TOPOLOGY

with $\int(\circ_\smile\circ)dx$

bit.ly/ca-205

↳ tut solution, etc.

- open sets
- closed sets
- unions and intersections
- bounded sets
- compact sets
- EVT
- path-connected

- All apples in this bag are good. → I

(negation)

- At least one apple in this bag is not good. → II

⇒ bag is non-empty

If my bag is empty, which of I or II is true? I

"Vacuously true"

I Many

II

Both ||

None

Let $U \subset \mathbb{C}$.

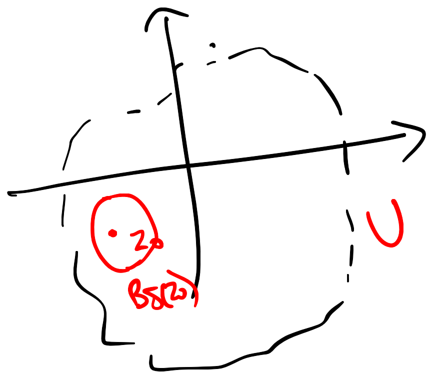
Let us say that $z_0 \in U$ is good (interior point)

if there exists some $\delta > 0$

such that

$$B_\delta(z_0) \subset U,$$

where $B_\delta(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \delta \}$.



Note that some δ should exist.

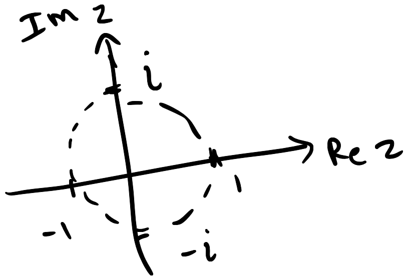
Def^n. A subset $U \subset \mathbb{C}$ is said to be open if all points of U are good.

In other words, for all $z_0 \in U$, $\exists \delta > 0$ s.t. $B_\delta(z_0) \subset U$.

Q. Is ϕ open? Yes. Why? Vacuously!

Q. Is $\mathbb{C}$ open? Yes. Why? Take any $z_0 \in \mathbb{C}$. $B_1(z_0) \subset \mathbb{C}$. i.e. $\delta = 1$ works.

Q. Is $D^2 = \{ z \in \mathbb{C} : |z| < 1 \}$ open?



Proof. Let $z_0 \in D^2$.

Let $\delta := 1 - |z_0|$.

Note that $\delta > 0$.

To show that: $\underline{\underline{B_\delta(z_0)}} \subset \underline{\underline{D^2}}$.

show

Let $z \in B_\delta(z_0)$.

I know: $|z - z_0| < \delta$

Now, observe that:

$$
\begin{aligned}
|z| &= |z - z_0 + z_0| \\
&\leq |z - z_0| + |z_0| \\
&< \delta + |z_0| = 1.
\end{aligned}
$$

$\therefore |z| < 1 \Rightarrow z \in D^2$. $\boxtimes$

Def^n. A subset $U \subset \mathbb{C}$ is said to be closed if $U^c = \mathbb{C} - U = \mathbb{C} \setminus U$ is open.

Remark. U is closed $\not\Leftrightarrow$ U is not open

U is not closed $\not\Leftrightarrow$ U is open

U is closed $\overset{\text{def}^n}{\Longleftrightarrow}$ U^c is open

- Is $\emptyset$ closed? Yes. Why? $\emptyset^c = \mathbb{C}$ is open.
- Is $\mathbb{C}$ closed? Yes. Why? $\mathbb{C}^c = \emptyset$ is open.

Q. Is $C = \{z \in \mathbb{C} : |z| > 1\}$ open?

Yes. Take $z_0 \in C$. Check that $\delta = |z_0| - 1$ works.

Q. Is $D^2 \cup C = \{z \in \mathbb{C} : |z| > 1 \text{ or } |z| < 1\}$ open in $\mathbb{C}$?

Yes. Use same argument as earlier:

If $z_0 \in D^2 \cup C$,

then either $z_0 \in D^2 \rightsquigarrow \delta = 1 - |z_0|$

or

$z_0 \in C \rightsquigarrow \delta = |z_0| - 1$.

Q. What is $\mathbb{C} \setminus (D^2 \cup C)$?

It is $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Q. Is S^1 closed? Yes! Its complement is open.

Q. Is S^1 open? No. ~~It is closed.~~

$\Updownarrow$

At least one point of S^1 is not good.

Let us $1+0i \in S^1$.

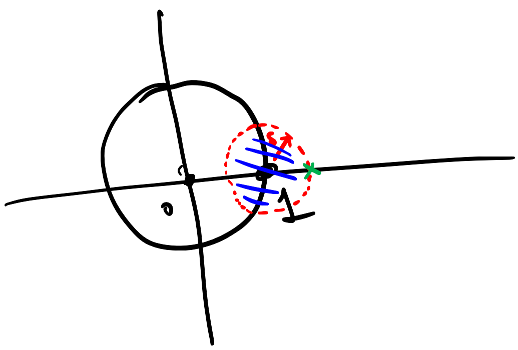
To show: It is not good.

That is, to show that

for every $\delta > 0$, we have that

$$\underbrace{B_\delta(1) \not\subset S^1}_{\text{This means that } \exists z \in B_\delta(1) \text{ s.t. } z \notin S^1}.$$

Proof. Let $\delta > 0$ be arbitrary.



Then, $1+\frac{\delta}{2} \in B_\delta(1)$. (Why?)

But $1+\frac{\delta}{2} \notin S^1$. (Why?)

Because

$$\left|1+\frac{\delta}{2}\right| = 1+\frac{\delta}{2} > 1.$$

Thus, for any $\delta > 0$, $B_\delta(1) \not\subset S^1$.

Thus, 1 is not good (for S')

Thus, S' is not open! $\boxed{\because}$

Q. Is $E = \{z \in \mathbb{C} : |z| \leq 1\}$.

Is E closed?

Yes. Why? E is not open, by similar argument.

This is true!

NO! ← Argument

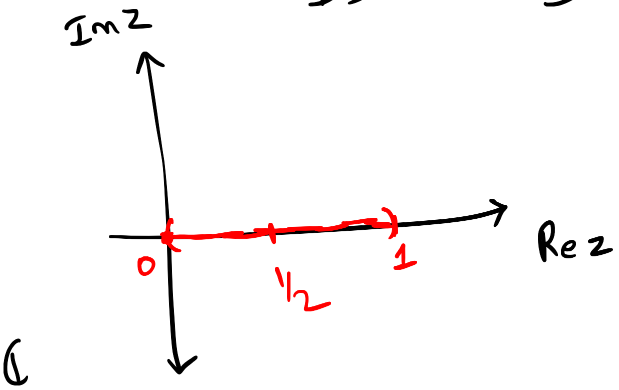
$\mathbb{C} \setminus E = \{z \in \mathbb{C} : |z| > 1\}$.

we have shown this is open!

Remark. There ARE subsets of $\mathbb{C}$ which are NEITHER open nor closed.

Q. Consider the interval $I = (0,1) \subset \mathbb{R} \subset \mathbb{C}$.

Is I open? No!



Why? Give one not good point.

$0.5 = \frac{1}{2}$

Note that for any $\delta > 0$

$$\frac{1}{2} + i\frac{\delta}{2} \in B_\delta\left(\tfrac{1}{2}\right) \text{ but}$$

$$\frac{1}{2} + i\frac{\delta}{2} \notin (0,1).$$

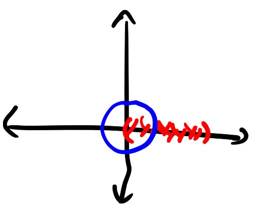
Thus, $(0,1)$ is not an open subset of $\mathbb{C}$.

Q. Is $(0,1)$ closed? No.

Let us look at

$\mathbb{C}\setminus(0,1) \rightarrow$ want to show this is not open.

Give me a not good point.



Claim: $0 \in \mathbb{C}\setminus(0,1)$ is not good.

[Want: $B_\delta(0) \not\subset \mathbb{C}\setminus(0,1)$]

$A \not\subset \mathbb{C}\setminus B$
$\Leftrightarrow A \cap B \neq \emptyset$

Proof. Let $\delta > 0$.

$\Updownarrow$

$\left[B_\delta(0) \cap (0,1) \neq \emptyset \right]$

works if $\delta < 1$

Claim: $\frac{\delta}{2} \in B_\delta(0)$ but $\frac{\delta}{2} \notin \mathbb{C}\setminus(0,1)$

or

$\underline{\frac{\delta}{2} \in B_\delta(0)}$ and $\underline{\frac{\delta}{2} \in (0,1)}$

Is this true? Not always.

Let $\boxed{x_0 := \min\left\{ \frac{1}{2}, \frac{\delta}{2} \right\}}$.

Now, clearly, $x_0 \in (0,1)$. (Why?)

Moreover, $x_0 \in B_\delta(0)$. (Why?)

$|x_0 - 0| = x_0 \leq \frac{\delta}{2} < \delta.$

Thus, for any $\delta > 0$, $B_\delta(0) \not\subset \mathbb{C}\setminus(0,1)$.

Conclusion. $(0,1)$ is a subset of $\mathbb{C}$ which is neither open nor closed.

Fact. $\emptyset$ and $\mathbb{C}$ are the only subsets of $\mathbb{C}$ which are <u>both</u> open and closed.

Def^n. A subset $S \subset \mathbb{C}$ is said to be bounded if there exists $M > 0$ s.t.

$$|z| \leq M \quad \text{for all} \quad z \in S.$$

(M is obviously independent of z.)

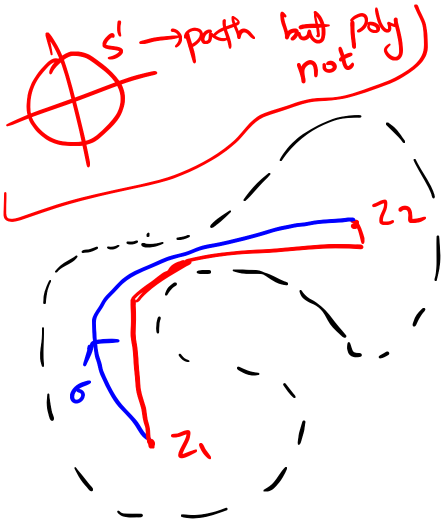
Def^n. A subset $K \subset \mathbb{C}$ is said to be compact if it is closed and bounded.

Fact. If $f : \overset{\text{compact}}{K} \to \mathbb{C}$ is continuous, then $\exists\, M > 0$ s.t. $|f(z)| \leq M \quad \forall\, z \in K$.

Def^n. A subset $P \subset \mathbb{C}$ is said to be path-connected, if for every $z_1, z_2 \in P$, there is a path in P that joins z_1 and z_2. (ϕ, by THIS def^n, is path-connected.)

To be precise, there is a continuous function $\sigma : [0,1] \to P$ such that

$$\sigma(0) = z_1 \quad \text{and} \quad \sigma(1) = z_2.$$



Defn. A subset $\Omega \subset \mathbb{C}$ is said to be a domain if Ω is open and path-connected.

Q. Let D^2 be as earlier. $\{z \in \mathbb{C} : |z| < 1\}$.

Let $A \subset D^2$ be at most countable. (finite or infinite)

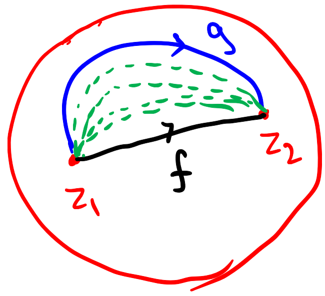
For this argument, you only need D^2 to be convex & open.

Claim: $D^2 \setminus A$ is path-connected.

Result is true in more generality.

↳ D^2 can be replaced with path-conn & open.

Proof. Let $z_1, z_2 \in D^2 \setminus A$.



First consider f, line segment in D^2 joining them

Similarly, consider an arc, g joining them.
in D^2

For every $\lambda \in \underbrace{[0,1]}_{\text{uncountable}}$, define the path

$$\sigma_\lambda := \lambda f + (1-\lambda) g.$$

$t \in [0,1]$ $\quad \sigma_\lambda(t) = \lambda f(t) + (1-\lambda) g(t).$

Claim 1. σ_λ is a path in D^2.

Claim 2. $\sigma_\lambda(0) = z_1$ & $\sigma_\lambda(1) = z_2$ $\forall \lambda \in [0,1]$

Claim 3. If $\lambda_1 \neq \lambda_2$ and $t \in (0,1)$
then, $\lambda_1(t) \neq \lambda_2(t)$

images of the paths are disjoint

Fact: $[0,1]$ is uncountable.

$\{\sigma_\lambda\}_{\lambda \in [0,1]}$ is uncount.

$\exists$ some λ_0 s.t. $\sigma_{\lambda_0}(t) \notin A$ for all $t \in [0,1]$.

In other words,

σ_{λ_0} is a path in $D^2 \setminus A$ starting at z_1 & ending at z_2.

Yes! ϕ is bounded. Any $M > 0$ works.

Yes. Vacuous.

$\phi \to$ open, closed, bdd, path conn.

$\mathbb{C} \to$ open, closed, NOT bdd, path conn.

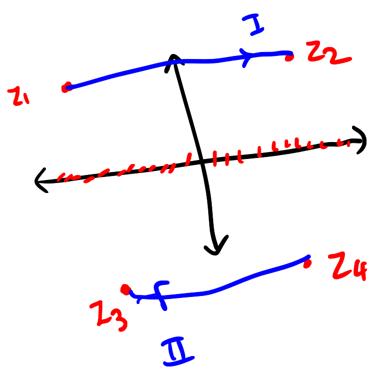
? $\to$ $\underbrace{\text{open, closed}}$, NOT path-conn.

No.

↳ only ϕ and $\mathbb{C}$.

Both are path-conn.

Q. $\mathbb{C} \setminus (\mathbb{R} \setminus \{0\})$ is path connected.



Three cases

(1) Both are in the same half plane.

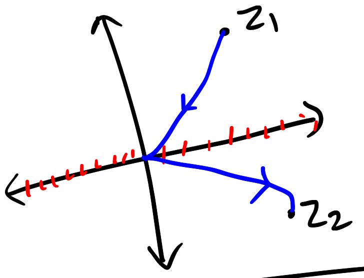
either both have $\operatorname{Im} > 0$ or both $\operatorname{Im} < 0$

↳ take the line segment joining them

(2) In different half planes.

WLOG, $\operatorname{Im} z_1 > 0$

$\operatorname{Im} z_2 < 0$

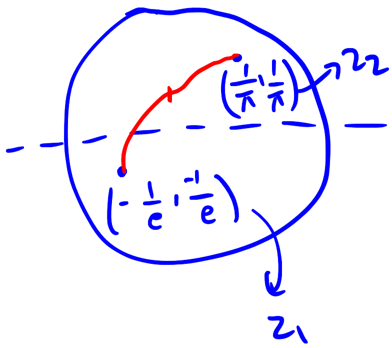


Join z_1 to 0 & 0 to z_2.

(3) one or both are 0.

↑ Trivial.

Q: Is $D^2 \setminus \big((\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q}) \big)$.



Suppose σ is a path from z_1 to z_2.

$$\sigma : [0,1] \to \mathbb{C}$$

s.t. $\sigma(0) = z_1$ & $\sigma(1) = z_2$.

$(a,b) \equiv a+bi$

Note that I have used $\mathbb{R}^2$ & $\mathbb{C}$ interchangeably.

Consider $\gamma : [0,1] \to \mathbb{R}$

given by $\gamma(t) = \mathrm{Re}(\sigma(t))$.

Is γ continuous? Yes. (why?)

↓

Moreover $\gamma(0) = -\dfrac{1}{e}$

$\gamma(1) = \dfrac{1}{\pi}$.

Thus, $\gamma(c) = 0$ for some $c \in (0,1)$

Thus, $\mathrm{Re}(\sigma(c)) = 0 \in \mathbb{Q}$.

$\rightarrow\leftarrow$

↳ The above question was

Take the disc $D^2 = \{ x \in \mathbb{C} : |x| < 1 \}$.

Remove all those points which have real part $\in \mathbb{Q}$.

Then remove all those points which have imag. part $\in \mathbb{Q}$

TS: The final set is NOT path-connected.

What I did: I gave you point $z_1, z_2 \in$ Set s.t. there is no path from z_1 to z_2

Q: Let $S \subset \mathbb{C}$ s.t. $S \neq \phi$ and $S^c \neq \phi$.

for every $\epsilon > 0$, can we find

$s \in S$ & $s' \in S^c$ s.t.

$|s - s'| < \epsilon$.

Answer: Yes.

Def^n. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of sets.

① A is some set. → called the indexing set.

② for each $\alpha \in A$, U_α is a set

Then,

$$\bigcup_{\alpha \in A} U_\alpha := \{ x : x \in U_\alpha \text{ for some } \alpha \in A \}.$$

This is defined even if $A = \phi$.

and

$$\bigcap_{\alpha \in A} U_\alpha := \{ x : x \in U_\alpha \text{ for all } \alpha \in A \}.$$

Only defined when $A \neq \phi$.

Theorem. If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open subsets of $\mathbb{C}$, then

$$U := \bigcup_{\alpha \in A} U_\alpha \text{ is open in } \mathbb{C}.$$

Proof. Let $z_0 \in U$. We want to show z_0 is good.

$\Downarrow$

$z_0 \in U_{\alpha_0}$ for some $\alpha_0 \in A$.

this is open

Thus, $\exists \delta > 0$ s.t. $B_\delta(z_0) \subset U_{\alpha_0}$.

However, $U_{\alpha_0} \subset U$.

Thus, $B_\delta(z_0) \subset U$. $\longrightarrow$ z_0 is good.

Thus, U is open.

Theorem. Let $U_1, \dots, U_n \subset \mathbb{C}$ be open sets. $(n \geq 1)$

Then, $U := U_1 \cap U_2 \cap \dots \cap U_n$ is open in $\mathbb{C}$.

Proof. Let $z_0 \in U$.

$\Downarrow$

$z_0 \in U_1$, $z_0 \in U_2$, ..., $z_0 \in U_n$

$\Downarrow$ each is open, $\exists \delta_i > 0$ for all $1 \leq i \leq n$

$B_{\delta_i}(z_0) \subset U_i \quad \forall\, 1 \leq i \leq n$

Take $\boxed{\delta = \min\{\delta_i : 1 \leq i \leq n\}}$. $\leftarrow$ finiteness

$\delta \leq \delta_i \quad \forall i$

$\Rightarrow B_\delta(z_0) \subset B_{\delta_i}(z_0) \subset U_i \quad \forall i$

$\Rightarrow B_\delta(z_0) \subset U$.

Example. Take $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Clearly, every element of A is > 0.

Does $\min A$ exist? No.

$\inf A$ exists and is $= 0$.

Thus, δ can't be chosen.

Take $U_n = B_{1/n}(0)$ for all $n \in \mathbb{N}$.

$\{U_n\}_{n \in \mathbb{N}}$ is a collection of open sets.

BUT $\bigcap_{n \in \mathbb{N}} U_n = \{0\} \longrightarrow$ NOT open.

De Morgan's Laws

$$\left(\bigcup_{\alpha \in A} U_\alpha \right)^c = \bigcap_{\alpha \in A} \left(U_\alpha^c \right)$$

$$\left(\bigcap_{\alpha \in A} U_\alpha \right)^c = \bigcup_{\alpha \in A} \left(U_\alpha^c \right)$$

Ex. Show that finite union of closed sets is closed.

Show that arbitrary inters. of closed sets is closed.