Q1.

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1. Locate and classify the type of singularities of:

(i) $\dfrac{\sin(1/z)}{1+z^4}$,

(ii) $\dfrac{z^5\sin(1/z)}{1+z^4}$,

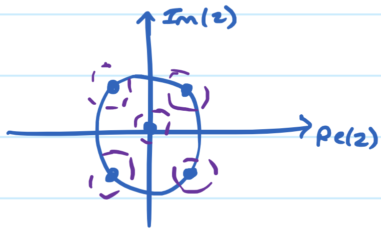
(iii) $\dfrac{1}{\sin(1/z)}$,

(iv) $\exp\left(\dfrac{1}{z}\right)$.

(i) $f(z) := \dfrac{\sin(1/z)}{1+z^4}$. $\longrightarrow$ not defined if $z = 0$ or $z^4 = -1 = e^{i\pi} = e^{3i\pi} = e^{5i\pi} = e^{7i\pi}$.

Step 1. Singularities $= \{0, e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}\}$.

Step 2. Which ones are isolated?



All. (WHY?! There are only finitely many.)

Step 3. Thus, we further classify all of them.

- Let us all look the 4^{th} roots of -1.

Let z_0 be s.t. $z^4 = 1$.

Then $\displaystyle\lim_{z\to z_0} \frac{1}{f(z)} = \lim_{z\to z_0} \frac{z^4+1}{\sin(1/z)} = 0.$

$\because \sin(1/z_0) \neq 0$

Thus, all the fourth roots are poles.

• Now, we look at $z_0 = 0$.

We have $f(z) = \dfrac{\sin(1/z)}{1+z^4}$

$$= \underbrace{\frac{1}{1+z^4}\left\{\frac{1}{2i}\right\}}_{\text{well behaved at } 0}\left\{e^{i/z} - e^{-i/z}\right\}.$$

If $z = iy$ and $y \to 0^+$ along $\mathbb{R}$, then

$$\lim_{y\to 0^+} \left|e^{1/y} - e^{-1/y}\right| = \infty.$$

But if $z = y$ and $y \to 0^+$ along $\mathbb{R}$, then

$$\lim_{y\to 0^+} \left|e^{i/y} - e^{-i/y}\right| \neq \infty$$ since the expression is bounded by 2.

Thus, $\lim_{z\to 0} |f(z)|$ is neither finite nor ∞.

Thus, 0 is essential.

(ii) $\dfrac{z^5 \sin(1/z)}{1+z^4}$. The answers are same as above.

For $z \in \mathbb{C}^{\times}$, we have

$$z^5 \sin\left(\frac{1}{z}\right) = z^5\left(\frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \dots\right)$$

$$= \left\{\frac{z^5}{z} - \frac{z^2}{3!} + \frac{1}{5!} - \frac{1}{7!\,z^2} + \dots\right\}$$

principal part has ∞

principal part has ∞ many terms

$\therefore$ $z^5 \sin(1/z)$ has ess. sing. at 0.

(iii) $\dfrac{1}{\sin(1/z)}$

Step 1. sing $= \{0\} \cup \left\{ \dfrac{1}{n\pi} : n \in \mathbb{Z}\setminus\{0\} \right\}$.

Step 2. Isolated ones : $\left\{ \dfrac{1}{n\pi} : n \in \mathbb{Z}\setminus\{0\} \right\}$ → for every n, find appropriate δ.

Nonisolated : $\{0\}$. → for every $\varepsilon > 0$, find $n > \frac{1}{\varepsilon}$.

Then $\dfrac{1}{n\pi} < \dfrac{1}{n} < \varepsilon$.

won't classify this any further!

Step 3. Classify the ISOLATED ones.

$\dfrac{1}{n\pi}$ is a pole $\forall n \in \mathbb{Z}\setminus\{0\}$.

(WHY?! $\displaystyle\lim_{z \to \frac{1}{n\pi}} \frac{1}{f(z)} = 0$)

(iv) $e^{1/z}$. ⟶ $\exp(1/z)$, $\sin(1/z)$, $\cos(1/z)$, $\sinh(1/z)$, $\cosh(1/z)$

$$\lim_{\substack{z \to 0^- \\ z \in \mathbb{R}}} e^{1/z} = 0.$$

$$\lim_{\substack{z \to 0 \\ z \in i\mathbb{R}}} \left| e^{1/z} \right| = 1.$$

Conclude that 0 is an essential sing.

Aliter: $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$

infinitely many terms in principal part!

Q2.

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2. Construct a meromorphic function on $\mathbb{C}$ with infinitely many poles.

f

only singularities of f in $\mathbb{C}$ are poles

Idea: Take entire $g \neq 0$ s.t. g has infinitely many zeroes and set $f = 1/g$.

(the zeroes will be isolated)

only isolated sing. They will all be poles.

Thus, $f(z) = \dfrac{1}{\sin(z)} = \operatorname{cosec}(z)$ works.

Other examples: $\dfrac{1}{e^z - 1}$, $\dfrac{1}{\cos(z)}$, $\tan(z)$, $\cot(z)$, $\sec(z)$.

Q3.

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3. Find Laurent expansions for the function $f(z) = \dfrac{2(z-1)}{z^2-2z-3}$ valid on the regions:

(i) $0 \leq |z| < 1$,

(ii) $1 < |z| < 3$,

(iii) $|z| > 3$.

idea is to use $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots$ for $|z|<1$.

$$f(z) = \frac{2(z-1)}{(z+1)(z-3)} = \frac{1}{z+1} + \frac{1}{z-3}.$$

(i) $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \quad |z|<1$

$$\frac{1}{z-3} = \frac{1}{(-3)}\frac{1}{\left(1-\frac{z}{3}\right)} = \left(-\frac{1}{3}\right)\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right)$$

$\left|\frac{z}{3}\right| < \frac{1}{3} < 1.$

$$\rightarrow f(z) = \sum_{n=0}^{\infty}\left((-1)^n + \left(-\frac{1}{3}\right)\left(\frac{1}{3^n}\right)\right) z^n.$$

$\uparrow$ Genuine pow series!

(ii) $1 < |z| < 3$

$$\frac{1}{1+z} \neq 1 - z + z^2 - \cdots$$

$$\sum_{n=-\infty}^{-1} (-z)^{-n-1}$$

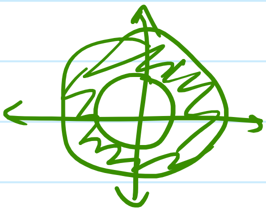
$$\frac{1}{1+z} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$|z| > 1 \Rightarrow \dfrac{1}{|z|} < 1.$

$|z| > 1 \Rightarrow \frac{1}{|z|} < 1.$

$$\frac{1}{z-3} = \left(-\frac{1}{3}\right)\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$\left|\frac{z}{3}\right| < \frac{3}{3} = 1.$



(iii) $|z| > 3$

$$\frac{1}{1+z} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$\because |z| > 3 \Rightarrow \frac{1}{|z|} < \frac{1}{3} \Rightarrow \frac{1}{|z|} < 1.$

$$\frac{1}{z-3} = \frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}\right) = \frac{1}{z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots\right)$$

$\left(\because |z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1\right)$

Wrap it up!

Q4.

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RRST: Riemann's Removable Singularity Theorem

4. Let Ω be a domain in $\mathbb{C}$, and let $z_0 \in \Omega$. Suppose that z_0 is an isolated singularity of f, and f is bounded in some punctured neighbourhood of z_0 (that is, there exists $M > 0$ and $\delta > 0$ such that $|f(z)| \leq M$ for all $z \in B_\delta(z_0) - \{z_0\}$). Show that f has a removable singularity at z_0.

Fix $\delta > 0$ and $M > 0$ s.t.

① f is holomorphic on $B_\delta(z_0) - \{z_0\}$, and

② $|f(z)| \leq M$ for $z \in \underbrace{B_\delta(z_0) - \{z_0\}}_{U}$.

Consider $g : U \to \mathbb{C}$ given by

$$g(z) = (z - z_0) f(z).$$

Then, ① g is holo on U,

② g has an isolated singularity at z_0,

③ $\lim_{z \to z_0} g(z) = 0$, (WHY?! Because f is bounded on U.)

④ thus, z_0 is a removable singularity for g AND defining

$$g(z_0) := 0$$

makes g holomorphic on $B_\delta(z_0)$.

Thus, we can write

$$g(z) = \cancel{a_0} + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

$a_0 = 0$

for $z \in B_\delta(z_0)$.

Thus, $g(z) = (z - z_0) \left[a_1 + a_2 (z - z_0) + \cdots \right]$.

$\parallel$

$(z-z_0) f(z)$

For $z \in U$, we get

$$f(z) = a_1 + a_2 (z-z_0) + \cdots.$$

Thus, (re)defining $f(z_0) := a_1$ does the job.

Q5.

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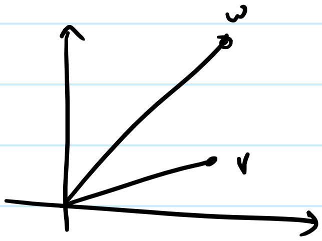
5. A complex-valued function f on $\mathbb{C}$ is called *doubly periodic* if there exist complex numbers $v, w \in \mathbb{C}$, which are linearly independent over $\mathbb{R}$, such that

$$f(z+v) = f(z) \quad \text{and} \quad f(z+w) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

Show that any doubly periodic entire function is constant.

(v, w are not the same line.)

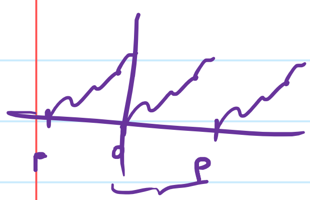


Thus, given any $z \in \mathbb{C}$, $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$z = \alpha v + \beta w.$$

(WHY?! $\mathbb{C}$ is 2-dimensional $\mathbb{R}$-vector space.)

Digression: If $g : \mathbb{R} \to \mathbb{R}$ is periodic with period $p > 0$, then the values that g takes on $\mathbb{R}$ is determined by the values it takes on $[0, p)$.



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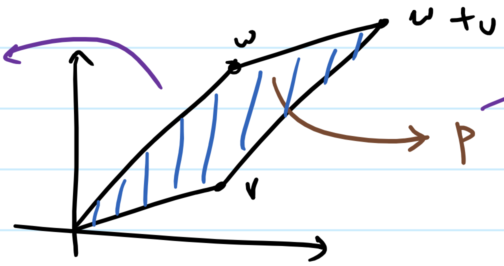

closed & bounded

$$P = \left\{ \alpha w + \beta v : \alpha \in [0,1], \beta \in [0,1] \right\}$$

Given: $f(z) = f(z+w) \quad \forall\, z \in \mathbb{C}.$

Thus, $f(z) = f(z+\omega) = f(z+2\omega = f(z+3\omega) = \cdots$
$= f(z-\omega) = f(z-2\omega) = f(z-3\omega) = \cdots$
$= f(z+n\omega) \quad \forall n \in \mathbb{Z}.$

Similarly, $f(z) = f(z+mv) \quad \forall m \in \mathbb{Z}.$

Thus, given $z \in \mathbb{C}$, first write it as

$$z = \alpha v + \beta \omega \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

Then, we have

floor (Greatest integer function) fractional part

$$z = \lfloor \alpha \rfloor v + \{\alpha\} v + \lfloor \beta \rfloor \omega + \{\beta\} \omega$$

$$\Rightarrow f(z) = f(\underbrace{\{\alpha\} v + \{\beta\} \omega}_{\in P})$$

since $\{\alpha\} \in [0,1)$
$\{\beta\} \in [0,1)$

Thus, $f(z) \in f(P)$. $\rightarrow$ bounded

(WHY?! P is compact and f is continuous.)

Thus, f is entire and bounded.
Thus, f is constant, by Liouville's test.

Q6.

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6. Show by transforming into an integral over the unit circle, that

$$\underbrace{\int_0^{2\pi} \frac{1}{a^2 - 2a\cos\theta + 1} \mathrm{d}\theta}_{=: I} = \frac{2\pi}{a^2-1},$$

where $a > 1$. Also compute the value when $a < 1$.

$\boxed{0 < a \neq 1}$

$$I = \int_0^{2\pi} \frac{1}{a^2 - a(e^{i\theta} + e^{-i\theta}) + 1} \cdot d\theta$$

$$= \int_0^{2\pi} \frac{1}{(a - e^{i\theta})(a - e^{-i\theta})} d\theta$$

$$= \frac{1}{i} \int_0^{2\pi} \frac{1}{(a - e^{i\theta})(a - e^{-i\theta})} \frac{ie^{i\theta}}{e^{i\theta}} d\theta$$

$$= \frac{1}{i} \int_0^{2\pi} \frac{1}{(a - e^{i\theta})(ae^{i\theta} - 1)} \cdot ie^{i\theta} d\theta$$

$$= \frac{1}{i} \int_{|z|=1} \frac{1}{(a-z)(az-1)} dz$$

$$= \frac{-1}{ia} \int_{|z|=1} \frac{1}{(z-a)(z-1/a)} dz.$$

two poles: a and $1/a$

$0 < a < 1$: the pole inside is a

$a > 1$: the pole inside the circle is $1/a$

inside
is a

the circle

$$= -\frac{1}{ia}(2\pi i)\left(\frac{1}{a-1/a}\right)$$

$$= \frac{2\pi}{1-a^2}.$$

$$= -\frac{1}{ia}\int \frac{\frac{1}{z-a}}{z-1/a}\,dz$$

$$= -\frac{1}{ia}(2\pi i)\left(\frac{1}{\frac{1}{a}-a}\right)$$

$$= -\frac{2\pi}{a}\left(\frac{a}{1-a^2}\right)$$

$$= \frac{2\pi}{a^2-1}.$$

Q7.

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7. Show that if $a_1, \ldots, a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n, for each $1 \le k \le n$ we have the formula:

$$\prod_{j \neq k} (a_k - a_j) = P'(a_k).$$

$$P(z) = \cancel{c}\,1 \cdot (z - a_1) \cdots (z - a_n)$$

WAY #1.

$$P'(a_k) = \lim_{z \to a} \frac{P(z) - \cancel{P(a_k)}^{0}}{z - a_k}$$

$$= \lim_{z \to a_k} \frac{\prod_{j=1}^{n} (z - a_j)}{z - a_k}$$

$$= \lim_{z \to a_k} \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - a_j)$$

$$= \prod_{j \neq k} (a_k - a_j).$$

WAY# 2. Fix $k \in \{1, \ldots, n\}$ and write

$$P(z) = (z - a_k)\, P_k(z).$$

diff.

$$P_k(z) = \prod_{j \neq k} (z - a_j).$$

$$P'(z) = (z - a_k)\, P_k'(z) + 1 \cdot P_k(z).$$

Put $z = a_k$ to get

$$P'(a_k) = P_k(a_k) = \prod_{j \neq k} (a_k - a_j). \boxtimes$$