n-1} z_0^{n-1} = n z_0^{n-1}.$$

Thus, f is (complex) diff. at each $z_0 \in \mathbb{C}$.

Thus, f is holo. on $\mathbb{C}$. (def^n)

Thus, f is real diff on $\mathbb{C}$. (Thm.)

② $n < 0$. Here f is defined on $\mathbb{C} \setminus \{0\}$.

On $\mathbb{C} \setminus \{0\}$, $f(z) = z^n$ is non-zero.

Thus, $\frac{1}{f}$ is diff at z_0 $\Leftrightarrow$ f is diff at z_0.

This, we know by case 1.

$$\left[\frac{1}{f(z)} = z^{-n} \text{ and } -n > 0. \right]$$

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④ $f(z) = \mathrm{Re}(z)$

$$f(z) = f(x,y) = u(x,y) + i\,v(x,y).$$

Here $u(x, y) = x$,

$v(x, y) = 0$.

Clearly, u_x, v_y, v_x, v_y exist everywhere and are continuous.

Thus, f is real diff. everywhere.

Holomorphic?
(Complex diff.)

$$\left.\begin{aligned} u_x(x_0, y_0) &= 1 \\ u_y(x_0, y_0) &= 0 \\ v_x(x_0, y_0) &= 0 \\ v_y(x_0, y_0) &= 0 \end{aligned}\right\} \text{ for all } (x_0, y_0) \in \mathbb{R}^2$$

$u_x(x_0, y_0) \neq v_y(x_0, y_0)$ for any $(x_0, y_0) \in \mathbb{R}^2$

Thus, the CR eq^n's hold NOWHERE.

Thus, f is not complex diff. anywhere.

Remark: f is cont. everywhere but diff. nowhere.

(5) $f(z) = |z|$

$f(x,y) = \sqrt{x^2+y^2} + i0$

$u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = 0$

First note: u_x, u_y don't exist at $(0,0)$.

$\hookrightarrow \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{|x|}{x}$.

Thus: f is NOT real diff at $(0,0)$.

On $\mathbb{R}^2 \setminus \{(0,0)\}$: u & v have cont. partial derivatives.

(Check!)

Thus, f is real diff on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Now, we check for holo:

$$V_x(x_0, y_0) = V_y(x_0, y_0) = 0 \quad \forall (x_0, y_0) \in \mathbb{R}^2.$$

$\mathbb{R}^2 \setminus \{(0,0)\}$: $u_x(x_0, y_0) = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} = V_y = 0$

$$u_y(x_0, y_0) = \frac{y_0}{\sqrt{x_0^2+y_0^2}}. = -V_x = 0$$

For CR eq^n's to hold, we need $x_0 = 0$ & $y_0 = 0$.

$\rightarrow\leftarrow$

To conclude: f is real diff on $\mathbb{R}^2 \setminus \{(0,0)\}$.

f is complex diff. NOWHERE.

5. Show that the CR equations take the form

$$u_r = \frac{1}{r} v_\theta \;\&\; v_r = -\frac{1}{r} u_\theta$$

in polar coordinates.

Imitate slides.

$f(r, \theta) = u(r, \theta) + i\, v(r, \theta)$.

$f(re^{i\theta}) =$

Consider $z_0 \neq 0$.

Thus, $z_0 = r_0 e^{i\theta_0}$ for some $r_0 > 0$ and $\theta_0 \in \mathbb{R}$.

Suppose f is diff at $z_0 = r_0 e^{i\theta_0}$.

Then,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

Compute $\nearrow$ in 2 diff. ways.

① Fix $\theta = \theta_0$ and let $r \longrightarrow r_0$.

$$f'(z_0) = \lim_{r \to r_0} \frac{f(r, \theta) - f(r_0, \theta_0)}{r e^{i\theta_0} - r_0 e^{i\theta_0}}$$

$$= e^{-i\theta_0} \lim_{r \to r_0} \left\{ \underbrace{\frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0}}_{\in \mathbb{R}} + i \underbrace{\left(\frac{\quad}{\quad} \right)}_{\in \mathbb{R}} \right\}$$

this must exist and by def^n, it $u_r(r_0, \theta_0)$

$$\boxed{f'(z_0) = e^{-i\theta_0} \left\{ u_r(r_0, \theta_0) + i v_r(r_0, \theta_0) \right\}.} \qquad (*)$$

② Fix $r = r_0$ and let $\theta \longrightarrow \theta_0$.

$$\frac{1}{r_0} \lim_{\theta \to \theta_0} \left\{ \underbrace{\frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}}} + i \left(\frac{\quad}{\quad} \right) \right\}$$

this must exist

$$\lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{\underbrace{\theta - \theta_0}_{= u_\theta(r_0, \theta_0)}} \cdot \overbrace{\frac{\theta - \theta_0}{\underbrace{e^{i\theta} - e^{i\theta_0}}}}^{\text{this exists } \& \neq 0}$$

this limit can be computed as

$$\frac{1}{i e^{i\theta_0}}$$

$$\boxed{f'(z_0) = \frac{1}{r_0}\left\{ \frac{u_\theta(r_0,\theta_0)}{ie^{i\theta_0}} + i\,\frac{v_\theta(r_0,\theta_0)}{ie^{i\theta_0}} \right\}} \quad (**)$$

Compare $(*)$ & $(**)$ to get the answer.

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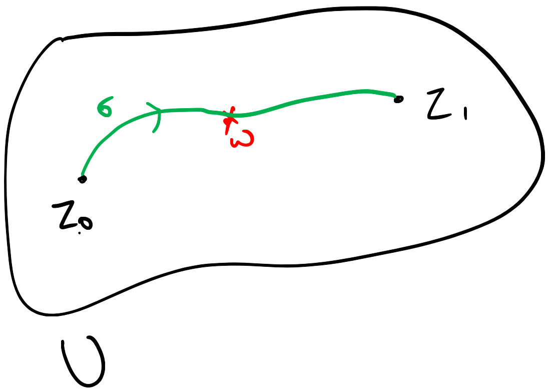
3. Show that if U is a path connected open set in $\mathbb{C}$, so is U minus finite set. (Try to prove it as rigorously as you can)

Sol^n: Note: If $w \in U$, then $U \setminus \{w\}$ is open. (Proof?)

If I show $U \setminus \{w\}$ is path-connected, then I am done. (Why?)



(Want: $\sigma : [0,1] \to U \setminus \{w\}$ continuous s.t. $\sigma(0) = z_0$, $\sigma(1) = z_1$)

Let $z_0, z_1 \in U \setminus \{w\}$.

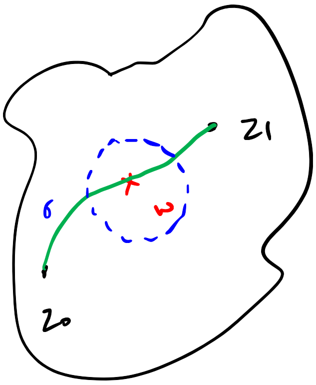
Since U is path-conn., $\exists$ cts. $\sigma : [0,1] \to U$ s.t. $\sigma(0) = z_0$, $\sigma(1) = z_1$

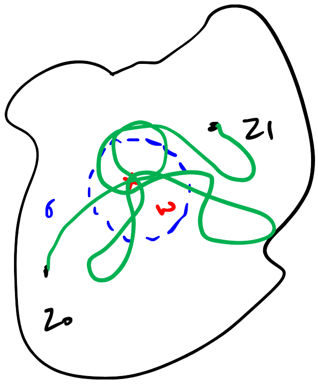
① $\sigma(x) \neq w$ for any $x \in [0,1]$.

Then done. (why?)

σ then actually maps into $U \setminus \{w\}$.

② $\sigma(x) = w$ for some $x \in [0, 1]$.





Using the fact that U is open,

I can find a closed ball

$$\overline{B} = \{ z \in \mathbb{C} : |z - w| \leq \delta \}.$$

$(\exists \delta > 0)$

s.t.

① $z_0 \notin \overline{B}$, ② $z_1 \notin \overline{B}$, ③ $\overline{B} \subset U$.

δ_1 happens δ_2 $\delta_3/2$

Why does such a $\overline{B}$ exist?

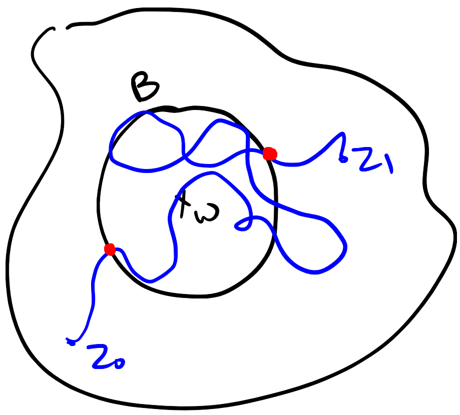
Since U is open, $\exists\, \delta > 0$ s.t. $B_\delta(w) \subset U$.

Then, $\overline{B_{\delta/2}}(w) \subset B_\delta(w) \subset U$.

Now, consider

$$C := \sigma^{-1}(\bar{B}) = \{x \in [0,1] : \sigma(x) \in B\}.$$

closed



Thus, C is closed. (why? Because σ is continuous)

$[C \subset [0,1].]$

[(Ex) If $f : \mathbb{R} \to \mathbb{C}$ is cts, then $f^{-1}(K)$ is closed whenever $K \subset \mathbb{C}$ is closed.]

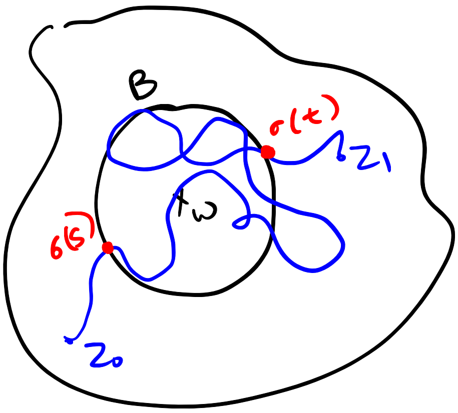
① C is closed.

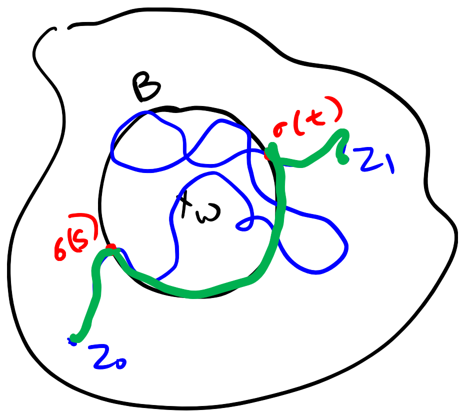
② C is bounded. (Why?)

$\Downarrow$

③ $\sup C$ & $\inf C$ exist
$\sup C = s$, $\inf C = t$

④ $s \in C$ and $t \in C$] $\to$ this is because C is closed





Thus, the green path is one such desired path!

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① Show that $f(x)$ has $\deg f(x)$ roots.

Assume FTA.

f has one root x_0.

Then, $f(x) = (x - x_0)\, g(x)$

$\deg f(x) - 1$

use induc.

Note: $f(x) = 0$

$\iff x = x_0$ or $g(x) = 0$

Thus, the complete (multi)set of roots is

$\underset{1}{\{x_0\}} \cup \overbrace{\{x_i : g(x_i) = 0\}}^{n-1}.$

1. A complex polynomial of degree n has exactly n roots. (Assuming fundamental theorem of algebra)

2. Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \ a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

② $f(x) \in \mathbb{R}[x]$. $\deg f(x) \geq 3$.

s.t. $f(x) = g(x)h(x)$

↳ ↲ non-const. real poly

① f has a root in $\mathbb{R}$. Done ✓ (why?)

$f(x) = (x - x_0)g(x)$

↳ non-const

② f has no root in $\mathbb{R}$.

(By FTA), f has a root $x_0 \in \mathbb{C}$.

Then, $f(\bar{x}_0) = 0$ as well. (why?)

$x - x_0$ & $x - \bar{x}_0$ are factors of $f(x)$

$$\Rightarrow f(x) = \underbrace{(x - x_0)(x - \bar{x}_0)}_{\in \mathbb{R}[x] \quad = g(x)} h(x)$$

↳ $\in \mathbb{C}[x]$

$f(x) = g(x)h(x)$

$f(x), g(x) \in \mathbb{R}[x]$

↳ deg = 2

Thus, $h(x) \in \mathbb{R}[x]$.

↳ deg $= n - 2 \geq 1$.

Poly. div. $f(x) = g(x)q(x) + r(x)$ in $\mathbb{R}[x]$

$\deg r(x) < 2$.

↳ in $\mathbb{C}[x]$ as well

- $r(x) = 0$ since $g(x) \mid f(x)$ in $\mathbb{C}[x]$