# Tutorial 2 - Recap

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Given a sequence $(a_n)$ of complex numbers, we get a sequence

$$S_n := \sum_{k=1}^{n} a_k .$$

We say that $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} S_n$ exists.

Otherwise, we say that it diverges.

in this case

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n .$$

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Given a sequence $(x_n)$ of reals, define

$$y_n := \sup \{ x_m : m \geq n \}.$$

$$x_1 \quad x_2 \quad \cdots \quad \underbrace{x_n \quad x_{n+1} \quad \cdots}_{\sup \to y_n}$$

Then, $(y_n)$ is a mono. decr$^n$ sequence and thus,

$\lim_{n\to\infty} y_n$ exists. (one or more $y_n$ can be $\infty$. The limit can also be $\infty$.)

We write $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} y_n$.

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**Thm.** Given a series of the form

$$\text{(PS)} \qquad \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $z_0 \in \mathbb{C}$ and $a_n \in \mathbb{C}$ $\forall n \in \mathbb{N} \cup \{0\}$,

there exists $R \in [0, \infty]$ s.t.

- (PS) converges for all $z$ s.t. $|z - z_0| < R$
- (PS) diverges for all $z$ s.t. $|z - z_0| > R$

Nothing is said for those $z$ s.t. $|z - z_0| = R$.

**Thm.** (Calculation of $R$)

Let $(a_n)$ be as above.

Define $\alpha := \limsup_{n \to \infty} \sqrt[n]{|a_n|}$.

Remark: lim sup of any real seq. always exists.

$(0 \leq \alpha \leq \infty)$

$\alpha = 0$ $\hookrightarrow R = \infty$

$\alpha = \infty$ $\hookrightarrow R = 0$

Then, $R = 1/\alpha$. ($R$ is the rad. of conv. as before.)

**Thm.** Let $(a_n)$ be as above.

Then, IF $\alpha := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists,

THEN $R = 1/\alpha$.

Remark. This limit $\alpha$ need not always exist.

**Remark.** (General) If $\lim_{n\to\infty} x_n$ exists, then

((xn) is a real seq)

$$\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

This can also be applied to the root test above.

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Note. In general, the equality

$$\limsup_{n\to\infty} (a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

IS NOT TRUE.

$a_n = (-1)^n$, $b_n = -a_n$. $\limsup a_n = 1 = \limsup b_n$

but $a_n + b_n \equiv 0$.

# Question 1

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1. If $u(X,Y)$ and $v(X,Y)$ are harmonic conjugates of each other, show that they are constant functions.
   Remark (my own): This is true iff $u$ and $v$ are defined on domains, that is, open and path-connected sets.

$v$ is a harmonic conjugate of $u$:

$$\underline{u_x = v_y} \quad \text{and} \quad \underbrace{u_y = -v_x}_{(*)}.$$

$u$ is a harmonic conjugate of $v$:

$$\underbrace{v_x = u_y}_{(*)} \quad \text{and} \quad \underline{v_y = -u_x}.$$

Red eq$^n$'s : $u_x = -u_x = v_y = -v_y$.

$$\therefore u_x = 0 = v_y.$$

(*) Blue eq$^n$'s: $v_x = -v_x = u_y = -u_y$

$$\therefore v_x = 0 = u_y.$$

Since $u_x \equiv 0 \equiv u_y$, $u$ is constant.

path-connected

Similarly $v$ is constant.

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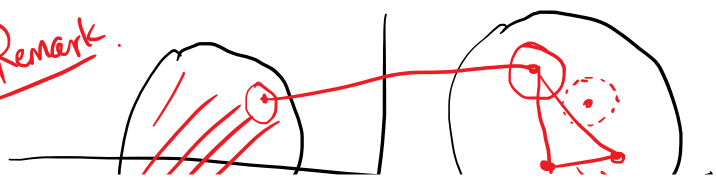
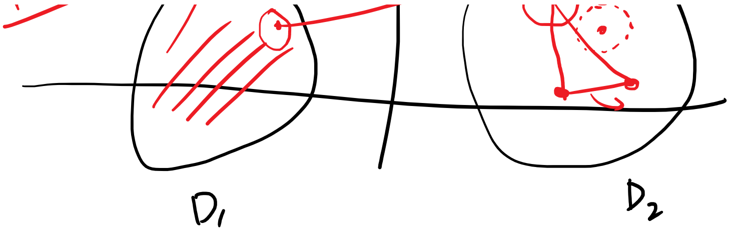
Remark.



Let $\Omega$ be the union of two disjoint open discs.




of two disjoint open discs.

Define $u : \Omega \to \mathbb{R}$ as

$$u(x,y) = \begin{cases} 1 ; & (x,y) \in D_1, \\ 2 ; & (x,y) \in D_2 . \end{cases}$$

Check: $u_x = u_y \equiv 0$.

But $u$ is not - constant.

**Ex.** Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined as $f(x,y) = \log(x^2 + y^2)$.

Show that: ① $f$ is harmonic,

② $f$ has no harmonic conj. on $\mathbb{R}^2 \setminus \{(0,0)\}$.

↳ IS path-conn. but not simply.

$$\left( \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right)$$

↳ this is not the grad of anything.

# Question 2

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2. Show that $u = XY - 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

Smart Way. Consider $f: \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = z^3 + \frac{1}{2} z^2.$$

Then, $u = \mathrm{Im} \circ f$.

Thus, $u$ is harmonic.

Moreover, $-\mathrm{Re} \circ f$ is a harmonic conjugate of $u$.

$$f(x,y) = x^3 - 3xy^2 + \frac{1}{2}(x^2 - y^2) + i(u(x,y))$$

Thus, $v = x^3 - 3xy^2 + \frac{1}{2}(x^2 - y^2)$

is a harmonic conjugate of $u$.

(It happened to work nicely here.)

# Question 3

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3. Find the radius of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} nz^n$,

(b) $\sum_{p \text{ prime}} z^p$,

(c) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

In each part, $a_n$ denotes the co-eff of $z^n$.

(a) Note that $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ itself exists.

(Root)

$$\left( \lim_{n\to\infty} n^{1/n} = 1 \right)$$

Thus, $\alpha = \limsup_{n\to\infty} n^{1/n} = 1$ and hence,

$$R = \alpha^{-1} = 1^{-1} = 1.$$

(b) $a_n = \begin{cases} 0 \;; & \text{if } n \text{ is not prime} \\ 1 \;; & \text{if } n \text{ is prime} \end{cases}$

Given any $n \in \mathbb{N}$, $\exists\, m \in \mathbb{N}$, $m \geq n$ s.t. $m$ is prime.

Thus, $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left( \sup \{ a_m : m \geq n \} \right)$

$= \lim_{n\to\infty} \left( \sup \{0, 1\} \right)$

$= \lim_{n\to\infty} (1) = 1.$

Here we HAD to look at lim sup. lim DNE.

Thus, $\alpha = 1$ and $R = \alpha^{-1} = 1$.

(c) (Ratio) Here, we have

$$a_n = \frac{n!}{n^n}. \quad \text{Thus,}$$

Note reciprocal

$$\frac{a_n}{a_{n+1}} = \frac{n!}{(n+1)!} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{\cancel{n+1}} \cdot \cancel{(n+1)} \cdot \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$\text{Thus,} \quad \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \Bigg[\underbrace{\left(1 + \frac{1}{n}\right)^n}_{= e}\Bigg]^{-1} = e^{-1}$$

Thus, $\alpha = e^{-1}$ and $\boxed{R = e}$.

Remark. Here the limit happened to exist. Thus, we could use the test.

NO LIMSUP TEST WITH RATIO FOR POW. SERIES!

# Question 4

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4. Show that $\boxed{L > 1}$ in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

this was $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Fact: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Thus, $\frac{1}{1^2} + \frac{2}{1^2} + \frac{1}{2^2} + \frac{2}{2^2} + \frac{1}{3^2} + \frac{2}{3^2} + \cdots$ converges.

$$\underset{a_1}{\frac{1}{1^2}} \quad \underset{a_2}{\frac{2}{1^2}} \quad \underset{a_3}{\frac{1}{2^2}} \quad \underset{a_4}{\frac{2}{2^2}} \quad \cdots$$

$$a_{2n-1} = \frac{1}{n^2} \quad ; \quad n \geq 1$$

$$a_{2n} = \frac{2}{n^2} \quad ; \quad n \geq 1$$

If $(x_n)$ is a seq. & $(y_n)$ is a subseq., then $\limsup x_n \geq \limsup y_n$

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq \limsup_{2n-1\to\infty} \left|\frac{a_{2n}}{a_{2n-1}}\right| = 2.$$

Thus, $L \geq 2 > 1$.

Hence. $l \to 1$ but the series

Hence, $L > 1$ but the series still converges.

# Question 5

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$f$ is non-zero means that $f$ is not identically zero.

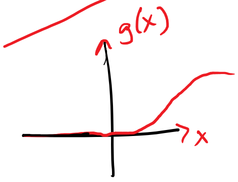
$\Updownarrow$

$f$ is non zero at SOME $x \in \mathbb{R}$.

5. ① Construct a infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. ② Show that there are no holomorphic functions which satisfy this property.

① Recall $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) := \begin{cases} e^{-1/x} ; & x > 0, \\ 0 ; & x \le 0. \end{cases}$$




We saw that $g$ was inf. diff. (but not analytic)

Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = g(x)\, g(1-x).$$

Then, ① $f(x) = 0$ if $x \le 0$. $(g(x))$

② $f(x) = 0$ if $x \ge 1$ $(g(1-x))$

Thus, $f$ is zero outside the bounded set $[0, 1]$. ← I don't care what happens here

Moreover, $f$ is non-zero since

$$f(1/2) = (g(1/2))^2 = e^{-4} \neq 0.$$

Also, $f$ is inf. diff. (Why?)

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② Claim. If $f : \mathbb{C} \to \mathbb{C}$ is holo. & vanishes outside a bounded set, then $f \equiv 0$.

($f$ is identically 0.)

Proof: We will show that the zeroes of $f$ do NOT form a discrete set.

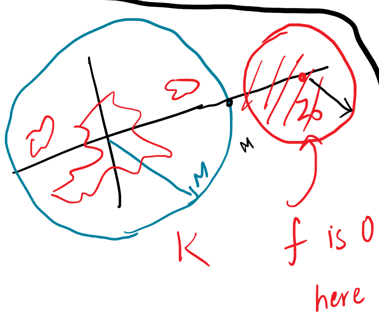
By hyp., we know there exists a bounded subset $K \subset \mathbb{C}$ s.t.

$$f(z) = 0 \quad \text{if} \quad z \notin K.$$

By def$^n$, $\exists M > 0$ s.t.

(Def$^n$ of being bounded.)

$$|z| \leq M \quad \text{for} \quad z \in K.$$




Let $z_0 := M + 43$ and take $\delta = 42$.

Then, $B_{42}(z_0) \cap B_M(0) = \emptyset$.

Thus, $f(z) = 0$ for all $z \in \underbrace{B_{42}(z_0)}$.

$\boxed{\text{not discrete}}$

$\boxed{\text{Let } f : \Omega \to \mathbb{C} \text{ be analytic where } \Omega \text{ is a domain.}}$

Theorem: Either f is identically zero or the zeroes of f form a discrete set.

(A or B. If $\neg B$, then A.)

$\boxed{\text{Thus, } f \text{ is identically zero.}}$

(Since holomorphic functions are analytic.)

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$\mathbb{R}$ is not a discrete subset of $\mathbb{C}$.

Q8) Thus, if $f : \mathbb{C} \to \mathbb{C}$ vanishes on $\mathbb{R}$ and is analytic, then $f \equiv 0$.

In particular, consider $f(z) = \cos^2 z + \sin^2 z - 1$.

We know that $f(z) = 0$ if $z \in \mathbb{R}$.

Thus, $f(z) = 0 \quad \forall z \in \mathbb{C}$.

Thus, $\cos^2 z + \sin^2 z = 1 \quad \forall z \in \mathbb{C}$.

# Question 6

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6. Show that $\exp : \mathbb{C} \to \mathbb{C}^{\times}$ is onto.

Let $z_0 \in \mathbb{C}^{\times}$. We wish to show that $\exists\, z \in \mathbb{C}$ s.t. $\exp(z) = z_0$.

Since $z_0 \in \mathbb{C}^{\times}$, $z_0 \neq 0$. Thus,

$$r_0 := |z_0| \neq 0.$$

Thus, $w_0 := \dfrac{z_0}{r_0}$ is well-defined.

Moreover, $|w_0| = \dfrac{|z_0|}{|r_0|} = \dfrac{|z_0|}{|z_0|} = 1.$

Thus, $w_0 = x_0 + iy_0$ for some $(x_0, y_0) \in \mathbb{R}^2$ satisfying $x_0^2 + y_0^2 = 1$.

Thus, $\exists\, \theta \in [0, 2\pi)$ s.t. $x_0 = \cos\theta$ and $y_0 = \sin\theta$.

Now, define $z := \log(r_0) + i\theta$.

$\hookrightarrow$ is the usual $\log : \mathbb{R}^{+} \to \mathbb{R}$.

Then, we have $\exp(z) = \exp(\log r_0 + i\theta)$

$$= \exp(\log r_0) \cdot \exp(i\theta)$$

$$= r_0 \cdot (\cos\theta + i\sin\theta) = r_0 w_0$$

$$= z_0 .$$ ☒

# Question 7

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7. Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and cosine which were bounded by $1$).

**Sin**

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Let $z_0 \in \mathbb{C}$. [We want to show: $\exists z \in \mathbb{C}$ s.t. $\sin z = z_0$]

Consider the quadratic

$$\frac{(t - 1/t)}{2i} = z_0 \quad (*)$$

$$\Updownarrow$$

$$t^2 - (2iz_0)t - 1 = 0.$$

Note: Quadratic equations always have a root. [Ex. Prove with FTA.]

Let $t_1$ be a root.

Clearly, $t_1 \neq 0$. $\therefore t_1 \in \mathbb{C}^\times$

Thus, by (Q 6), $\exists z \in \mathbb{C}$ s.t. $e^z = t_1$

Consider $z' = z/i \in \mathbb{C}$.

Then, $e^{iz'} = t_1$. Put this back in the quadratic. (*)

Thus,

$$\frac{e^{iz'} - (e^{iz'})^{-1}}{2i} = z_0.$$

$$\Downarrow$$

$$\sin(z') = z_0.$$

# Question 8

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8. Show that for any complex number $z$, $\sin^2(z) + \cos^2(z) = 1$.

① Brutely compute.

$$(\sin z)^{} = \frac{e^{iz} - e^{-iz}}{2i}$$

$$(\cos z)^{} = \frac{e^{iz} + e^{-iz}}{2}.$$

② Look at remark at the end of Q5.