Tutorial 1

07 August 2021 14:20

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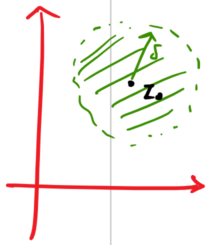
Lecture 1

Definition 1 (Some notation)

Given $z_0 \in \mathbb{C}$ and $\delta > 0$, the δ-neighbourhood of z_0, denoted by $B_\delta(z_0)$ is the set

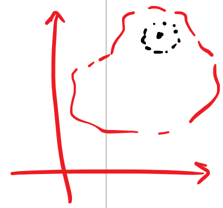
$$B_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$



Definition 2 (Open sets)

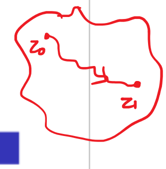
A set $U \subset \mathbb{C}$ is said to be open if:
for *every* $z_0 \in \mathbb{C}$, there exists *some* $\delta > 0$ such that

$$B_\delta(z_0) \subset U.$$



Definition 3 (Path-connected sets)

A set $P \subset \mathbb{C}$ is said to be path-connected if any two points in P can be joined by a path in P. (A continuous function from $[0, 1]$ to P.)



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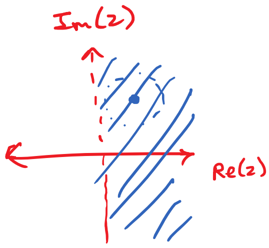
Examples.

① $B_\delta(z_0)$ are open for any $z_0 \in \mathbb{C}$ and $\delta > 0$.

② $\mathbb{C}$ is open. $\emptyset$ is open.

③ Strict right half plane $\mathbb{H}$ is open
$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.



④ $\{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$ is NOT open.

Lecture 1

Definition 4 (Differentiable)

Let $\Omega \subset \mathbb{C}$ be open. Let

$$f : \Omega \to \mathbb{C}$$

be a function. Let $z_0 \in \Omega$. f is said to be *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is denoted by $f'(z_0)$.

$f : (a, b) \to \mathbb{R}$

$f'(x_0) := \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}.$

$\Omega = \mathbb{C}$, $\quad z, z^2, \quad z^n, \ldots$

exp, sin, cos, ... ?

Non-diff: $\quad |z|, \quad \bar{z}, \ldots$

Lecture 1

Definition 5 (Holomorphic)

1. A function f is said to be holomorphic on an open set Ω if it is differentiable at every $z_0 \in \Omega$.
2. A function f is said to be holomorphic at z_0 if it is holomorphic on some neighbourhood of z_0.

Remark 1

A function may be differentiable at z_0 but not holomorphic at z_0. For example, $f(z) = |z|^2$ is differentiable only at 0. Thus, it is differentiable at 0 but holomorphic nowhere.

For sets, however, there is no difference.

Points: Holo. $\Rightarrow$ Diff but $\not\Leftarrow$

Notation

From this point on, Ω be always denote an open subset of $\mathbb{C}$.
Whenever I write some complex number z as $z = x + \iota y$, it will be assumed that $x, y \in \mathbb{R}$.
Similarly for $f(z) = u(z) + \iota v(z)$.

Lecture 2: CR Equations

$$\left[\begin{array}{ccc} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 \\ x+\iota y & \longleftrightarrow & (x, y) \end{array}\right]$$

Let $f : \Omega \to \mathbb{C}$ be a function. We can decompose f as

$$f(z) = \underline{u(z)} + \iota v\underline{(z)},$$

where $u, v : \underset{\subseteq \mathbb{C} \cong \mathbb{R}^2}{\Omega} \to \mathbb{R}$ are real valued functions.

The idea now is to consider u and v as functions of two variables. We can do so by simply considering $u(x, y) = u(x + \iota y)$ and similarly for v. Now, if we know that f is holomorphic, then we have the following result.

$$\left[u, v : \Omega^{\subset \mathbb{R}^2} \longrightarrow \mathbb{R} \right] \quad \underline{\text{MA 109, 111}}$$

u_x, u_y, v_x, v_y make sense.

Lecture 2: CR Equations

Cauchy – Riemann

Theorem 1 (CR equations)

Let $f : \Omega \to \mathbb{C}$ be differentiable at a point $z_0 \in \Omega$. Let $z_0 = x_0 + \iota y_0$.
Then, we have

$$① \quad u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, we have

$$② \quad f'(z_0) = u_x(x_0, y_0) + \iota v_x(x_0, y_0).$$

$f' = u_x + i v_x$

Existence of u_x, u_y, v_x, v_y is part of the theorem.

Note the subscript is x for both in the above.
Also note that all the equalities are only at the point z_0. In particular, we are only assuming differentiability at z_0.

Test $f(z) = z = x + iy$ to see what the signs should be.

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Converse? What is the converse? Is it true?

$\boxed{\text{converse}}$ No!

CR equations hold at the point (x_0, y_0)

$\Downarrow$?

f is differentiable at z_0?

No. The converse is not true.

An example for you to check is

$$f(z) := \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Check that u and v satisfy the CR equations at $(0,0)$ but f is not differentiable at $0 + 0\iota$. (Page 23 of slides.)

Lecture 2: CR Equations

We recall MA ~~105~~ 109 + 111 now.

Definition 6 (Total derivative)

If $f : \Omega \to \mathbb{C}$ is a function, we may view it as a function

$$f : \Omega \overset{\subseteq \mathbb{R}^2}{} \to \mathbb{R}^2.$$

Recall that f is said to be real differentiable at $(x_0, y_0) \in \Omega \subset \mathbb{R}^2$ if there exits a 2×2 real matrix A such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(x_0 + h, y_0 + k) - f(x_0, y_0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

The matrix A was called the *total derivative of f at* (x_0, y_0).

Theorem 2

If f is (complex) differentiable at a point $z_0 = x_0 + \iota y_0$, then f is real differentiable at (x_0, y_0).

Once again, this is only talking about differentiability at a point.
The converse is again not true.
Take the example $f(z) = \bar{z}$. Thus, we have seen two sufficient conditions for complex differentiability so far. Neither is individually sufficient. However, together, they are.

$$\left.\begin{aligned} u(x,y) &= x \\ v(x,y) &= -y \end{aligned}\right\} \quad \begin{aligned} u_x &\equiv 1 \\ v_y &\equiv -1 \end{aligned}$$

Lecture 2: CR Equations

CD $\Rightarrow$ CR + RD

CR $\not\Rightarrow$ CD, RD $\not\Rightarrow$ CD

Theorem 3

Let $f : \Omega \to \mathbb{C}$ be a function and let $z_0 = x_0 + \iota y_0 \in \Omega$. If the CR equations hold at the point (x_0, y_0) and if f is real differentiable at the point (x_0, y_0), then f is complex differentiable at the point z_0.

(CR + RD) $\Rightarrow$ CD

Recall from MA 109, 111:

If $f : \Omega \to \mathbb{R}^2$ is a function s.t.

$f = (u, v)$

u_x, u_y, v_x, v_y are continuous on Ω,

then f is real diff. on Ω.

Lecture 2: CR Equations

Note : if $f : \Omega \to \mathbb{R}^2$, then f_x, f_y, etc. are meaningless.

Definition 7 (Harmonic functions)

Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function. u is said to be *harmonic* if $u_{xx} + u_{yy} = 0$.

Proposition 1

The real and imaginary parts of a holomorphic function are harmonic.

$u_x = v_y \Rightarrow u_{xx} = v_{yx}$ $\quad$ $u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$ $\quad$ but $v_{xy} = v_{yx}$ by assumption of u, v being C^2

Suppose u and v are harmonic on Ω. v is said to be a harmonic conjugate of u if $f = u + \iota v$ is holomorphic on Ω.
If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.
Check the second last slide of this lecture to find the algorithm for finding a harmonic conjugate.

Harmonic Conjugate need not exist.

Example. Consider $\Omega = \mathbb{R}^2 - \{(0, 0)\}$ and $u : \Omega \to \mathbb{R}$ defined as

$$u(x, y) = \frac{1}{2}\log(x^2 + y^2).$$

If u had a harmonic conjugate v, then

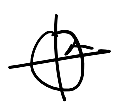
$$v_y(x, y) = \frac{x}{x^2+y^2} \quad \text{and} \quad v_x(x, y) = -\frac{y}{x^2+y^2}.$$

But $\nexists\, v : \Omega \to \mathbb{R}$ s.t.

$$\nabla v = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

(Proof?)

$\oint$



$\emptyset \qquad \emptyset$

Claim 1. Arbitrary union of open sets is open.

Proof. Let $\{U_i : i \in I\}$ be a collection of open sets.

Define $U := \bigcup_{i \in I} U_i$

$$= \{x : x \in U_i \text{ for some } i \in I\}.$$

TS: U is open.

Proof. Let $x \in U$ be arbitrary.

Then, $\exists i_0 \in U$ s.t. $x \in U_{i_0}$.

Since U_{i_0} is open, $\exists \delta > 0$ s.t.

$$B_\delta(x) \subseteq U_{i_0}.$$

But $U_{i_0} \subseteq U$. Thus, $B_\delta(x) \subseteq U$.

Thus, U is open. ∎

Claim 2. Finite intersection of open sets is open.

Proof. It suffices to prove that intersection of two open sets is open.

$A_1, \dots, A_n \to$ open

$\Downarrow$

$(A_1 \cap A_2), A_3, \dots, A_n \to$ open

$\Downarrow$

$A_1 \cap A_2 \cap A_3, \dots, A_n \to$ open

$\Downarrow$

$\vdots$

$\Downarrow$

$A_1 \cap \dots \cap A_n \to$ open

Let U_1 and U_2 be open and $x \in U_1 \cap U_2$.

Let U_1 and U_2 be open and $x \in U_1 \cap U_2$.

$\because U_1 \& U_2$ are open $\begin{cases} \exists \delta_1 > 0 \text{ s.t. } B_{\delta_1}(x) \subseteq U_1 \text{ and} \\ \exists \delta_2 > 0 \text{ s.t. } B_{\delta_2}(x) \subseteq U_2. \end{cases}$

Pick $\delta := \min(\delta_1, \delta_2) > 0$.

Then, $B_\delta(x) \subseteq B_{\delta_1}(x) \subseteq U_1$ and

$B_\delta(x) \subseteq B_{\delta_2}(x) \subseteq U_2$.

$\therefore B_\delta(x) \subseteq (U_1 \cap U_2)$. $\blacksquare$

"Dual" statements for closed sets.

$U_1, U_2 \to$ open You can say: $U_1 \cup U_2$ and $U_1 \cap U_2$ are open

$U_1, U_2, \dots, U_n \to$ open $\Rightarrow$ $U_1 \cup \dots \cup U_n$ & $U_1 \cap \dots \cap U_n$ are open.

$U_1, U_2, U_3, \dots \to$ open $\Rightarrow$ $\bigcup_{i=1}^{\infty} U_i$ is open but $\bigcap_{i=1}^{\infty} U_i$ may not be.

$$\mathbb{C} - \left(\bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (\mathbb{C} - U_i)$$

closed $\Leftrightarrow$ complement is open.

$\boxed{U_i := B_{1/i}(0).}$

$\bigcap_{i \in \mathbb{N}} U_i = \{0\}$ $\uparrow$ not open.

Tutorial 2

14 August 2021 11:55

Lecture 3: Power Series

Definition 8 (Convergence of series)

A series of the form

$(a_n)_{n\geq 0} \to$ sequence in $\mathbb{C}$

$$\to \sum_{n=0}^{\infty} a_n$$

of complex numbers is said to converge if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges (to a finite complex number).

The sequence of partial sums is just the following sequence:

$$\underline{a_0}, \underline{a_0 + a_1}, \underline{a_0 + a_1 + a_2}, \ldots .$$

"Divergent" is simply used to mean "not convergent."
Check that $\sum(-1)^n$ and $\sum n$ both diverge.

Similarly

$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ diverges

$1+2+3+4+\cdots$ diverges

$a_n = (-1)^n$ for $n\geq 0$

$1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$

partial sums $1, 0, 1, 0, 1, 0, 1, 0 \ldots$

↳ Diverges

Lecture 3: Power Series

Definition 9 (limsup)

Given a sequence (x_n) of real numbers, we may define a new sequence (y_n) as

$$y_n = \sup\{x_m : m \geq n\}.$$

The limit of this sequence always exists and we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

$x_0, x_1, x_2, x_3, x_4, \ldots$

sup $= y_4$

↳ sup $= y_3$

↳ sup $= y_2$

↳ sup $= y_1$

Remark 2

Each y_n might be ∞. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.

Each y_n might be ∞. That is allowed.
The limsup might be $\pm\infty$. This is also allowed.
If $\lim_{n\to\infty} x_n$ itself exists, then it equals the lim sup as well.

I know: $\lim_{n\to\infty} n^{1/n} = 1$.

Thus, $\limsup_{n\to\infty} n^{1/n} = 1$.

$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

$\Downarrow$

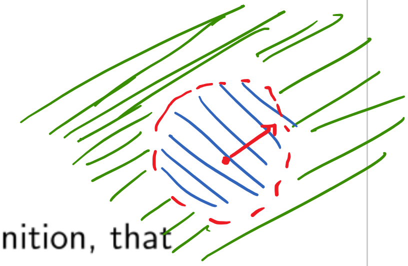
$\limsup_{n\to\infty} \frac{\sin(n)}{n} = 0$

lim sup of a real sequence always exists!

Lecture 3: Power Series

We will be interested in discussing radius of convergence of *power series*. We all know what that is. It is a series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n (z - z_0)^n} \qquad (*)$$



where $z_0 \in \mathbb{C}$ and each $a_n \in \mathbb{C}$.

What is the radius of convergence, though? (The definition, that is.)

Theorem 4 (Radius of convergence)

Given any power series as $(*)$, there exists $R \in [0, \infty]$ such that
1. $(*)$ converges for any z with $|z - z_0| < R$, and → absolutely
2. $(*)$ diverges for any z with $|z - z_0| > R$.

This R is called the radius of convergence.

→ on the circle, behaviour is weird.
↳ may converge for some
↳ may diverge for others

Note the brackets.

$\sum_{n=1}^{\infty} \frac{z^n}{n} \rightsquigarrow$ RoC $= 1$
at -1 : converges
at 1 : diverges

$\sum z^n \rightsquigarrow$ RoC $= 1$
↳ diverges for all z with $|z| = 1$

Lecture 3: Power Series

We would now like to be able to calculate the radius of convergence.

Theorem 5 (Root test)

Let $(*)$ be as earlier. Define

$$\boxed{\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.}$$

ALWAYS WORKS.

Then, $\boxed{R = \alpha^{-1}}$ is the radius of convergence.

This test *always works.* We had no assumptions of any kind on $(*)$.
Note that $^{-1}$.
If $\alpha = 0$, then $R = \infty$ and vice-versa.

$\sum_{n=1}^{\infty} \frac{z^n}{n}$. $\alpha = \limsup \left(\frac{1}{n}\right)^{1/n} \overset{?}{=}$

We know $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \frac{1}{1} = 1$. $\therefore \limsup \left(\frac{1}{n}\right)^{1/n} = 1$

Note: limit rules of $+$, $\cdot$, $(\ )^{-1}$ do not apply to limsup.

$\limsup (a_n + b_n) \le \limsup (a_n) + \limsup (b_n)$

Lecture 3: Power Series

We have another test. This is simpler (to calculate) but mightn't always work.

Theorem 6 (Ratio test)

Let $(*)$ be as earlier.
Assume that the limit

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$\rightarrow$ can apply to $\sum \frac{z^n}{n}$ or $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

exists. (Possibly as ∞.)
Then, R is the radius of convergence.

$$\frac{\left(\frac{1}{n!}\right)}{\frac{1}{(n+1)!}} = n+1 \longrightarrow \infty.$$

Note that here we assume that the limit does exist. This may not always be true.
Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Note that I'm not taking any inverse here but also note the way the ratio is taken. We have a_n/a_{n+1}.

Take $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Then, $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$.

R.O.C = 1

Converges at 1 does not converge at 1.

Lecture 3: Power Series

Differentiability of power series is what one should expect.

Theorem 7 (Differentiability)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. On the open disc of radius R, let $f(z)$ denote this sum. Then, on this disc, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Boundary: NO COMMENTS.

Note that this is again a power series with the same radius of convergence. Thus, we may repeat the process indefinitely. In other words, power series are infinite differentiable.

$\limsup |a_n|^{1/n} = \limsup n^{1/n} |a_n|^{1/n}$

Digression. Let $(x_n)_{n \geq 0}$ be a real sequence.

Consider

$E := \{\text{limits of all possible convergent subsequences}\} \subseteq \mathbb{R} \cup \{\pm\infty\}$.

Then, $\limsup_{n \to \infty} x_n = \sup E$.

Lecture 4: Exponential function

I shall just recall the facts from the lecture.

Definition 10 (Exponential function)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

by ratio test

converges on all of $\mathbb{C}$. This sum is denoted by $\exp(z)$.

Theorem 8 (Facts)

1. $\exp'(z) = \exp(z)$,
2. $\exp'(bz) = b\exp(bz)$, for $b \in \mathbb{C}$,
3. $\exp(z) \cdot \exp(-z) = 1$ for all $z \in \mathbb{C}$,
4. $\exp(z)$ is always nonzero.

Lecture 4: Exponential function

Now, we some "converse" facts.

Theorem 9 (Characterisations)

1. If $f'(z) = bf(z)$, then $f(z) = a\exp(bz)$ for some $a, b \in \mathbb{C}$,
2. If $f' = f$ and $f(0) = 1$, then $f(z) = \exp(z)$.

Theorem 10 (Final fact)

Let $z, w \in \mathbb{C}$, then

$$\exp(z + w) = \exp(z) \cdot \exp(w).$$

$\exp : \underset{+}{\mathbb{C}} \longrightarrow \underset{\cdot}{\mathbb{C}^\times}$ is a group homomorphism.

$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

Lecture 4: Exponential function

Definition 11 (Domain)

A subset $\Omega \subset \mathbb{C}$ is said to be a *domain* if it is open and path-connected.

We had one very nice result on the zeroes of a analytic functions.

Theorem 11 (Zeroes are isolated)

Let Ω be a domain and $f : \Omega \to \mathbb{C}$ be a non-constant analytic function. Let $z_0 \in \Omega$ be such that $f(z_0) = 0$. Then, there exists $\delta > 0$ such that f has no other zero in $B_\delta(z_0)$.

no

The above is saying that around every zero of f, we can draw a (sufficiently small) circle such that f has no other zero in that disc. This is the same as saying that the set of zeroes is *discrete*.

Logarithm

We discuss logarithm a bit.

Definition 14 (Branch of the logarithm)

Let $\Omega \subset \mathbb{C}$ be a domain. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\exp(f(z)) = z, \qquad \text{for all } z \in \Omega.$$

Then, f is called a *branch of the logarithm.*

Theorem 21 (Uniqueness of branches)

Assume that $f, g : \Omega \to \mathbb{C}$ are two branches of the logarithm. Then, $f - g$ is a constant function. Moreover, this constant is an integer multiple of $2\pi\iota$.

The last theorem also assumed that Ω is a domain.

Branch of log may not exit on a given domain.

no branch on $\mathbb{C}$. Also, there is no branch on $\mathbb{C}^\times$.

Tutorial 3

20 August 2021 22:02

Lecture 5: Integration

Definition 12

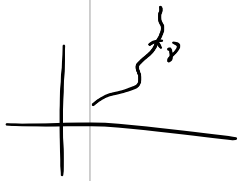
Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Writing $f = u + \iota v$ as usual, we define

$\in \mathbb{R}$

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + \iota \int_a^b v(t)\mathrm{d}t.$$

This is naturally what one would have wanted to define. Now, we define integration over a *contour*. (What is a contour?)

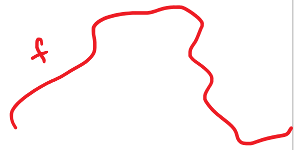
↳ continuous and piece wise diff.



Definition 13

Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \to \Omega$ be a contour. We define

$$\int_\gamma f(z)\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$



Lecture 5: Integration

We have a useful inequality called the ML inequality.

Theorem 12 (ML Inequality)

length

Let γ be a contour of length L and f be a continuous function defined on the image of γ.
Suppose that

$$\underline{|f(\gamma(t))| \leq M}, \qquad \text{for all } t \in [a, b].$$

Then, we have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq ML.$$

"Real analogue": If $f : [a,b] \to \mathbb{R}$ is bounded by M, then $\int_a^b f \leq M(b-a)$.

Lecture 5: Integration

"FTC"

Theorem 13 (Primitives and integrals)

Suppose $f : \Omega \to \mathbb{C}$ has a *primitive* on Ω. That is, there exists a function $F : \Omega \to \mathbb{C}$ such that $F' = f$. (The complex derivative.)
hol.
Then, we have

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

If γ is *closed*, that is, if $\gamma(b) = \gamma(a)$, then

$\gamma(a) = \gamma(b)$

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Primitive on Ω $\Downarrow$ Integral over closed curves in Ω is 0.

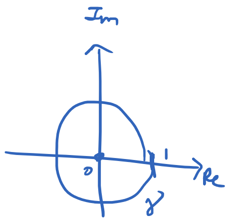
Note: Ω need not contain interior of γ. Thus, we can conclude $\int_\gamma \frac{1}{z^2} dz = 0$ over the unit circle.

Existence of a primitive is a strong condition, by the way. A holomorphic function need not have a primitive on all of Ω.

Consider $\Omega = \mathbb{C} - \{0\}$.

Let $f : \Omega \to \mathbb{C}$ be $f(z) := \frac{1}{z}$.

Then, f has no primitive on Ω.



Proof. Let $\gamma(t) := e^{2\pi i t}$, $t \in [0, 1]$.

Then, $\int_\gamma f = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 (e^{-2\pi i t})(2\pi i)(e^{2\pi i t})dt$

$= (2\pi i) \int_0^1 dt = 2\pi i \neq 0$

Lecture 5: Integration

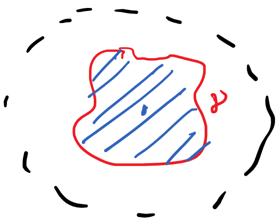
Now, we come to Cauchy's theorem.

Theorem 14 (Cauchy's Theorem)

Let γ be a simple, closed contour and let f be a holomorphic function defined on an open set Ω containing γ as well as its interior. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

If Ω is simply-connected, then the interior condition is automatically met. This gives us the next result.



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Lecture 5: Integration

Theorem 15 ("General" Cauchy Theorem)

Let Ω be a simply-connected domain. Let $\gamma : [a, b] \to \mathbb{C}$ be a simple, closed contour and $f : \Omega \to \mathbb{C}$ holomorphic. Then,

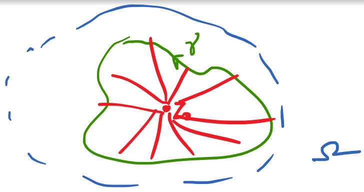
$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Lecture 6: CIF and Consequences

Theorem 16 (Cauchy Integral Formula)

Let f be holomorphic everywhere on an open set Ω. Let γ be a simple closed curve in Ω, oriented positively. If z_0 is interior to γ and Ω contains the interior of γ, then

$$f(z_0) = \frac{1}{2\pi \iota} \int_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z.$$



"mean value"

Lecture 6: CIF and Consequences

We then saw a consequence of CIF which I state as a theorem below.

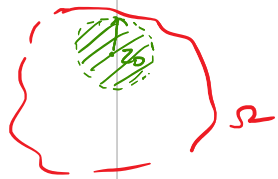
Theorem 17 (Holomorphic $\implies$ Analytic)

Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Pick any $z_0 \in \Omega$. Let $R > 0$ be the largest such that $B_R(z_0) \subset \Omega$. (The case $R = \infty$ is allowed. That just means $\Omega = \mathbb{C}$.) Then, on the disc $B_R(z_0)$, we may write $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

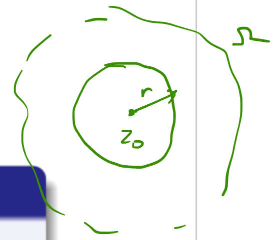
where each a_n is given by

$$a_n = \frac{1}{2\pi \iota} \int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w.$$



Lecture 6: CIF and Consequences

The above also gives us (what I call) the "generalised" Cauchy Integral Formula.



Theorem 18 ("Generalised" CIF)

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0),$$

where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

Remark 3

Note that, as usual, we require f to be holomorphic within the circle as well.

Just keep the "Generalised CIF" in mind, in case all these various theorems are too confusing! It will let you derive everything else quite simply!

In fact, the simplest thing is Cauchy's residue theorem which is the best generalisation of all these results, which we'll see later in the course and everything else becomes a very direct corollary of it.

Lecture 7: CIF and Consequences

Theorem 19 (Cauchy's estimate)

Suppose that f is holomorphic on $|z - z_0| < R$ and bounded by $M > 0$ on this disc. Then,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

An easy application of this give us:

Theorem 20 (Liouville's Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then f is constant!

Logarithm

The previous theorem from last tutorial talked about uniqueness of branches (up to a constant) assuming the existence of such a branch. Now, we see when a branch is actually possible.

Short : Simply connected domain without 0 has a log.

Theorem 22 (Existence of a branch)

Let Ω be a simply-connected domain in $\mathbb{C}$. Assume that $1 \in \Omega$ and $0 \notin \Omega$.
There exists a unique function $F : \Omega \to \mathbb{C}$ such that

1. $F(1) = 0$,
2. $F'(z) = 1/z$,
3. $\exp(F(z)) = z$ for all $z \in \Omega$,
4. $F(r) = \log(r)$ for all $r \in \Omega \cap \mathbb{R}^+$.

F is a branch of log

The log in the last point is the usual log for real numbers as seen in ~~105~~ 109/111. The above F is then denoted by log.

$F(r) = \log(r)$ for all $r \in \Omega \cap \mathbb{R}^+$.

The log in the last point is the usual log for real numbers as seen in ~~105~~. The above F is then denoted by log.

109/111

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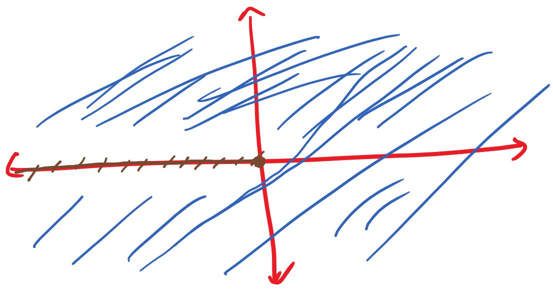
Special example: $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$= \mathbb{C} \setminus (-\infty, 0]$



Tutorial 5

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Lecture 8: Singularities

Definition 15 (Singularities)

Let $f : \Omega \to \mathbb{C}$ be a function. A point $z_0 \in \mathbb{C}$ is said to be a singularity of f if

1. $z_0 \notin \Omega$, i.e., f is not defined at z_0, or
2. $z_0 \in \Omega$ and f is not holomorphic at z_0.

$\frac{\sin(z)}{z}$ at 0 Ω

"bad"

Definition 16 (Isolated singularity)

A singularity $z_0 \in \mathbb{C}$ is said to be *isolated* if there exists *some* $\delta > 0$ such that f is holomorphic on $B_\delta(z_0) \setminus \{z_0\}$.

no other singularity

The above is saying that "f is holomorphic on some *punctured disc* around z_0."

Compare this "isolation" with what we saw earlier when we said that "zeroes are isolated."

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Lecture 8: Singularities

Definition 17 (Non-isolated singularity)

A singularity which is not an isolated singularity is called a non-isolated singularity.

The floor is made of floor.

Note that if f has only finitely many singularities, then all the singularities are isolated.

We classify isolated singularities into three types: ← only one of interest

1. Removable singularities,
2. Poles,
3. Essential singularities.

will see an example today of non-isolated.

$\frac{1}{\sin(\frac{1}{z})}$ singularities

$\{0\} \cup \{\pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \ldots\}$

Remark 4

The above classification is only for isolated singularities.

Lecture 8: Singularities

Definition 18 (Removable singularity)

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable.

$f(z_0) :=$ (re)define

These are characterised by the following theorem.

(RRST)

Theorem 23 (Riemann's Removable Singularity Theorem)

An isolated singularity

z_0 is a removable singularity of f iff $\lim_{z \to z_0} f(z)$ exists.

Moreover, in such a case, (re)defining $f(z_0)$ as this limit makes f holomorphic.

In the above, we mean that it exists as a (finite) complex number.

$$f(z) = \frac{\sin z}{z}$$

defined on $\mathbb{C} \setminus \{0\}$ has 0 as a removable singularity.

Lecture 8: Singularities

Definition 19 (Pole)

An isolated singularity z_0 is said to be a pole if $|f(z)| \to \infty$ as $z \to z_0$. (No matter how you approach it.)

0 is not a pole for $f(z) = e^{1/z}$. $z \to 0$ along negative real axis.

Theorem 24

An isolated singularity z_0 is a pole of f iff $\boxed{\lim_{z \to z_0} \frac{1}{f(z)} = 0.}$ ☆

Theorem 25 (Order of a pole)

If z_0 is a pole of f, then there exists an integer $m > 0$ such that

$$f(z) = (z - z_0)^{-m} f_1(z)$$

on a punctured neighbourhood of z_0, for some function f_1 which is holomorphic on the complete neighbourhood. The smallest such integer m is called the *order* of the pole.
If the order is 1, then z_0 is said to be *simple* pole.

$$f(z) = \frac{f_1(z)}{(z-z_0)^m}$$

$$= \frac{f_1(z)(z-z_0)}{(z-z_0)^{m+1}}$$

$$f(z) = \frac{f_1(z)}{z-z_0}.$$

Lecture 8: Singularities

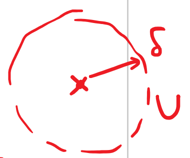
Definition 20 (Essential singularity)

An isolated singularity is called an essential singularity if it is neither a removable singularity nor a pole.

Theorem 26 (Casorati-Weierstrass Theorem)

If z_0 is an isolated singularity, then it is essential iff the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

z_0 : $\exists \delta > 0$



$f(U)$ is DENSE in $\mathbb{C}$.

Recall: $A \subseteq \mathbb{C}$ is DENSE in $\mathbb{C}$ if for every $w \in \mathbb{C}$, $\exists$ a sequence $(a_n)_n$ in A s.t. $a_n \to w$.

$(a_n)_n$ in A s.t. $a_n \to w$.

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Thus, by looking at lim_{z → z_0} f(z) and lim_{z → z_0} 1/f(z), we can completely deduce

TEST for an isolated singularity z_0:

→ ① Compute $\lim_{z \to z_0} f(z)$.

If this is a finite complex number, then → removable.

→ ② Else: compute $\lim_{z \to z_0} \frac{1}{f(z)}$.

If this is 0, then → pole.

→ ③ Else: essential.

Theorem 27 (Modified CIF)

Suppose that z_0 is an isolated singularity of f. Consider an annulus of the form

$$A = \{z : r < |z - z_0| < R\},$$

where $0 \leq r < R \leq \infty$. Assume that f is holomorphic on this open annulus A. Then, CIF takes the form

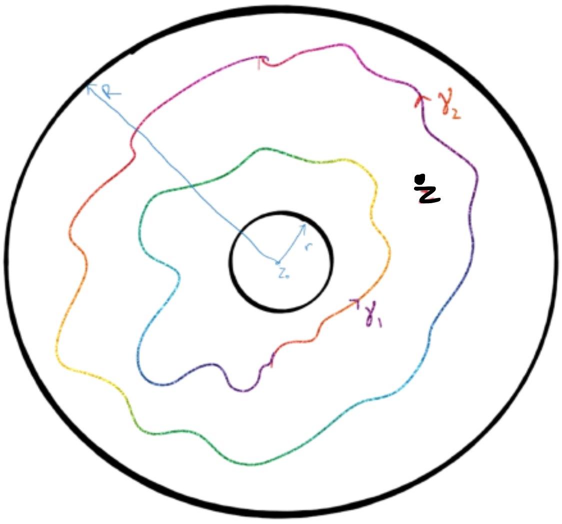
$$f(z) = \frac{1}{2\pi\iota}\int_{|w-z_0|=R'} \frac{f(w)}{w-z}\mathrm{d}w - \frac{1}{2\pi\iota}\int_{|w-z_0|=r'} \frac{f(w)}{w-z}\mathrm{d}w,$$

where $r < r' < |z| < R' < R$.

Just like how the usual CIF gave us the power series, this CIF gives us the Laurent series.

Lecture 9: Laurent Series

Allowing deformations and assuming $0 < r < R < \infty$, here's the general picture to keep in mind:



Assume f is holo on A.

A

$\boxed{\text{Pick } z \in A.}$

Pick γ_1 and γ_2 as shown.

$$f(z) = \frac{1}{2\pi\iota}\left(\int_{\gamma_2} - \int_{\gamma_1}\right)\frac{f(w)}{w-z}\mathrm{d}w.$$

Lecture 9: Laurent Series

Theorem 28 (Laurent Series)

With the same setup as earlier, for $z \in A$, we can write $f(z)$ as

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,}$$

is NOT assumed to be an iso. sing.

where each a_n is given, as before, by

$$a_n = \frac{1}{2\pi\iota} \int_{|w-z_0|=r_0} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w,$$

where $r < r_0 < R$.

$n=-1$

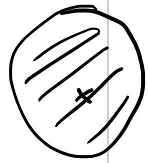
$$a_{-1} = \frac{1}{2\pi i} \int_{|w-z_0|=r_0} f(w)\,dw$$

Note that the above is valid for $n < 0$ as well.

Lecture 9: Laurent Series

Definition 21 (Laurent series expansion at z_0)

If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $\{z : 0 < |z - z_0| < r\}$ for some $r > 0$. The Laurent series expansion on this annulus is called the Laurent series expansion **at** z_0.



Definition 22 (Principal part)

Let $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be the Laurent series expansion *at* z_0. Its principal part is

$$\boxed{\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.}$$

The most interesting coefficient of the principal part is the -1^{st} one. When we integrate a Laurent series along a circle centered at z_0 (which contains no other singularity), only a_{-1} remains (with a factor of $2\pi\iota$). This is given by

$$a_{-1} = \frac{1}{2\pi\iota}\int_{|z-z_0|=r_0} f(w)\mathrm{d}w.$$

This is what is usually called the *residue* and written as

$$\boxed{a_{-1} = \operatorname{Res}(f; z_0).}$$

Lecture 9: Laurent Series

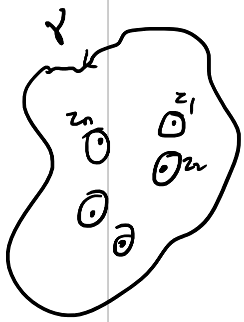
With residues, calculation of integrals becomes easier.

Theorem 29 (Cauchy's Residue Theorem)

Suppose f is given and has finitely many singularities $z_1, \ldots, z_n$ within a simple closed contour γ.Then, we have

oriented positively

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\iota\sum_{i=1}^{n}\mathrm{Res}(f; z_i).$$



Note that the above is implicitly implying that f is holomorphic at all other points within γ.

Lecture 9: Laurent Series

Recall that given an isolated singularity, we can expand the function as a Laurent series *around* that point on a punctured neighbourhood. We had defined the principal part of this series to be the part containing the negative powers of $z - z_0$. We now see how they are related to the nature of the isolated singularity.

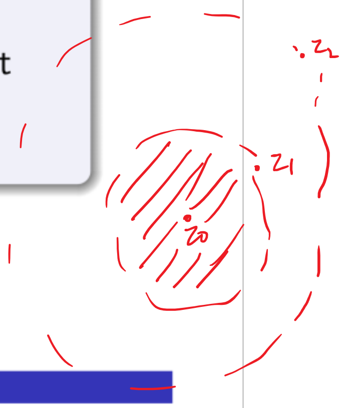
$$\sum_{n=-\infty}^{-1} a_n (z-z_0)^n$$

Theorem 30 (Isolated singularities and their principal parts)

The isolated singularity z_0 is

By "term", we mean "nonzero" term.

1. removable iff the principal part has no terms,
2. a pole iff the principal part has finitely many (and at least one) terms, and
3. essential iff the principal part has infinitely many terms.



In particular, the residue at a removable singularity is 0.

$a_{-1} = 0$

Lecture 9: Laurent Series

Now, we see how one can calculate residue at a pole.
By the previous theorem, we know that f can be written as

$$f(z) = \boxed{\frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots ,}$$

for some integer $m > 0$. $\quad a_{-m} \neq 0, \quad m \geq 1$

Thus,

$$g(z) = (z - z_0)^m f(z) = a_{-m} + \cdots + \boxed{a_{-1}(z-z_0)^{m-1}} + a_0(z-z_0)^m + \cdots$$

is holomorphic at z_0 (after redefining; note that z_0 is a removable singularity for g) and

$$\boxed{a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(z_0).}$$