Various forms of Cauchy

Recall: If $\gamma : [a, b] \to \mathbb{C}$ is sufficiently nice and f is continuous on γ, then

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

If $\gamma(t) = x(t) + iy(t)$, (x, y are real f^n_s)

then $\gamma'(t) = x'(t) + iy'(t)$.

① "FUNDAMENTAL THEOREM"

If $\Omega \subseteq \mathbb{C}$ is open and $\gamma : [a, b] \to \Omega$ is a curve and $f : \Omega \to \mathbb{C}$ admits a primitive,

nice

$\hookrightarrow$ $\exists F : \Omega \to \mathbb{C}$ s.t. $F' = f$.

then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, then

$$\int_\gamma f = 0.$$

Note that no condition on Ω and interior of γ.

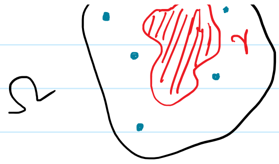
② **Theorem**

Let C be a simple closed contour and let f be a holomorphic function defined on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.

γ is simple if γ is 1-1 on $[a, b)$.

γ is closed if $\gamma(a) = \gamma(b)$.

If $\text{int}(\gamma) \subset \Omega$, then $\int_\gamma f = 0$.



[Note that I don't know, a priori, if f has a primitive.]

③ **Theorem**

(More general form of Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then* $\int_C f(z)dz = 0$

$\Rightarrow$ if γ is in Ω, then so is $\text{int}(\gamma)$.

Not really more general in the sense that (2) $\Rightarrow$ (3).

④ **Theorem**

(More General form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. If γ and γ' are two closed contours in Ω which can be "continuously deformed" into each other, then* $\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$.

⑤ **Theorem (Cauchy Integral Formula)**

Let f be holomorphic everywhere on an open set Ω. Let γ a simple closed curve in Ω (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - z_0}.$$

and $\text{int}(\gamma) \subset \Omega$

anti-clockwise ↺

using this, we derived holo $\Rightarrow$ analytic

For example, take $\Omega = \mathbb{C}\setminus\{0\}$

and $f : \Omega \to \mathbb{C}$ defined as

$$f(z) = \frac{1}{z}.$$

① f is holo on Ω.

② f has no primitive on Ω.

That is, there is no $F : \Omega \to \mathbb{C}$ s.t. $F' = f$.

$F' = f$.

Proof. $\displaystyle\int_{|z-1|=0} \frac{1}{z}\,dz = 2\pi i \neq 0$

$\displaystyle \overset{"}{=} \int_0^{2\pi} \frac{1}{(\cos t + i\sin t)} (-\sin t + i\cos t)\,dt = i\int_0^{2\pi} 1\,dt = 2\pi i.$

IF $\exists F : \Omega \to \mathbb{C}$ s.t $F' = f$, THEN

by Thm 1, $\displaystyle\int_{|z-1|=0} f = 0.$

But we know $\displaystyle\int_{|z-1|=0} f = 2\pi i \neq 0$. Thus, f admits no primitive.

Extra stuff

Won't make sense without context

$\Omega = \mathbb{C} \setminus \{0\} \longrightarrow$ hole

$f : \Omega \to \mathbb{C}$ be $z \mapsto \frac{1}{z^2}$.

Laurent

$F(z) = -\frac{1}{z}$ is a primitiver on Ω.

$\displaystyle\sum_{n=-k}^{\infty} a_n (z-z_0)^n$

$-n, \dots -2, -1, \quad 0, \dots$

$(\sum a_n z^n)\ (\sum b_n z^n)$

conv at $z = z_0$ div.

$\| $

$\sum \left(a_0 b_0 + (a_0 b_1 + b_0 a_1) z + \dots\right)$

$\displaystyle\sum^{\infty} n^n z^n \to$ conv iff

$$\sum_{n=1}^{\infty} n^n z^n \longrightarrow \text{conv. iff } z = 0.$$

$$\alpha = \limsup_{n\to\infty} (n^n)^{1/n} = \limsup_{n\to\infty} n = \infty$$

$$\boxed{R = 0}$$

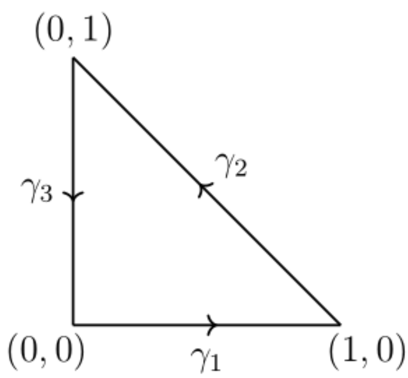
Question 1

08 September 2020 08:51 AM

1. Let γ be the boundary of the triangle

$$\{0 < y < 1 - x; 0 \le x \le 1\}$$

taken with the anticlockwise orientation.



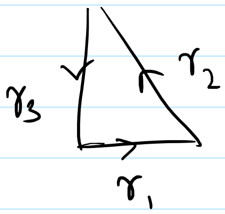
Evaluate:

(a) $\displaystyle\int_\gamma \Re(z)dz$

(b) $\displaystyle\int_\gamma z^2 \mathrm{d}z. = 0$ since z^2 admits a primitive on $\mathbb{C}$ and γ is closed. Aliter: Thm (2)

(a) Note $\Re(z)$ is not holo.
None of the theorems will help us :(

Brute Calculation! :(



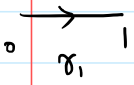
Along γ_3: $\displaystyle\int_{\gamma_3} = 0$ (why?)

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$

Along γ_1:

① Parameterise.

$\gamma_1(t) = t + 0i$ for $t \in [0,1]$.



① Parameterise.

$$\gamma_1(t) = t + 0i \quad \text{for } t \in [0,1].$$

(Orientation matches)

② Solve

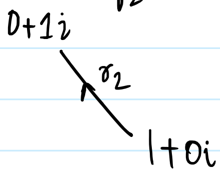
$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Here, $\gamma_1'(t) = 1 + 0i$

Thus,
$$\int_{\gamma_1} R(z)\,dz = \int_0^1 R(\gamma_1(t))\,\gamma_1'(t)\,dt$$

$$= \int_0^1 R(t+0i)\,(1+0i)\,dt$$

$$= \int_0^1 t\,dt = \boxed{1/2}.$$



γ_2: ① Param.

$$\gamma_2(t) = (1-t) + it \quad \text{for } t \in [0,1]$$

$$\gamma_2'(t) = -1 + i$$

② Solve

$$\int_{\gamma_2} R(z)\,dz = \int_0^1 (1-t)\,\underline{(-1+i)}\,dt$$

$$= (-1+i)\left\{1 - \frac{1}{2}\right\} = \frac{1}{2}(i-1)$$

Thus, $$\int_{\gamma} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$

$$= \frac{1}{2} + \frac{1}{2}(i-1) + 0$$

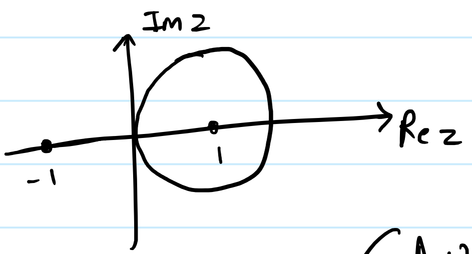
$$= i/2$$

Question 2

08 September 2020 08:51 AM

2. Compute $\displaystyle\int_{|z-1|=1} \frac{2z-1}{z^2-1}\mathrm{d}z.$

Remark (my own): If nothing is specified, we assume that the integral is in the counterclockwise sense.



CAUCHY INTEGRAL FORMULA!

Thought process

If I take $f(z) = \frac{2z-1}{z^2-1}$, can I apply Theorem (2)?

defined on $\mathbb{C}\setminus\{-1,1\}$.

$\mathrm{Int}(\gamma)$ contains 1.

Thus, Thm (2) is NOT applicable. :-(

So, now what?

What about $f(z) = \frac{2z-1}{z+1}$. quotient of holo. f^n.

defined on $\mathbb{C}\setminus\{-1\}$.

Now, we can use Thm (5) which is CIF with $z_0 = 1$.

γ is a param. of $|z-1|=1$

$$f(1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-1}\,dz = \frac{1}{2\pi i}\underbrace{\int_\gamma \frac{2z+1}{z^2-1}\,dz}_{\text{what we want}}$$

$|z-1|=1$

γ

γ

what we want

$$\Rightarrow \int_\gamma (\ )\,dz = 2\pi i\, f(1) = 2\pi i \left(\frac{2-1}{1+1}\right) = \pi i$$

Theorem (Cauchy Integral Formula)

Let f be holomorphic everywhere on an open set Ω. Let γ a simple closed curve in Ω (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi \imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

Question 3

08 September 2020 08:51 AM

3. ① Show that if γ is a simple closed curve traced counterclockwise, the integral $\int_\gamma \bar{z}\,dz$ equals $2\imath\,\mathrm{Area}(\gamma)$.

② Evaluate $\int_\gamma \bar{z}^m\,dz$ over a circle γ centered at the origin.

① $\gamma : [a, b] \longrightarrow \mathbb{C}$

$\gamma(t) = x(t) + iy(t)$ for real valued f^n $x, y : [a,b] \to \mathbb{R}$.

$\gamma'(t) = x'(t) + iy'(t)$

$$\int_\gamma \bar{z}\,dz = \int_a^b \overline{\gamma(t)}\,\gamma'(t)\,dt$$

$$= \int_a^b \big(x(t) - iy(t)\big)\big(x'(t) + iy'(t)\big)\,dt$$

$$= \int_a^b x \cdot x' + y \cdot y' \quad + i \int_a^b \big[x(t)y'(t) - y(t)x'(t)\big]\,dt$$

$$= \int_\gamma x\,dx + y\,dy \quad + \quad i \int_\gamma x\,dy - y\,dx$$

Green's.

$$\int_\gamma (M\,dx + N\,dy) = \iint_{\mathrm{Int}(\gamma)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d(x, y)$$

if γ is a (nice-enough) closed curve oriented counterclockwise. (Here is where we have used orientation.)

$$= \iint_{\mathrm{int}(\gamma)} (0 - 0)\,d(x,y) \quad + i \iint_{\mathrm{Int}(\gamma)} \big[1 - (-1)\big]\,d(x,y)$$

$$= 2i \iint_{\mathrm{Int}(\gamma)} 1 \, d(x,y)$$

$$= \boxed{2i \, \mathrm{Area}(\gamma)}$$

② $\int_\gamma \bar{z}^m \, dz$ γ is a circle around 0.

Let $r > 0$ be its rad.

$(m \in \mathbb{Z})$

Then, $\gamma(t) = r(\cos t + i \sin t)$ $t \in [0, 2\pi]$

$$\gamma'(t) = i \gamma(t)$$

Thus,

$$\int_\gamma \bar{z}^m \, dz = \int_0^{2\pi} \left(\overline{\gamma(t)}\right)^m (i \gamma(t)) \, dt$$

$$= i \int_0^{2\pi} \left(\overline{\gamma(t)}\right)^{m-1} \underbrace{\overline{\gamma(t)} \gamma(t)}_{|\gamma(t)|^2 = r^2} \, dt$$

$$= r^2 i \int_0^{2\pi} r^{m-1} \left[\cos((m-1)t) + 2 \sin((m-1)t)\right] dt$$

$$= r^{m+1} i \int_0^{2\pi} \left[\cos((m-1)t) + 2 \sin((m-1)t)\right] dt$$

$m \neq 1$

$$\left[\sin(m-1)t - i \cos(m-1)t\right]^{2\pi} = 0$$

$$\left[\frac{\sin(m-1)t}{(m-1)} - \frac{i\cos(m-1)t}{m-1}\right]_0^{2\pi} = 0$$

Thus, $\displaystyle\int_\gamma \bar{z}^m \, dz = 0$ if $m \neq 1$.

If $m=1$:

$$\int_\gamma \bar{z}\, dz = r^2 i \int_0^{2\pi} \left[(\cos 0t) + i\sin(0t)\right] dt$$

$$= 2\pi r^2 i$$

$$= 2i\,(\pi r^2) = 2i \text{ Area}(\gamma).$$

Question 4

08 September 2020 08:51 AM

4. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Re(z) > 0\}$ be the (strict) open right half plane. Construct a non-constant function f which is holomorphic on $\mathbb{H}$ such that $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$.
Note that the coloured part is my addition.

$$f(z) = \prod_{n \in \mathbb{N}} \left(z - \frac{1}{n}\right) \longrightarrow \text{convergence?}$$

take $z = 10$

$$f(10) = (10-1)\left(10 - \frac{1}{2}\right)\left(10 - \frac{1}{3}\right)\cdots$$

all are ≥ 9

infinite product diverges!

Define $f : \mathbb{H} \to \mathbb{C}$ as

$$f(z) = \sin\left(\frac{\pi}{z}\right).$$

Note that $0 \notin \mathbb{H}$. Thus, f is well-defined AND holomorphic?

$\frac{\pi}{z}$ is holo. on $\mathbb{C} \setminus \{0\}$

$\sin\left(\frac{\pi}{z}\right)$ is composite of holo. f^n's

Thus, f is a holo f^n.

Is f non-constant?

Yes, $f(2) = \sin\left(\frac{\pi}{2}\right) = 1$

$$\neq$$

$$f(1) = \sin(\pi) = 0$$

Finally, is $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$?

Yes. $\sin\left(\frac{\pi}{(1/n)}\right) = \sin(n\pi) = 0 \quad \forall n \in \mathbb{N}$.

Question 5

08 September 2020 08:51 AM

5. Let f be a holomorphic function on $\mathbb{C}$ such that $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$. Show that f is constant.

Difference here is in the domain.
Earlier $0 \notin \mathbb{H}$ but here, $0 \in \mathbb{C}$.

domains

A way to show that f is constant is to show that the zeroes of f are not discrete.

Here, we already know that f is zero on

$$\{z \in \mathbb{C} : f(z) = 0\} = Z(f) \supseteq \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

This IS discrete.

However, using the above zeroes, I can construct a new zero of f!

Note that $0 = \lim_{n \to \infty} \frac{1}{n}$ $\quad$ f is continuous

$$\Rightarrow f(0) = \lim_{n \to \infty} \left[f\left(\frac{1}{n}\right) \right]$$

This made sense because $0 \in \mathrm{dom}(f)$.

$$= \lim_{n \to \infty} 0$$

$$= 0.$$

Thus, we have a found a new zero.

Thus, $\underbrace{\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}}_{=: S} \subseteq Z(f)$.

Note that S is not discrete!

Then, we can conclude that

$$f \equiv 0.$$

Proof.

Definition 3.1 (Discrete Set). A set $S \subset \Omega$ is said to be *discrete* if for every $s \in S$, there exists some $\delta > 0$ such that

$$B_\delta(s) \cap S = \{s\}.$$

In other words, for every $s \in S$, there exists some $\delta > 0$ such that the δ neighbourhood of s contains no other point of S.

Consider $0 \in S$. Then, given any $\delta > 0$

$$B_\delta(0) \cap S \neq \{0\},$$

it contains another point of S.

$\frac{1}{n} < \delta$

$\frac{1}{\delta} < n$

For example: $1/(\lfloor 1/\delta \rfloor + 1) \in S \cap B_\delta(0)$.

Thus, S is not discrete.

Question 6

08 September 2020 08:51 AM

6. Expand $\frac{1+z}{1+2z^2}$ into a power series around 0. Find the radius of convergence.

Power series expansion around a point is unique!

How do you expand $\frac{1}{1+2z^2}$ as a pow. series?

Geometric series

$$\frac{1}{1+(2z^2)} = 1 - (2z^2) + (2z^2)^2 - (2z^2)^3 + \cdots \quad (*)$$

This has radius of convergence (RoC) = ?

This conv. if $|2z^2| < 1$ or $|z| < \frac{1}{\sqrt{2}}$.

Is this enough to conclude that RoC $= 1/\sqrt{2}$?

No.

But we do know that $(*)$ diverges if $|z| > 1/\sqrt{2}$.

Thus, we may conclude RoC $= \frac{1}{\sqrt{2}}$.

Thus, the PS expansion of $\frac{1}{1+2z^2}$ around 0 has RoC $= 1/\sqrt{2}$

Thus,

$$\frac{1+z}{1+2z^2} = (1+z)\left(1-(2z^2)+(2z^2)^2+\cdots\right)$$
$$= 1 + z - 2z^2 - 2z^3 + 4z^4 + 4z^5 + \cdots$$

also has RoC $= 1/\sqrt{2}$.

also has RoC $= 1/\sqrt{2}$.

Does not conv. at -1.

Proof. Use root test.

Look at the partial sums

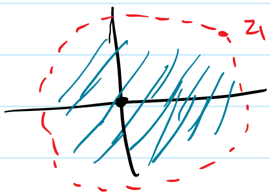
$1, 0, -2, 0, 4, 0, -8, 0, \ldots \not\to 0$

Thm. If $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_1 \neq 0$,

then it converges

ABSOLUTELY for

all z s.t. $|z| < |z_1|$.



bit.ly/ca-205

link to notes of MA 412

$\mathbb{C}$ Analysis