Q1.

07 September 2021 19:45

1. Evaluate $I = \int_0^{2\pi} \frac{\cos^2(3x)}{5 - 4\cos(2x)}\, dx.$

$$I = \int_0^{2\pi} \frac{\cos^2(3\theta)}{5 - 4\cos(2\theta)}\, d\theta$$

$$= \int_0^{2\pi} \frac{\frac{1}{4}\left(e^{3i\theta} + e^{-3i\theta}\right)^2}{5 - 2\left(e^{2i\theta} + e^{-2i\theta}\right)}\, d\theta$$

$z = e^{i\theta}$

"$dz = i e^{i\theta} d\theta$"

$$= \int_{|z|=1} \frac{\frac{1}{4}\left(z^3 + 1/z^3\right)^2}{5 - 2\left(z^2 + 1/z^2\right)} \frac{1}{iz}\, dz$$

Use CRT and finish it!

$$I = \frac{1}{4} \int_{|z|=1} \frac{\frac{1}{z^6}\left(z^6 + 1\right)^2}{\frac{1}{z^2}\left\{5z^2 - 2(z^4+1)\right\}} \frac{1}{iz}\, dz$$

$$= \frac{1}{4i} \int_{|z|=1} \frac{1}{z^5} \frac{\left(z^6 + 1\right)^2}{(-2)\left\{z^4 - \frac{5z^2}{2} + 1\right\}}\, dz$$

$$= -\frac{1}{8i} \int_{|z|=1} \frac{\left(z^6 + 1\right)^2}{z^5 \underbrace{\left(z^2 - 2\right)\left(z^2 - 1/2\right)}_{= f(z)}}\, dz$$

Use CRT and finish it.

$$\overset{\|}{f(z)}$$

↳ use CRT and finish it.

Step 1. Poles?

$$z = 0, \quad \pm\sqrt{2}, \quad \pm \frac{1}{\sqrt{2}}.$$

Step 1.5. Which of them are inside?

$$z = 0, \pm 1/\sqrt{2}.$$

Step 2. Residues.

- $\pm 1/\sqrt{2}$ are simple. Calculate them yourself.

  Check: $\operatorname{Res}(f; \pm 1/\sqrt{2}) = -27/8$.

- $\operatorname{Res}(f; 0) = ?$ One way: Compute $\left.\frac{d^4}{dz^4}\left(z^5 f(z)\right)\right|_{z=0}$.

  ↳ No way! (V. tough)

Way #2: Compute Laurent series directly.

$$\frac{1}{z^5}\left\{\frac{(z^6+1)^2}{1-\left(5z^2/2 - z^4\right)}\right\}$$

$$= \frac{1}{z^5}\left\{ (z^6+1)^2 \underbrace{\left(1 + \left(\frac{5z^2}{2} - z^4\right) + \left(\frac{5z^2}{2} - z^4\right)^2 + \cdots\right)}_{\text{need co-efficient of } z^4} \right\}$$

↳ $(\underset{=}{1} + 2z^6 + z^{12})$

↳ only contributing term

..

Only need co-eff of z^4 in

$$1 + \left(\frac{5z^2}{2} - z^4 \right) + \left(\frac{5z^2}{2} - z^4 \right)^2 + \cdots$$

-1 $\qquad$ $\frac{25}{4}$ $\qquad$ none here

Thus, $\operatorname{Res}(f; 0) = \frac{25}{4} - 1 = \frac{21}{4}$.

Thus, $$I = -\frac{1}{8i} \int_{|z|=1} f = -\frac{2\pi i}{8i}\left(\frac{21}{4} - \frac{27}{8} - \frac{27}{8} \right)$$

$$= -\frac{2\pi i}{8i}\left(-\frac{6}{4} \right)$$

$$= \frac{3\pi}{8}.$$

Q2.

07 September 2021 19:45

2. Evaluate $\displaystyle\int_{|z-2|=4} \frac{2z^3+z^2+4}{z^4+4z^2}\,dz.$ $= 2\pi i \displaystyle\sum_{\substack{\text{sing. within} \\ |z-2|=4}} \text{residue.}$

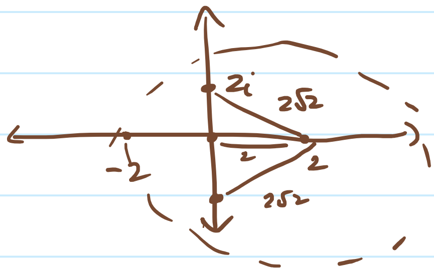
CRT, CRT, CRT, ...

Step 1. (Singularities?) Poles?

$z = \pm 2i,\ 0$

Step 2. Compute residues. (For those which are within the curve.)

all of them



- $\operatorname{Res}(f;\ 2i) = \lim_{z\to 2i} (z-2i)\,f(z)$

$= 1.$

- $\operatorname{Res}(f;\ -2i) = 1.$

- $\operatorname{Res}(f;\ 0) = \ldots\ ?$

Note that 0 is not a simple pole.
It is a pole of order 2.

$f(z) = \frac{1}{z^2} \cdot \frac{2z^3+z^2+4}{(z^2+4)}$

$= \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots$

Thus, $a_{-1} = \frac{d}{dz}\left(z^2 f(z)\right)\Big|_{z=0}$

$z^2 f(z) = \frac{2z^3+z^2+4}{2\ldots}.$

$$\overline{z^2+4}$$

$$\therefore \quad \frac{d}{dz}\left(z\, f(z)\right) = \frac{(6z^2+2z)(z^2+4) - 2z(2z^3+z^2+4)}{(z^2+4)^2}$$

$$\Big\downarrow \text{put } z=0$$

$$Res(f;0) = 0$$

Thus,

$$\int_{|z-2|=4} \left(\quad \right) = 2\pi i\,(1+1+0) = 4\pi i$$

Q3.

07 September 2021 19:45

$f: X \to Y$ (top. spaces) is open if $f(U)$ is open in Y whenever U is open in X.

3. Show with and without the $\underbrace{\text{open mapping theorem}}_{\text{OMT}}$ that if f is a holomorphic function on a domain Ω with $|f|$ is constant, then f is constant.

#1. With OMT.

If $f: \Omega \to \mathbb{C}$ is holomorphic and non-constant, then $f(U)$ is open whenever $U \subseteq \Omega$ is open.
In particular, $f(\Omega)$ is open.

Digression: ① Consider $f: \mathbb{C} \to \mathbb{C}$ to be $f(z) := |z|$.
Then, f is not an open map.
(But it is continuous.)
non constant

② Consider $f: \mathbb{R} \to \mathbb{R}$ to be $f(x) := \dfrac{x}{1+x^2}$,

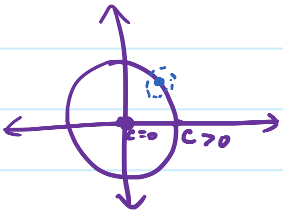
then $f(\mathbb{R})$ is not open.
(Note that f is smooth, even analytic.)

Given: $|f|$ is constant.
To show: f is constant
can change goal to
Can do: Show that $f(\Omega)$ is not open.

$$\left(\begin{array}{l} p + q \Rightarrow r \\ \quad \equiv \\ p + \neg r \Rightarrow \neg q \end{array}\right) \quad \left(\begin{array}{l} \text{Holo.} + \text{nonconstant} \Rightarrow f(\Omega) \text{ open} \\ \text{Thus, holo} + f(\Omega) \text{ not open} \Rightarrow \text{constant} \end{array}\right)$$

But what is $f(\Omega)$? It is a subset of a circle.

To elaborate: $|f|$ is constant, say c.
Then $|f(z)| = c \quad \forall z \in \Omega$



To elaborate: $|f|$ is constant, say c.

Then, $|f(z)| = c \quad \forall z \in \Omega$.

Thus,

$$\phi \neq f(\Omega) \subseteq \{w \in \mathbb{C} : |w| = c\}.$$

But such a circle has no nonempty open subset.

Thus, we are done.

There is no ($\neq \phi$) subset of the circle which is open in $\mathbb{C}$.

#2 Without OMT.

Idea: CR equations.

As before, suppose $|f| \equiv c$.

If $c = 0$, then $f \equiv 0$ and we are done.

Assume $c \neq 0$.

Write $f = u + iv$ as usual.

We know

$$u^2 + v^2 \equiv c^2.$$

$\frac{\partial}{\partial x}$

$$uu_x + vv_x \equiv 0 \quad \text{— (1)}$$

$\partial/\partial y$

$$uu_y + vv_y \equiv 0$$

CR

$$-uv_x + vu_x \equiv 0 \quad \text{— (2)}$$

(1) and (2) give:

$$\begin{bmatrix} u(x,y) & v(\ ) \\ v(x,y) & -u(\ ) \end{bmatrix} \begin{bmatrix} u_x(\ ) \\ v_x(\ ) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This holds for all $(x,y) \in \Omega$.

This holds for all $(x,y) \in \Omega$.

Claim:

This matrix is invertible for all $(x,y) \in \Omega$.

Proof. $\det \begin{bmatrix} u(x,y) & v(x,y) \\ v(x,y) & -u(x,y) \end{bmatrix} = (u^2+v^2)(x,y) = c \neq 0$.

Thus, $\begin{bmatrix} u_x(x,y) \\ v_x(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $(x,y) \in \Omega$.

$\therefore$ $u_x = v_x \equiv 0$. By CR, $u_y = v_y \equiv 0$.

Since Ω is a domain, the above implies that u and v are constant. Thus, f is so. ∎

Q4.

07 September 2021 19:45

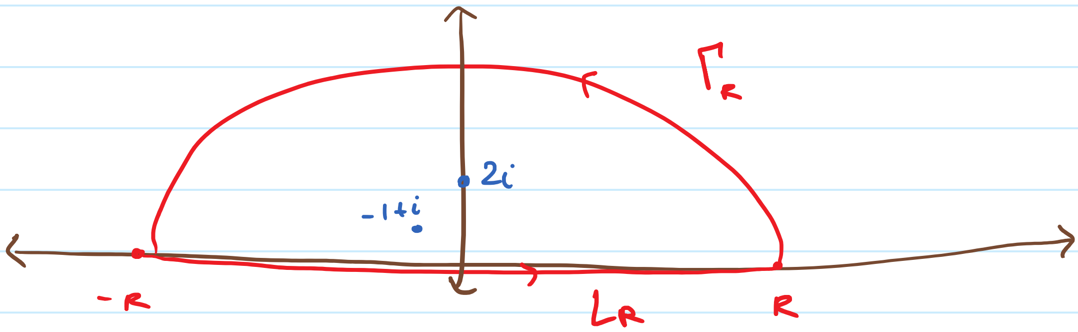
4. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx = -\frac{\pi}{10}$.

Consider f defined as

$$f(z) := \frac{z}{(z^2+2z+2)(z^2+4)}$$

on $\mathbb{C} - \{\text{zeroes of denominator}\}$.

The poles are : $\pm 2i,\ -1 \pm i$.



We wish to compute $\displaystyle\lim_{R\to\infty} \int_{L_R} f$.

By CRT, we know

$$\int_{L_R} f + \int_{\Gamma_R} f = 2\pi i \sum_{\substack{\text{singularities} \\ \text{with the} \\ \text{semicircle}}} (\text{residue}).$$

If $R > 2$, then the sing. in the semicircle are $2i,\ -1+i$.

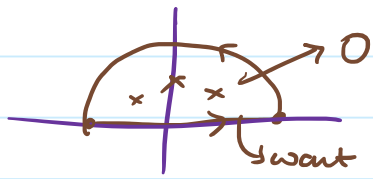
— $\lim_{R\to\infty} \int_{L_R} f$ want.

— Residues, we have. (Can compute.)

— $\lim_{R\to\infty} \int_{\Gamma_R} f$ ← comes down to this.
Can we compute this (limit)?
Yes! It is 0 since
2 + deg (num) ≤ deg(den).

Thus,

$$\lim_{R\to\infty} \int_{L_R} f + 0 = 2\pi i \left(\text{Res}(f; 2i) + \text{Res}(f; -1+i) \right).$$



• Residue at $2i$: Note $2i$ is a simple pole.

Thus, $\text{Res}(f; 2i) = \lim_{z\to 2i} (z-2i) f(z)$

$$= \lim_{z\to 2i} \frac{z}{(z^2+2z+2)(z+2i)}$$

$$= -\frac{1}{20} - \frac{i}{10}.$$

Similarly, $\text{Res}(f; -1+i) = \frac{1}{20} + \frac{3i}{20}$.

Thus, desired integral $= 2\pi i \left(\frac{2i}{20} \right)$

$$= -\frac{\pi}{10}.$$

Q5.

07 September 2021 19:45

5. Compute the number of zeroes of the polynomial $z^5 + z^2 - 6z + 3$ in the annulus $\frac{1}{3} < |z| < 1$ using Rouché's theorem.

Theorem 35 (Rouché's Theorem)

Let $f, g : \Omega \to \mathbb{C}$ be holomorphic. Let γ be closed curve in Ω, such that Ω contains the interior of γ.
Suppose that

(*) $$|f(z) - g(z)| < |f(z)|,$$

for all z on the image of γ.

Then,

$$N_\gamma(f) = N_\gamma(g).$$

number of zeroes within γ (counted with multiplicity)

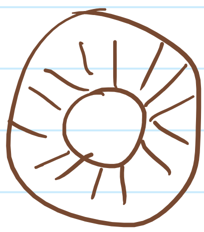
Q1. What about number of zeroes on γ?

Ans. If (*) holds, then neither f nor g can have a zero on γ. (WHY?!)

Here, we wish to wish to count the number of zeroes of

$$g(z) := z^5 + z^2 - 6z + 3$$

on $A := \{z \in \mathbb{C} : \frac{1}{3} < |z| < 1\}$.



Strategy: Count the number of zeroes in $\{z : |z| < 1\}$ and subtract the number of zeroes in $\{z : |z| \leq 1/3\}$.

Note: $A = \{z : |z| < 1\} \setminus \{z : |z| \leq 1/3\}$.

- Within $|z| = 1$:
  Take $f(z) = -6z$. → exactly one root (with multiplicity)

Then, $|f(z)| = 6$ on $|z|=1$.

On the other hand,

$$|f(z) - g(z)| = |z^5 + z^2 + 3|$$
$$\leq |z^5| + |z^2| + |3| = 5.$$

Thus, $|f(z) - g(z)| < |f(z)|$ for $|z|=1$.

Thus, $g(z)$ has 1 root on $\{|z| < 1\}$.

- Within $|z| = 1/3$.

  Take $f(z) = 3$. → 0 zeroes (with mult.)

$$|f(z) - g(z)| \leq |z^5| + |z^2| + |6z|$$
$$= \left(\frac{1}{3}\right)^5 + \left(\frac{1}{3}\right)^2 + 2$$

on $|z| = 1/3$

$$< \frac{1}{2} + \frac{1}{2} + 2 = 3.$$

Thus, $|f(z) - g(z)| < |f(z)|$ for $|z| = 1/3$.

Thus, g has no zeroes on $\{|z| \leq 1/3\}$.

how do we get $\leq$ here?

See first question.

- Thus, g has exactly 1 zero (with multiplicity) on

$$\{z \in \mathbb{C} : \tfrac{1}{3} < |z| < 1\}.$$

☐

Q6.

07 September 2021 19:45

6. ① Show that the function $u(x,y) := \log(x^2+y^2)$ is harmonic on the annulus $1 < |z| < 2$. ② Does u have a harmonic conjugate?

$\to u_{xx} + u_{yy} \equiv 0$

① Let A denote the annulus.
For $(x,y) \in A$, we see that

$$u_x(x,y) = \frac{2x}{(x^2+y^2)}, \text{ and thus,}$$

$$u_{xx}(x,y) = -\frac{(2)(x^2-y^2)}{(x^2+y^2)^2}.$$

Similarly, $u_y(x,y) = \dfrac{2y}{x^2+y^2}$ and $u_{yy}(x,y) = \dfrac{(2)(x^2-y^2)}{(x^2+y^2)^2}$.

Thus, $u_{xx} + u_{yy} \equiv 0$ on A. Thus, u is harmonic on A.

② We now show that u has no harmonic conjugate on A.

Proof: Suppose not. Let $v : A \to \mathbb{R}$ be a harmonic conjugate of u. Then,

$$\begin{pmatrix} u_x = v_y \\ u_y = -v_x \end{pmatrix}$$

$$v_x(x,y) = -u_y(x,y) = \frac{-2y}{x^2+y^2}.$$

Similarly, $v_y(x,y) = \dfrac{2x}{x^2+y^2}$.

Thus, $(\nabla v)(x,y) = \left(\dfrac{-2y}{x^2+y^2}, \dfrac{2x}{x^2+y^2}\right)$.

(— Had seen this is not the

↳ Had seen this is not the grad of anything on $\mathbb{R}^2 - \{(0,0)\}$. We will show the same for A as well. (Same idea.)

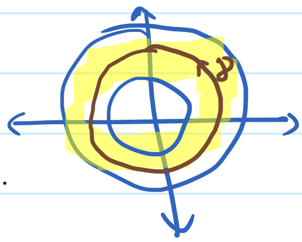
<u>Recall</u>: If a vector field $F : \Omega \to \mathbb{R}$ is a grad field, then

$$\oint_\gamma F = 0$$

for any simple closed curve γ lying in Ω.

In our case here, consider γ to be the circle of radius $R = 1.5$ centered at $(0,0)$.

Parameterise it as: $\gamma(t) := (R\cos(t), R\sin(t))$ for $t \in [0, 2\pi]$.



Now, we have

dot product

$$\oint_\gamma (\nabla v) = \int_0^{2\pi} \left(-\frac{2R\sin(t)}{R^2}, \frac{2R\cos(t)}{R^2} \right) \cdot (-R\sin t, R\cos t)\,dt$$

$$= \int_0^{2\pi} 2\,dt = 4\pi \neq 0.$$

The above is a contradiction since the integral of a grad field along a closed curve must be 0. ∎

Q7.

07 September 2021 19:45

7. Show that if f is a nonzero polynomial, then $g(z) := e^z f(z)$ has an essential singularity at ∞.

def^n ↪ $z \mapsto g\left(\frac{1}{z}\right)$ has an essential singularity at 0

Suffices to show: $\lim_{|z| \to \infty} g(z)$ does not exist (either in $\mathbb{C}$ or as ∞).

Idea: Use different paths to get different answers.

- If $z \to \infty$ along $\mathbb{R}^-$, we get

$$\lim e^z f(z) = 0.$$

$z \to -\infty$ $\left\{ \begin{array}{l} |z| \to \infty \\ z \in \mathbb{R}^- \end{array} \right.$

(WHY?! we know $\lim_{z \to -\infty} e^z z^k = 0$ $\forall k \in \{0, 1, 2, \ldots\}$. Take linear combination)

- If $z \to \infty$ along $\mathbb{R}^+$, we get

$$\lim_{\substack{|z| \to \infty \\ z \in \mathbb{R}^+}} |e^z| \, |f(z)| = \infty.$$

for this, we know it is ∞

for this, it is ∞ or a nonzero constant since it is a polynomial

Thus, $\lim_{z \to \infty} g(z)$ does not exist and hence, ∞ is an ess. sing. for g. ∎