Tutorial 4 - Recap

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Singularity → "where things go 'bad'"

Let $f : \Omega \to \mathbb{C}$ be a function.

Let $z_0 \in \mathbb{C}$. z_0 is said to be a singularity if:

→ (i) $z_0 \notin \Omega$
(ii) $z_0 \in \Omega$ but f is not holo. at z_0.

For example, consider ① $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) := |z|$.
Any $z_0 \in \mathbb{C}$ is a singularity.

② $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined as

$$f(z) = \frac{\sin z}{z}.$$

0 is a sing. since f is not defined as 0.

③ $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

$$f(z) = \frac{1}{z}.$$

Again is a singularity.

④ $f : \mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\} \to \mathbb{C}$

$$f(z) = \frac{z}{\sin z}.$$

Each $n\pi \in \mathbb{C}$ $(n \in \mathbb{Z})$ is a sing.

⑤ $f : \quad \to \mathbb{C}$

⑤ $f: \quad \longrightarrow \mathbb{C}$

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$

$z = 0$ is a sing.

Solutions of $\sin\left(\frac{1}{z}\right) = 0$ are sing.

$\Updownarrow$

$z \in \left\{ \frac{1}{n\pi} : n \in \mathbb{Z} \setminus \{0\} \right\}$.

All the singularities

A singularity z_0 of f is said to be ISOLATED if f is holomorphic on some deleted nbd of z_0.

$\exists \delta > 0$ s.t. f is holo. on $B_\delta(z_0) \setminus \{z_0\}$.

<u>Remark.</u> If the set of sing. is finite, then each sing. is isolated.

Classification of isolated singularities

Let $f : \Omega \to \mathbb{C}$.

① Removable singularity.

$z_0 \in \mathbb{C}$ is said to be a rem. sing. if $\exists c \in \mathbb{C}$ s.t. the function

$$\boxed{\begin{gathered} g : \Omega \cup \{z_0\} \to \mathbb{C} \\ g(z) = \begin{cases} c \; ; & z \neq z_0 \\ f(z) \; ; & z \neq z_0 \end{cases} \end{gathered}}$$

g is holomorphic on some nbd of z_0.

PRST. z_0 is a rem. sing. of f

iff

$\lim_{z \to z_0} f(z)$ exists. (as a finite complex number)

② Poles

$\underline{\underline{z_0}}$ is said to be a pole of f if
iso. sing.

① $\lim_{z \to z_0} f(z) = \infty$

② $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

③ $\exists m \in \mathbb{N}$ s.t. $\lim (z - z_0)^m f(z)$ exists

$z \to z_0$

AND is non-zero.

Ex. 0 is a pole for f given by $f(z) = 1/z$.

3 Essential sing.

Neither 1 nor 2.

Question 1

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1. Show that there is a strict inequality

New Q. [assume that $n+1 < m$. Prove that $\int = 0$]

$$\left|\int_{|z|=R} \frac{z^n}{z^m - 1}\,dz\right| < \frac{2\pi R^{n+1}}{R^m - 1}; \quad R > 1,\ m \geq 1,\ n \geq 0.$$

$$= (2\pi R) \cdot \frac{R^n}{R^m - 1}$$

Theorem 2: The Stronger ML Inequality

Let $f : \Omega \to \mathbb{C}$ be a continuous function and $\gamma : [a, b] \to \Omega$ be a curve.
Let $M > 0$ be such that

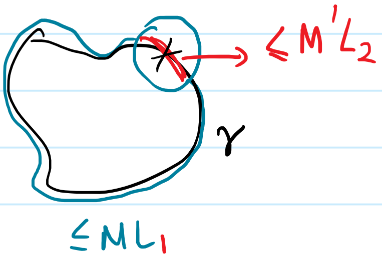
$$M \geq |f(\gamma(t))|, \quad \text{for all } t \in [a, b].$$

If $|f|$ is not constant, then stronger ML is applicable

Also, suppose that $|f(t)| < M$ for some $t \in [a, b]$.
Then,

$$\left|\int_\gamma f(z)\,dz\right| < ML,$$

where L is the length of the curve, as usual.



where $M' < M$

$$\int_\gamma f$$

If $|f| \leq M$ on γ, that is (continuous)

$$|f(\gamma(t))| \leq M \quad \forall\, t \in [a,b]$$

then $$\left|\int_\gamma f\right| \leq M \int_a^b |\gamma'(t)|\,dt = ML.$$

STRONGER.

[Digression to MA101 $(a < b)$: If $f : [a, b] \to [0, \infty)$ is cont. & $\int_a^b f(t)\,dt = 0$, then $f \equiv 0$.]

Now, if $|z| = R$, then

$$\left|\frac{z^n}{z^m-1}\right| = \frac{R^n}{|z^m-1|} \geq \frac{R^n}{||z|^m-1|} = \frac{R^n}{|R^m-1|}$$

$$= \frac{R^n}{R^m-1}$$

Thus, $M = \dfrac{R^n}{R^m-1}$ is a candidate.

$z^m = -R^m$

Now, take $z = R\exp\left(\frac{i\pi}{m}\right)$, then

$$\left|\frac{z^n}{z^m-1}\right| = \frac{R^n}{|z^m-1|} = \frac{R^n}{|-R^m-1|} = \frac{R^n}{R^m+1} < \frac{R^n}{R^m-1}.$$

Thus, Strong ML applies.

$$\Rightarrow \int_{|z|=R} \frac{z^n}{z^m-1}\,dz < \frac{R^n}{R^m-1}\cdot(2\pi R) = \frac{2\pi R^{n+1}}{R^m-1}.$$

Question 2

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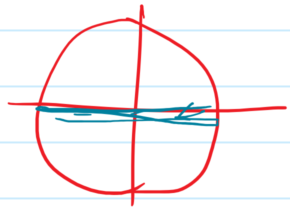
$\rightarrow D = $ disc of conv.

2. A power series with center at the origin and positive radius of convergence, has a sum $f(z)$. If it is known that $f(\bar{z}) = \overline{f(z)}$ for all values of z within the disc of convergence, what conclusions can you draw about the power series?

$f(z) = \sum a_n z^n$?

$$\overline{\left(\sum_{n=0}^{\infty} a_n z^n\right)} = \sum_{n=0}^{\infty} a_n (\bar{z})^n.$$



Claim. $a_n \in \mathbb{R}$ for each $n \in \mathbb{N} \cup \{0\}$.

Note that $a_n = \dfrac{f^{(n)}(0)}{n!}$ for $n \geq 0$.

Note: if $x \in D \cap \mathbb{R}$, then

$$f(x) \underset{x \in \mathbb{R}}{=} f(\bar{x}) \underset{\text{given}}{=} \overline{f(x)}.$$

Thus, $f(x) \in \mathbb{R}$.

Claim 1. $f'(x_0)$ is real for all $x_0 \in D \cap \mathbb{R}$.

Note that we know f' exists. Thus, we may compute it however.

$$f'(x_0) = \lim_{\substack{z \to x_0 \\ z \in \mathbb{R} \cap D}} \frac{f(z) - f(x_0)}{z - x_0}$$

$$= \lim_{\substack{x \to x_0 \\ x \in D \cap \mathbb{R}}} \frac{f(x) - f(x_0)}{x - x_0}$$

real

real

$f'(x_0) \in \mathbb{R}$.

Since $x_0 \in D \cap \mathbb{R}$ was arbit, $f'(x)$ is real for all $x \in D \cap \mathbb{R}$.

<u>Claim 2</u> $f''(x_0)$ is real for all $x_0 \in \mathbb{R} \cap D$.

$\vdots$

Induction!

<u>Claim n.</u> $f^{(n)}(x_0)$ —— " ——.

Thus, $a_n = \frac{1}{n!} f^{(n)}(0) \in \mathbb{R}$ for all $n \geq 0$

$(\because 0 \in D \cap \mathbb{R})$ $\boxed{\smile}$

Replace the condition as: f(x) \in \Bbb R whenever x is real. Conclude that f(z*) = (f(z))*.

Question 3

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3. This is called Taylor series with remainder:

$$\underbrace{f(z) = f(0) + zf'(0) + \cdots + \frac{z^N}{N!} f^{(N)}(z)(0)}+ \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N f^{(N+1)}(tz)\mathrm{d}t$$

Use this to prove the following inequalities:

(a) $\left| e^z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| \leq \frac{|z|^{N+1}}{(N+1)!}; \Re z \leq 0.$ $\quad |\exp(z)| = \exp(\mathcal{R} z)$

(b) $\left| \cos(z) - \sum_{n=0}^{N} (-1)^n \frac{z^{2n}}{(2n)!} \right| \leq \frac{|z|^{2N+2} \cosh R}{(2N+2)!}; |\Im z| \leq R.$

(b) If $f(z) = \cos(z)$

Note that $f^{(2N+1)}(0) = \pm \sin^{(2N+1)}(0)$
$= 0.$

Thus,

$$\left| \cos(z) - \sum_{n=0}^{N} (-1)^n \frac{z^{2n}}{(2n)!} \right|$$

$$= \left| \frac{z^{2N+2}}{(2N+2)!} \int_0^1 (1-t)^{2N+1} f^{(2N+2)}(tz)\, dt \right|$$

Let $I(z) = \int_0^1 (1-t)^{2N+1} f^{(2N+2)}(tz)\, dt$

Note that $f^{(2N+2)} = \begin{cases} \cos \\ -\cos \end{cases}$

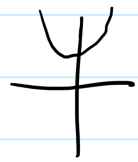
$$
\begin{aligned}
|\cos(z)| &= \frac{1}{2}\left|e^{\iota z} + e^{-\iota z}\right| \\
&\leq \frac{1}{2}\left(\left|e^{\iota z}\right| + \left|e^{-\iota z}\right|\right) \\
&= \frac{1}{2}(e^{y} + e^{-y}) \\
&= \cosh y.
\end{aligned}
$$

$$|f^{(2N+2)}(tz)| = |\cos(tz)| \leq \cosh(\mathcal{I}(tz))$$

$y = \text{Im}\, z$

$$= \cosh(ty)$$

$\cosh y$ is incr. in $|y|$.



Thus, if $t \in [0,1]$, then

$|ty| \leq |y|$, then

$\cosh ty \leq \cosh y \leq \cosh R.$

$$|I(z)| = \left| \int_0^1 (1-t)^{2N+1} f^{(2N+2)}(tz)\, dt \right|$$

$$\leq \int_0^1 \left| (1-t)^{2N+1} f^{(2N+2)}(tz) \right| dt$$

$$\leq \int_0^1 \left| (1-t)^{2N+1} \underbrace{f^{(2N+2)}(tz)} \right| dt$$

$$\leq \int_0^1 \left| (1-t)^{2N+1} \right| \cosh(R)\, dt$$

$$\leq \int_0^1 \cosh(R)\, dt = \cosh(R).$$

Complete!

Question 4

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4. By computing

$$I_1 = \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \mathrm{d}z,$$

show that

$$\int_0^{2\pi} \cos^{2n}\theta \mathrm{d}\theta = \frac{2\pi}{4^n} \cdot \frac{(2n)!}{(n!)^2}.$$

$$I_1 = \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} \, dz$$

$$\hookrightarrow (z - 0)^{2n+1}$$

Solution. Recall the "generalised" Cauchy integral formula[3] which tells us that

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w = \frac{2\pi \iota}{n!} f^{(n)}(z_0)$$

where f is a function which is holomorphic on an open disc $D(z_0, R)$ and $r < R$.

$$\Rightarrow I_1 = \frac{2\pi i}{(2n)!} \left. \frac{d^{2n}}{dz^{2n}} (z^2+1)^{2n} \right|_{z=0}$$

Note that $\displaystyle (z^2+1)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} z^{2r}$

$$\Rightarrow \left. \frac{d^{2n}}{dz^{2n}} (z^2+1)^{2n} \right|_{z=0} = (2n)! \binom{2n}{n}$$

$$\Rightarrow I_1 = (2n)! \binom{2n}{n} \cdot \frac{2\pi i}{(2n)!}$$

$$\left(\ \ n\ \right) \qquad (2n)!$$

$$= (2\pi i) \cdot \binom{2n}{n}.$$

Now, parameterise $|z| = 1$ in the usual way.

$$\gamma(t) = e^{it}, \qquad t \in [0, 2\pi].$$

$$I_1 = \int_0^{2\pi} \left(e^{it} + \frac{1}{e^{it}}\right)^{2n} \underbrace{\frac{1}{e^{it}} \cdot \gamma'(t)}_{i} \, dt$$

$$= i \int_0^{2\pi} (2\cos t)^{2n} \, dt = (2\pi i) \binom{2n}{n}$$

$$\Downarrow$$

$$\int_0^{2\pi} \cos^{2n} t \, dt = \frac{2\pi}{4^n} \cdot \binom{2n}{n}$$

Point : Solve the integral in 2 ways $\rightarrow$ Generalised CIF; $\rightarrow$ Parameterise & solve

Question 5

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5. Locate and classify the singularities of the following:

(a) $\dfrac{z^5 \sin(1/z)}{1+z^4}$, $\dfrac{(z-1)z^2}{z}$ ← formally, 0 is a sing but it is removable

(b) $\dfrac{1}{\sin(1/z)}$, $\dfrac{z^3-6z^2+11z-6}{z-1}$ ← 1 is a rem. sing.

(c) $\dfrac{z^2+z+1}{z^3-11z+13}$. → sing. are roots of $z^3-11z+13$.

↳ check that they are poles

↳ because z^2+z+1 and den. share no factors

(a) $\dfrac{z^5 \sin(1/z)}{1+z^4}$, $= f(z)$

Singularities : $S = \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm i), 0 \right\}$.

↳ these are points at which f is not defined

Note: f is holo on $\mathbb{C} \setminus S$.

- All are isolated. (Why?)

$\delta = 1/100$ works for all

Aliter: S is finite

- Claim. If $\zeta \in S$ is such that $\zeta^4 = -1$, then ζ is a pole

Proof. $\displaystyle \lim_{z \to \zeta} \frac{1}{f(z)} = \lim_{z \to \zeta} \frac{z^4+1}{z^5 \sin(1/z)}$

Note that $\zeta \neq 0$. $\displaystyle = \frac{\zeta^4+1}{\zeta^5 \sin(1/\zeta)} = 0.$ ($\zeta^4 = -1$)

Also, $\sin\left(\frac{1}{\zeta}\right) \neq 0$.

Thus, $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$ are all poles of f.

- Claim. 0 is an essential singularity.

(1) 0 is not a rem. sing.

(*) $\lim_{z\to 0} \frac{z^5 \sin(1/z)}{1+z^4}$ DNE.

$$\sin\left(\frac{1}{z}\right) = \frac{1}{2i}\left(e^{i/z} - e^{-i/z}\right)$$

In (*), let $z \to 0$ along the pos. im. axis.

$$\lim_{y\to 0^+} \frac{(iy)^5 \sin(1/iy)}{1+(iy)^4} = \frac{1\cdot \cancel{i}}{2\cancel{i}} \lim_{y\to 0^+} y^5\left(e^{1/y} - e^{-1/y}\right)$$

$$= \frac{1}{2}\left(\lim_{y\to 0} y^5 e^{1/y}\right)$$

does not exist as a finite complex no.

Thus, 0 is not a rem. sing.

(2) 0 is not a pole.

$$\lim_{\substack{z\to 0\\ z\in\mathbb{R}}} f(z) = \lim_{x\to 0} \frac{x^5 \sin(1/x)}{1+x^4} = 0.$$

Thus, 0 is removable

(b) $\frac{1}{\sin(1/z)}$, Sing: $\{0\} \cup \left\{ \frac{1}{n\pi} : n \in \mathbb{Z}\setminus\{0\} \right\}$

will not classify

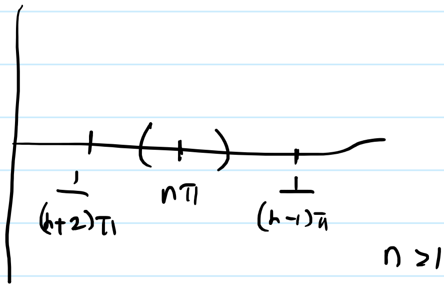
- 0 is not isolated.

Given any $\delta > 0$, $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n\pi} < \delta$.

Thus, $\frac{1}{n\pi} \in B_\delta(0)\setminus\{0\}$.

singularity

- All others are isolated.



$$\delta := \min\left\{ \frac{1}{(n-1)\pi} - \frac{1}{n\pi},\ \frac{1}{n\pi} - \frac{1}{(n+1)\pi} \right\}.$$

works $n > 1$

Similarly, choose δ for $n = 1$, $n < -1$, $n = -1$.

- All others are poles. Why?

Because

$$\lim_{z \to \frac{1}{n\pi}} \frac{1}{f(z)} = \sin(n\pi) = 0. \checkmark$$