Question 1

29 September 2020 12:00 PM

1. Evaluate $\int_0^{2\pi} \frac{\cos^2(3x)}{5-4\cos(2x)} \mathrm{d}x.$

$2\cos(n\theta) = \underline{e^{in\theta}} + e^{-in\theta}$ ← idea

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2(3\theta)}{5-4\cos(2\theta)} \mathrm{d}\theta &= \frac{1}{4}\int_0^{2\pi} \frac{(2\cos(3\theta))^2}{5-2(2\cos(2\theta))} \mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{2\pi} \frac{(e^{3\iota\theta}+e^{-3\iota\theta})^2}{5-2(e^{2\iota\theta}+e^{-2\iota\theta})} \mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{2\pi} \frac{(e^{3\iota\theta}+e^{-3\iota\theta})^2}{5-2(e^{2\iota\theta}+e^{-2\iota\theta})} \frac{\iota e^{\iota\theta}}{\iota e^{\iota\theta}} \mathrm{d}\theta \\
&= \frac{1}{4}\int_{|z|=1} \frac{(z^3+z^{-3})^2}{5-2(z^2+z^{-2})} \frac{1}{\iota z} \mathrm{d}z \\
&= -\frac{1}{8\iota}\int_{|z|=1} \frac{(z^6+1)^2}{z^5(z^4-5z^2/2+1)} \mathrm{d}z
\end{aligned}$$

parameterise $|z|=1$ as $z(\theta) = e^{i\theta}$

Want to use CRT

$$f(z) := \frac{(z^6+1)^2}{z^5\left(z^4 - \frac{5z^2}{2} + 1\right)}$$

$z = \pm\frac{1}{\sqrt{2}}, \pm\sqrt{2}$ are the roots

Poles of f: $\underbrace{0, \pm\frac{1}{\sqrt{2}}}_{\text{these that matter}}$, $\pm\sqrt{2}$ → outside curve

$$\int_\gamma f = 2\pi\iota \sum_{z}{}' \operatorname{Res}(f; z)$$

z → poles within γ

Residue at 0 : $f(z) = \dfrac{(z^6+1)^2}{z^5(z^2-2)(z^2-\frac{1}{2})}$

0 is a pole of order 5.

Thus, define $g(z) := z^5 f(z)$ and $\text{Res}(f;0) = \frac{1}{4!} g^{(4)}(0)$.

Ugly (but correct)

We instead compute Laurent series around 0.

$(1-w)^{-1} = 1 + w + w^2 + \cdots \quad |w|<1$

$$\frac{(z^6+1)^2}{z^5(z^4-5z^2/2+1)} = \frac{1}{z^5}\frac{(z^6+1)^2}{1-(5z^2/2-z^4)}$$

$$= \frac{1}{z^5}(z^6+1)^2\left[1+\left(\frac{5z^2}{2}-z^4\right)+\left(\frac{5z^2}{2}-z^4\right)^2+\cdots\right]$$

$$= \frac{1}{z^5}\left\{[z^{12}+2z^6+1]\left[1+\left(\frac{5z^2}{2}-z^4\right)+\underbrace{\left(\frac{5z^2}{2}-z^4\right)^2}_{=\frac{25z^4}{4}-5z^6+z^8}+\cdots\right]\right\}$$

Res $(f;0)$ is coefficient of $\frac{1}{z}$ here

= Co-eff of z^4 in $\{\cdots\}$.

$$= -1 + \frac{25}{4} = \frac{21}{4}$$

$\text{Res}(f;0) = \frac{21}{4}$

$\text{Res}(f;\frac{1}{\sqrt{2}})$:

$f(z) = \dfrac{(z^6+1)^2}{z^5(z^2-2)(z-\frac{1}{\sqrt{2}})(z+\frac{1}{\sqrt{2}})}$

$$z^5(z^2-2)\left(z-\frac{1}{\sqrt{2}}\right)\left(z+\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \operatorname{Res}\left(f; \frac{1}{\sqrt{2}}\right) = \lim_{z\to\frac{1}{\sqrt{2}}} \left(z-\tfrac{1}{\sqrt{2}}\right) f(z) = \lim_{z\to\frac{1}{\sqrt{2}}} \frac{(z^6+1)^2}{z^5(z^2-2)\left(z+\frac{1}{\sqrt{2}}\right)}$$

$$//$$

$$\frac{\left(\left(\frac{1}{\sqrt{2}}\right)^6+1\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^5\left(\left(\frac{1}{\sqrt{2}}\right)^2-2\right)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)} = \frac{\left(\frac{1}{8}+1\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^5\left(\frac{1}{2}-2\right)\left(\frac{2}{\sqrt{2}}\right)}$$

$$= -\frac{\frac{81}{64}}{\left(\frac{1}{4\sqrt{2}}\right)\left(\frac{3}{2}\right)(\sqrt{2})}$$

$$= -\frac{27}{8}$$

$$\therefore \operatorname{Res}\left(f; \frac{1}{\sqrt{2}}\right) = -\frac{27}{8}$$

Similarly, $\operatorname{Res}\left(f; -\frac{1}{\sqrt{2}}\right)$ //

$$\int_\gamma f = 2\pi i\left(\frac{21}{4} - \frac{27}{8} - \frac{27}{8}\right)$$

$$= -3\pi i$$

Thus, desired integral is: $-\frac{1}{8i}(3\pi i)$

$$= \frac{3\pi}{8}$$

Real integral $\longrightarrow$ integral over unit circle

$\downarrow$

Use CRT

Question 2

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2. Evaluate $\displaystyle\int_{|z-2|=4} \frac{2z^3+z^2+4}{z^4+4z^2}\,\mathrm{d}z.$

Use CRT! let $f(z) := \dfrac{2z^3+z^2+4}{(z^4+4z^2)}$

$(z^4+4z^2) = z^2(z^2+4) = z^2(z+2i)(z-2i)$

all lie within the curve

Poles : $0, \pm 2i$

0: order 2

$\pm 2i$: order 1 (of each)

Residue at 0:

Define $g(z) = z^2 f(z)$.

Compute $g'(0)$.

Then, $\operatorname{Res}(f; 0) = \dfrac{1}{1!} g'(0)$

In general:

z_0, pole of order m of f

$g(z): z^m f(z)$

Then $\operatorname{Res}(f; z_0) = \dfrac{1}{(m-1)!} g^{(m-1)}(z_0)$.

$$g(z) = \frac{2z^3+z^2+4}{z^2+4}$$

$g'(0) = 0$ (Check!)

Thus, $\operatorname{Res}(f; 0) = 0$.

Res. at $2i$:

$2i$ is a simple pole.

Thus, $\operatorname{Res}(f; 2i) = \displaystyle\lim_{z\to 2i} (z-2i) f(z)$

$$= \lim_{z\to 2i} \frac{2z^3+z^2+4}{z^2(z+2i)}$$

$$\begin{aligned}\lim_{z\to 2\imath}(z-2\imath)f(z) &= \lim_{z\to 2\imath}\frac{2z^3+z^2+4}{z^2(z+2\imath)}\\ &= \frac{2(2\imath)^3+0}{(2\imath)^2(2\imath+2\imath)}\\ &= \frac{-16\imath}{-4(4\imath)}\\ &= 1.\end{aligned}$$

Simi., $\operatorname{Res}(f; -2i) = 1.$

Thus, $\displaystyle\int_{|z-2|=4} f = 2\pi i(0+1+1)$

$= \boxed{4\pi i}$

Question 3

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3. Show with and without the open mapping theorem that if f is a holomorphic function on a domain Ω with $|f|$ is constant, then f is constant.

Without OMT: (CR equations)

Write $f = u + iv$ as usual.

We are given: $\exists c \geq 0$: $u^2(x,y) + v^2(x,y) = c$ for all $(x,y) \in \Omega$.

Case 1. $c = 0$. forces $u \equiv 0 \equiv v$. Thus, $f \equiv 0$ and we are done.

Case 2. $c \neq 0$.

$$u^2 + v^2 = c$$

$\partial/\partial x$: $uu_x + vv_x = 0$ (1)

$\partial/\partial y$: $uu_y + vv_y = 0 \xrightarrow{CR} -uv_x + vu_x = 0$ (2)

(1) and (2): $\underbrace{\begin{bmatrix} u & v \\ v & -u \end{bmatrix}}\begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

↳ this has determinant $-c \neq 0$.

Thus $\begin{bmatrix} u(x,y) & v(x,y) \\ v(,) & -u(,) \end{bmatrix}$ is inv. for all $(x,y) \in \Omega$.

$$\Rightarrow \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for all points in } \Omega.$$

$$\Rightarrow u_x \equiv 0 \equiv v_x$$

$$u_x = v_y, \qquad v_x = -u_y$$

Thus, u and v are constant. ($\because$ Ω was a domain.)

With OMT:

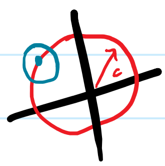
Recall: If Ω is a domain (open & (path-)connected) and $f: \Omega \to \mathbb{C}$ is hdo & non-constant, then $f(\Omega)$ is open.
(OMT actually gives: $f(U)$ is open for all open $U \subset \Omega$.)

Sol^n. Given: $|f|$ is constant.
To show: f is constant.

Proof: Assume not. That is, f is not const.
Then, $f(\Omega)$ must be open. However,

$$f(\Omega) \subset \{ z \in \mathbb{C} : |z| = c \} \qquad (c = |f|)$$

This has no open subset. Thus, we get a contradiction.



Question 4

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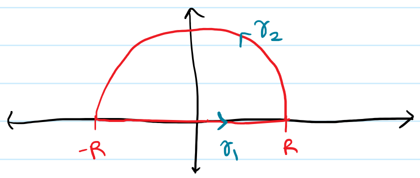
4. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\mathrm{d}x = -\frac{\pi}{10}$.



Let $f(z) = \dfrac{z}{(z^2+2z+2)(z^2+4)}$ | $(z+1)^2+1=0$

Poles of f: $-1 \pm i$, $\pm 2i$

Take $R > 2$.

The poles enclosed within the drawn contour: $-1+i$, $2i$

CRT:

$$\int_{\gamma_1} f + \int_{\gamma_2} f = 2\pi i\left(\mathrm{Res}(f; -1+i) + \mathrm{Res}(f; 2i)\right)$$

$$\lim_{R\to\infty} \int_{\gamma_1} f = \int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx$$

$$\lim_{R\to\infty} \int_{\gamma_2} f = 0.$$

Use ML inequality

$\exists C > 0$

$$\frac{|z|}{|z^2+2z+2|\cdot|z^2+4|} \le \frac{C}{|z|^3}$$

for $|z|$ sufficiently large

$\exists C > 0$ s.t. for $|z|$ suff. large we have

$$|z^n + a_{n-1}z^{n-1} + \dots + a_0| \geq C|z|^n$$

$$\Updownarrow$$

$$\Big|\underbrace{1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}}_{\to 1 \text{ as } |z|\to\infty}\Big| \geq C$$

We can find $R > 0$ s.t. $\left|1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right| \geq \frac{1}{2}$ for $|z| > R$.

Thus, desired integral $= 2\pi i\left(\mathrm{Res}(f; -1+i) + \mathrm{Res}(f; 2i)\right)$

both are simple

do it!

Question 5

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5. Show that any injective entire function is of the form $az + b$ for some $a \neq 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Obs. f cannot be constant.

(1) We show f is a poly.

Assume not.

Then, $a_n \neq 0$ for inf. many $n \geq 0$.

$\Rightarrow$ 0 is an essential sing. of $z \mapsto f\left(\frac{1}{z}\right)$. $(*)$

Define $\Omega_1 = \{z : |z| < 1\}$ and $\Omega_2 = \{z : |z| > 1\}$.

By OMT, $f(\Omega_1)$ is open.

Note that $a_0 \in f(\Omega_1)$. $(\star)$ $\left(\because a_0 = f(0) \text{ \& } 0 \in \Omega_1\right)$

By $(*)$, 0 is an ess. sing. of $f(1/z)$.

Thus, by Casorati-Weierstrass, $\exists$ a sequence (z_n) s.t.

$z_n \to 0$ and $f\left(\frac{1}{z_n}\right) \to a_0$.

Note: $z_n \to 0 \Rightarrow \exists N_1 \in \mathbb{N}$ s.t. $|z_n| < 1$ for all $n \geq N_1$.

$\Rightarrow \frac{1}{z_n} \in \Omega_2$ for $n \geq N_1$.

$\Rightarrow f\left(\frac{1}{z_n}\right) \in f(\Omega_2)$ for $n \geq N_1$. (1)

Since $a_0 \in \Omega_1$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(a_0) \subset \Omega_1$. (2)

Since $f\left(\frac{1}{z_n}\right) \to a_0$, $\exists N_2 \in \mathbb{N}$ s.t. $f\left(\frac{1}{z_n}\right) \in B_\varepsilon(a_0)$ for all $n \geq N_2$. (3)

For $N = \max\{N_1, N_2\}$.

Then, by (1), $f\left(\frac{1}{z_N}\right) \in f(\Omega_2)$.

(3), $f\left(\frac{1}{z_N}\right) \in B_\varepsilon(a_0) \subset f(\Omega_1)$.

Thus, $f\left(\frac{1}{z_N}\right) \in f(\Omega_1) \cap f(\Omega_2)$

$\Rightarrow f(\Omega_1) \cap f(\Omega_2) \neq \phi$.

contradicts 1-1

since $\Omega_1 \cap \Omega_2 = \phi$.

Thus, f is a polynomial.

2. $f(z) = a_0 + \cdots + a_n z^n$ for some $n \geq 1$

$a_n \neq 0$.

Assume $n > 1$. Then, $f(z) = 0$ has n roots.
Injectivity forces all equal.

Thus, $f(z) = k(z - z_0)^n$

for some $z_0 \in \mathbb{C}$
& $k \in \mathbb{C}^\times$.

But then, $f(z) = 1$ has n distinct roots

But $n > 1$ contradicts injectivity.

Thus, $n = 1$. Hence, $f(z) = a_1 z + a_0$ & $a_1 \neq 0$