MA 205: $\mathbb{C}$omplex Analysis

Extra questions

Aryaman Maithani

https://aryamanmaithani.github.io/tuts/ma-205

Last Update: August 21, 2021

Contents

§0. Notations

1. $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers.
2. $\mathbb{Z}$ is the set of integers.
3. $\mathbb{Q}$ is the set of rational numbers.
4. $\mathbb{R}$ is the set of real numbers.
5. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
6. $A \subset B$ is read as "A is a subset of B." In particular, note that $A \subset A$ is true for any set A.
7. $A \subsetneq B$ is read "A is a *proper* subset of B."
8. $\supset$ and $\supsetneq$ are defined similarly.
9. Given a function $f : X \to Y$, $A \subset X$, $B \subset Y$, we define

   $$f(A) = \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \subset Y,$$
   $$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X.$$

   (Note that this f^{-1} is different from the inverse of a function. In particular, this is always defined, even if f is not bijective. However, the f and f^{-1} above need not be "inverses.")
10. A *domain*, as a subset of $\mathbb{C}$ will always refer to a set which is open and path connected.
    (Note that this is different from domain of a function.)

§1. Topology

1. Is the interval $(0, 1)$ open as a subset of $\mathbb{C}$?
   HIDDEN: No

2. Is the interval $(0, 1)$ closed as a subset of $\mathbb{C}$?
   HIDDEN: No

3. Consider the following four properties that a subset of $\mathbb{C}$ can have:

   (a) Open

   (b) Closed

   (c) Bounded

   (d) Path connected

   Thus, we can classify all the subsets of $\mathbb{C}$ into 2^4 classes on the basis of what properties they have (and what they don't).
   Give an example of each or a proof that some certain class cannot have anything.
   You may assume that $\varnothing$ and $\mathbb{C}$ are the only subsets of $\mathbb{C}$ which are both open and closed.

4. Let $U \subset \mathbb{C}$ be open and nonempty. Show that U is not countable.

5. Let $U \subset \mathbb{C}$ be open and K be countably infinite. Give examples to show that $U \setminus K$ may or not be open.

§2. Cauchy Riemann Equations

1. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined as
$$f(z) = \bar{z}.$$
Show that f is continuous at each point.
Show that f is differentiable at no point.
(This has given us a very easy example of a function which is continuous everywhere but differentiable nowhere. On the contrary, one has to put a lot more effort to construct an example in the case of real analysis.)

2. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as
$$f(x, y) = (x, -y)$$
is differentiable in the sense that you saw in MA 105. (That is, its total derivative exists at every point.)
Compare this with the previous question.

3. Let Ω be open (and not necessarily path-connected).
Let $f : \Omega \to \mathbb{C}$ be holomorphic such that $f'(z) = 0$ for all $z \in \Omega$.
Show that it is *not* necessary that f is constant.

Show that if Ω is also assumed to be path-connected (that is, Ω is a domain), then it *is* necessary that f is constant.

4. Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose
$$f(z) \in \mathbb{R} \quad \text{for all } z \in \Omega.$$
Show that f is constant. (That is, if a complex differentiable function takes only real values, then it must be constant on path-connected sets.)

5. Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose that $|f|$ is constant. Show that f is constant.

§3. Series

1. (Cauchy criterion for series.) "Recall" Cauchy criterion for convergence from MA 105. (Prove or assume that the analogous thing holds for complex sequences as well.)

   Let (a_n) be a sequence of complex numbers. Show that $\sum_{n=1}^{\infty} a_n$ converges iff for every $\epsilon > 0,$ there exists $N \in \mathbb{N}$ such that

   $$\left|\sum_{k=n}^{m} a_n\right| < \epsilon, \qquad \text{for all } m \geq n \geq N.$$

2. Let (a_n) be a sequence of complex numbers such that $\sum |a_n|$ converges. Use the above Cauchy criteria to show that $\sum a_n$ converges.

3. Let (a_n) and (b_n) be complex sequences such that $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$. Show that if $\sum |b_n|$ converges, then so does $\sum |a_n|$ and hence, so does $\sum a_n$. Show that you can weaken the "for all $n \in \mathbb{N}$" condition to "for all n sufficiently large." (Formulating what we mean by "sufficiently large" is part of the exercise.)

4. Use the above to show that

   $$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

   converges for all $z \in \mathbb{C}$ satisfying $|z| = 1$.

5. Show that

   $$\sum_{n=1}^{\infty} \frac{1}{n}$$

   diverges.
   HIDDEN: Compare it with the sequence $1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, \ldots$.

6. Let (a_n) be a sequence of real numbers and (b_n) a sequence of complex numbers satisfying

   (a) (a_n) is monotonic,

   (b) $\lim_{n\to\infty} a_n = 0,$

   (c) there exists $M \geq 0$ such that

   $$\left|\sum_{n=1}^{N} b_n\right| \leq M$$

for every $N \in \mathbb{N}$.

Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Here's an outline of what you can do:

(a) Define the partial sums $S_n = \sum_{k=1}^{n} a_k b_k$ and $B_n = \sum_{k=1}^{n} b_k$.

Show that

$$S_n = a_n B_n + \sum_{k=1}^{n-1} B_k (a_k - a_{k+1}).$$

(This is called summation by parts.)

(b) Note that B_n is bounded by M and $a_n \to 0$. Conclude that the first term $\to 0$ as $n \to \infty$.

(c) Note that give any k, we have $|B_k(a_k - a_{k+1})| \leq M|a_k - a_{k+1}|$.

(d) Using (a_n) is monotonic, conclude that

$$\sum_{k=1}^{n-1} |a_k - a_{k+1}| = \sum_{k=1}^{n-1} |a_1 - a_n|.$$

(e) Conclude that $\lim_{n\to\infty} S_n$ exists.

The above is called **Dirichlet's test**.

7. Let $z \in \mathbb{C}$ be such that $|z| = 1$ and $z \neq 1$. Define the sequences (a_n) and (b_n) as

$$a_n := \frac{1}{n}, \quad b_n := z^n.$$

Show that (a_n) and (b_n) satisfy the hypothesis of Dirichlet's test. Conclude that

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges.

8. Compute the radius of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

These should come out to be 1. By the previous questions, conclude that the first converges everywhere on the boundary of the disc except at 1. However, the second one converges everywhere on the boundary.
Do the same for the power series

$$\sum_{n=1}^{\infty} z^n.$$

HIDDEN: You should get that it converges nowhere on the boundary.
(Note that these series are (more or less) derivatives and anti-derivatives of each other on the *open* disc. However, they show very different behaviour on the boundary of the disc.)

9. Let (a_n) and (b_n) be sequences of complex numbers such that the power series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

have radii of convergence R_1 and R_2 respectively.
Show that if $R_1 < R_2$, then the radius of convergence of

$$\sum_{n=0}^{\infty} (a_n + b_n) z^n$$

is R_1.
Show that if $R_1 = R_2$, then all that we can conclude is that the radius of convergence of the sum is at least R_1.
(The possibilities of radii being 0 or ∞ should not be excluded.)
At this point, I'll remark that you should recall that the radius of convergence being R not only says that it converges for all $|z| < R$ but also that it *diverges* for all $|z| > R$.

§4. Properties of holomorphic functions

1. Let $\mathbb{H} = \{z \in \mathbb{C} : \Re z > 0\}$ be the open half right plane.
   Construct a non-constant holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

   $$f\left(\frac{1}{n}\right) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

   (Does this contradict what we saw in slides? Why not?)

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that

   $$f\left(\frac{1}{n}\right) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

   Show that f is constant (and that the constant is 0).
   Compare this with the previous question.

3. Suppose that the domain in the previous question was replaced by an arbitrary domain Ω such that $\{n^{-1} : n \in \mathbb{N}\} \subset \Omega$.
   Characterise Ω precisely such that the above $f(1/n) = 0$ condition ensures that f is constant. (That is, come up with a rule such that if Ω follows that rule, then f has to be constant and that if Ω does not follow the rule, then f may be non-constant.)
   HIDDEN: The rule should be (equivalent to): $0 \in \Omega$.

4. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions which are nonzero everywhere. Suppose that f and g satisfy

   $$\left(\frac{f'}{f}\right)\left(\frac{1}{n}\right) = \left(\frac{g'}{g}\right)\left(\frac{1}{n}\right), \qquad \text{for all } n \in \mathbb{N}.$$

   (The LHS is the function f'/f is evaluated at $1/n$ and similarly for the RHS.)
   Find a simpler relation between f and g. (Yes, "simpler" is subjective.)

5. Consider the principal branch $\log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$. Choose the point $z_0 = -3 + 4\imath$ in the domain and expand $\log$ as a power series around this point.
   Show that the radius of convergence of this power series is 5 and not 4.

§5. Picard, Rouché, Cauchy's estimates, Liouville, MMT

1. Show that $\exp(z) = z$ has a solution in $\mathbb{C}$.
2. Let f, g be entire functions such that $\exp f + \exp g = 1$. Show that f and g are constant.
3. Let f be a non-vanishing entire function. (That is, f is never zero.) Show that there exists an entire function g such that $f = \exp \circ g$.
4. Let f be a non-vanishing entire function. (That is, f is never zero.) Show that there exists an entire function g such that $f = g^2$. (That is, $f(z) = (g(z))^2$ for all $z \in \mathbb{C}$.)
5. **Minimum Modulus Theorem.**
   Let Ω be open and connected and $f : \Omega \to \mathbb{C}$ be non-constant and non-vanishing. Show that $|f|$ attains no minimum.
6. Without using Little Picard, show that there is no entire non-constant function such that the image is contained in the upper half plane.
   HIDDEN: Consider $z \mapsto \dfrac{z-i}{z+i}$.
7. Let $P(z)$ and $Q(z)$ be polynomials with real coefficients such that $\deg Q(z) \geq \deg P(z) + 2$.
   Moreover, assume that Q has no real root.
   (a) Show that there exist constants $C, R > 0$ such that
   $$\left| \frac{P(z)}{Q(z)} \right| \leq \frac{C}{|z|^2}$$
   for all $z \in \mathbb{C}$ with $|z| > R$.
   (b) Conclude the improper integrals
   $$\int_{-\infty}^{-R} \frac{P(x)}{Q(x)} \mathrm{d}x \quad \text{and} \quad \int_{R}^{\infty} \frac{P(x)}{Q(x)} \mathrm{d}x$$
   exist.
   (c) Argue that the integral
   $$\int_{-R}^{R} \frac{P(x)}{Q(x)} \mathrm{d}x$$
   also exists.

(d) Conclude that the integral

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \mathrm{d}x$$

exists.

(e) Let γ_r denote the semicircle (without the diameter) in the upper half plane with ends $-r$ and r. Show that

$$\lim_{r\to\infty} \int_{\gamma_r} \frac{P(z)}{Q(z)} \mathrm{d}z = 0.$$

(f) Use Cauchy residue theorem to conclude that $\dfrac{1}{2\pi\iota}\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \mathrm{d}x$ is equal to the sum of the residues of $P(x)/Q(x)$ at the poles in the upper half plane.

8. Let $f : \mathbb{C}^{\times} \to \mathbb{C}$ be a holomorphic function such that

$$|f(z)| \leq \sqrt{|z|} + \frac{1}{\sqrt{|z|}}$$

for all $z \in \mathbb{C}^{\times}$.

(a) Show that 0 is a removable singularity of f. Conclude that f can be made entire.

(b) Show that ∞ is a removable singularity of f. Conclude that f is bounded.

(c) Conclude that f is constant.

9.

Lemma 5.1 (Schwarz Lemma)**.** Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. (Recall $\mathbb{D}$ from Section 0.)
Assume that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ and $f(0) = 0$.

Then,

$$|f(z)| \leq |z| \quad (z \in \mathbb{D}), \tag{1}$$
$$|f'(0)| \leq 1. \tag{2}$$

Moreover, if equality holds in (1) for some $z \in \mathbb{D} \setminus \{0\}$, or if equality holds in (2), then $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

This was there in the slides but here's an outline you can follow to prove it:

(a) Show that $g(z) := f(z)/z$ has a removable singularity at 0.

(b) For $0 < r < 1$, define $\mathbb{D}_r := \{z \in \mathbb{C} : |z| \leq r\}$. By MMT, conclude that there exists $z_r \in \mathbb{D}_r$ such that $|g(z)| \leq g(z_r)$ for all $z \in \mathbb{D}_r$.

(c) Given any $z \in \mathbb{D}$, note that there exists r such that $z \in \mathbb{D}_r$. Thus, for any $z \in \mathbb{D}$, conclude that
$$|g(z)| \leq |g(z_r)| \leq \frac{1}{r}.$$

(d) Let $r \to 1$ to conclude (1) and (2). (Hint for (2): What is $g(0)$?)

(e) Now, if either of the appropriate equalities hold, then $|g(z)| = 1$ at some point of $\mathbb{D}$. Use MMT to conclude that g must be a constant with modulus 1. Conclude the last statement of the lemma.

10. For $\alpha \in \mathbb{D}$, define the function φ_α by
$$\varphi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha} z}.$$
This function is defined and holomorphic on $\mathbb{C} \setminus \{\bar{\alpha}^{-1}\}$. In particular, it is holomorphic on $\mathbb{D}$.

(a) If $\alpha \in \mathbb{D}$, show that $-\alpha \in \mathbb{D}$. Show that $\varphi_{-\alpha}(\varphi_\alpha(z)) = z$ for all z in the domain. Conclude that φ_α is one-one.

(b) Show that if $|z| = 1$, then $|\varphi_\alpha(z)| = 1$.

(c) Show that φ_α is nonconstant. (You have actually already done the work for this. Do you see how?)
Conclude that if $z \in \mathbb{D}$, then $\varphi_\alpha(z) \in \mathbb{D}$.
HIDDEN: Use MMT.

(d) The above shows that $\varphi_\alpha(\mathbb{D}) \subset \mathbb{D}$. By considering $\varphi_{-\alpha}$, show that the equality $\varphi_\alpha(U) = \mathbb{D}$ is true. Conclude that $\varphi_\alpha|_{\mathbb{D}}$ is a bijection from $\mathbb{D}$ onto itself.

(e) Show that
$$\varphi_\alpha'(0) = 1 - |\alpha|^2, \quad \varphi_\alpha'(\alpha) = \frac{1}{1 - |\alpha|^2}.$$

11. Continue the same notations from the previous question. We see there that φ_α are biholomorphisms[1] from $\mathbb{D}$ onto itself. Let us now answer the following extremal question.

[1]A biholomorphism is a bijective holomorphic function whose inverse is also holomorphic.

> Suppose α and β are complex numbers, $|\alpha| < 1$, and $|\beta| < 1$. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function bounded by 1. How large can $|f'(\alpha)|$ if $f(\alpha) = \beta$?

To answer this, do the following:

(a) Put $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$.

(b) Since $\varphi_{-\alpha}$ and φ_β map $\mathbb{D}$ onto $\mathbb{D}$, we see that g is a holomorphic function on $\mathbb{D}$ bounded by 1.

(c) Use Schwarz lemma to conclude that $|g'(0)| \le 1$. Use chain rule and the previous question to conclude

$$|f'(\alpha)| \le \frac{1 - |\beta|^2}{1 - |\alpha|^2}. \tag{3}$$

(Along the way, you'll also use that $f(\alpha) = \beta$.)

(d) Observe that this solves our problem by showing that equality can actually be achieved in (3).
Also show that in that case, we have

$$f(z) = \varphi_{-\beta}(\lambda \varphi_\alpha(z)) \quad (z \in \mathbb{D})$$

for some constant λ with $|\lambda| = 1$.

12. We still continue with the earlier notation. Here, we finish off the discussion by finding *all* biholomorphisms from $\mathbb{D}$ to itself. We had already seen that φ_αs are some of these. We now show that these are essentially all.

Theorem 5.2. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a biholomorphism. Let $\alpha \in \mathbb{D}$ be such that $f(\alpha) = 0$. Then, there is a constant λ with $|\lambda| = 1$ such that

$$f(z) = \lambda \varphi_\alpha(z),$$

for all $z \in \mathbb{D}$.

In other words, f is simply obtained by composing φ_α with a rotation.

Remark. In the above, we have assumed that f is a biholomorphism but one can actually only assume that f is bijective and holomorphic, the holomorphicity of its inverse comes for free.

Prove this via the following steps:

(a) Let g be the inverse of f. In particular, $g(f(z)) = z$ for all $z \in \mathbb{D}$.

(b) Use chain rule to conclude that

$$g'(0)f'(\alpha) = 1. \tag{4}$$

(c) Note that $f(\alpha) = 0$ and $g(0) = \alpha$. Now, use the conclusion of the previous extremal problem to conclude that

$$|f'(\alpha)| \le \frac{1}{1 - |\alpha|^2}, \quad |g'(0)| \le 1 - |\alpha|^2.$$

Use (4) to conclude that equalities hold above.
In turn, use the final conclusion of the extremal problem solution to conclude the theorem.

13. Suppose f, g are entire functions and $|f(z)| \le |g(z)|$ for every $z \in \mathbb{C}$. What conclusion can you draw about f and g?
HIDDEN: If g is not identically zero, then its zeroes are isolated. Show that all zeroes of g are actually removable singularities of f/g. Thus, conclude that f/g is entire. Finish it from that.

14. Suppose f is an entire function and there exist constants $A, B > 0$ and $k \in \mathbb{N}$ such that
$$|f(z)| \le A + B|z|^k$$
for all $z \in \mathbb{C}$. Show that f is a polynomial of degree at most k.

15. **Fractional Residue Theorem.**
Let f have a simple pole at z_0. Let $\delta > 0$ be such that f is holomorphic on the punctured neighbourhood $B_\delta(z_0) \setminus \{z_0\}$.
Fix $\alpha \in (0, 2\pi]$ and $\alpha_0 \in [0, 2\pi)$.
For $0 < r < \delta$, define $\gamma_r(\theta) := z_0 + re^{\iota(\theta + \alpha_0)}$ for $\theta \in [0, \alpha]$. (Draw a picture to see that this is an arc centered at z_0 subtending angle α and having radius r.)

Let $l := \text{Res}(f; z_0)$.

(a) Show that $g(z) := f(z) - \dfrac{l}{z - z_0}$ is holomorphic on $B_\delta(z_0)$.
(More correctly: show that z_0 is a removable singularity of g.)

(b) Conclude that there exists M such that $|g(z)| \le M$ for $z \in B_\delta(z_0)$.

(c) Conclude that

$$\lim_{r \to 0} \int_{\gamma_r} g(z) \mathrm{d}z = 0.$$

(d) Conclude that

$$\lim_{r \to 0} \int_{\gamma_r} f(z) \mathrm{d}z = \lim_{r \to 0} \int_{\gamma_r} \frac{l}{z - z_0} \mathrm{d}z.$$

(e) Show that the RHS is $\alpha\iota \operatorname{Res}(f; z_0)$ and conclude the fractional residue theorem.

16. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Recall that a fixed point of f is a point $z_0 \in \Omega$ such that $f(z_0) = z_0$. Suppose that Ω contains the closed unit disc. Moreover, assume that $|f(z)| < 1$ for $|z| = 1$. Show that f has exactly one fixed point in the open unit disc.

17. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and Ω contains the closed unit disc. Suppose that $f(0) = 1$ and $|f(z)| > 2$ if $|z| = 1$. Then, show that f has at least one zero in the open unit disc.
HIDDEN: Minimum modulus theorem.