Q1.

28 August 2021 12:08

$f : [0,1] \to [0, \infty)$ continuous.

Q. If $\int_0^1 f(t)\,dt = 0$, then what can you say?

Then, $f \equiv 0$.

Show that there is a strict inequality

$$\left| \int_{|z|=R} \frac{z^n}{z^m - 1} \mathrm{d}z \right| < \frac{2\pi R^{n+1}}{R^m - 1}; \quad R > 1,\ m \geq 1,\ n \geq 0.$$

Let us do the dumb obvious ML bound:

$L = 2\pi R.$

$$\left| \frac{z^n}{z^m - 1} \right| = \frac{|z|^n}{|z^m - 1|} = \frac{R^n}{|z^m - 1|}$$

$$\leq \frac{R^n}{|z|^m - 1} = \frac{R^n}{R^m - 1}.$$

Thus, $ML = 2\pi \dfrac{R^{n+1}}{R^m - 1}$.

Theorem 2: The Stronger ML Inequality

Let $f : \Omega \to \mathbb{C}$ be a continuous function and $\gamma : [a, b] \to \Omega$ be a curve.
Let $M > 0$ be such that

$$M \geq |f(\gamma(t))|, \quad \text{for all } t \in [a, b].$$

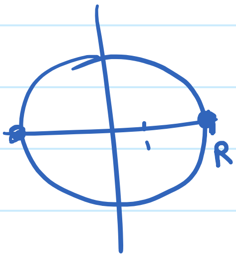
Also, suppose that $|f(\gamma(t))|$ $\cancel{|f(t)|} < M$ for some $t \in [a, b]$.
Then,

$$\left| \int_\gamma f(z)\mathrm{d}z \right| < ML,$$

where L is the length of the curve, as usual.

Assuming the theorem, we are done.
WHY ?! Because we had already the dumb bound.
We just need to show that the inequality is strict.
For that, we need to show that

$|z^n| \ldots R^n$



For that, we need to show that

$$\left|\frac{z^n}{z^m-1}\right| < \frac{R^n}{R^m-1} \quad \text{for } \underline{\underline{\text{some}}}\ z$$

with $|z|=R$.

(Take z s.t. $z^m = -R^m$.
(You can take $z = R\exp\left(\frac{i\pi}{m}\right)$.)
Note $|z| = R$ and $z^m = -R^m$.)

Now, we only need to prove the Theorem.

Note that $\displaystyle\int_a^b \left[M - |f(\gamma(t))|\right] |\gamma'(t)|\, dt \geq 0.$

If for some t, we have $|f(\gamma(t))| < M$,
then for that t, we have

$$\left[M - |f(\gamma(t))|\right] \underline{\underline{|\gamma'(t)|}} > 0.$$

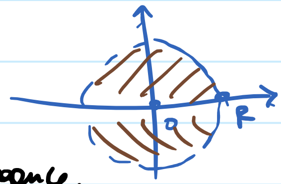
can be assumed to be nonzero, since $\gamma' \neq 0$.

Then, $\displaystyle\int_a^b \left[M - |f(\gamma(t))|\right] |\gamma'(t)|\, dt > 0.$ ∎

Q2.

28 August 2021 12:08

A power series with center at the origin and positive radius of convergence R, has a sum $f(z)$. If it is known that $\boxed{f(\bar{z}) = \overline{f(z)}}$ for all values of z within the disc of convergence, what conclusions can you draw about the power series?



Conclusion. All the coefficients are real.

Let $D = B_R(0)$ be the open disc of convergence.

Justification. (To show: all coefficients are real.)

Observation 1. $f(x) \in \mathbb{R} \quad \forall x \in \mathbb{R} \cap D.$

WHY?! Since $x \in \mathbb{R}$, then $\bar{x} = x$.

Thus, $f(x) = f(\bar{x}) = \overline{f(x)}$.

$\Rightarrow f(x) \in \mathbb{R}$.

Observation 2. To prove the claim, it suffices to show that $f^{(n)}(0)$ are real for all $n \geqslant 1$.

WHY?! The coefficients are simply $\dfrac{f^{(n)}(0)}{n!}$.

(And $n! \in \mathbb{R}$.)

Observation 3. We know all derivatives of exist on D.

It suffices to show that $f^{(n)}(x) \in \mathbb{R}$ $\forall x \in \mathbb{R} \cap D$ and $n \geqslant 1$.

Observation 4. It suffices to show that $f'(x) \in \mathbb{R}$ $\forall x \in \mathbb{R} \cap D$.

WHY?! Induct!

WHY?! Induct!

<u>Claim</u>. If $g : D \longrightarrow \mathbb{C}$ is holomorphic s.t. $g(x) \in \mathbb{R}$ whenever $x \in \mathbb{R} \cap D$, then $g'(x) \in \mathbb{R}$ whenever $x \in \mathbb{R} \cap D$.

<u>Proof</u>. Fix $x_0 \in \mathbb{R} \cap D$. We know $g'(x_0)$ exists. So, we can compute it along any path we want. Choose the path along $\mathbb{R}$. Then, we get

$$g'(x_0) = \lim_{\substack{x \to x_0 \\ x \in \mathbb{R} \cap D}} \frac{\overbrace{g(x) - g(x_0)}^{\in \mathbb{R}}}{\underbrace{x - x_0}_{\in \mathbb{R}}}.$$

Since both the num. and deno. are real, so is their limit. ∎

To conclude: We have shown that

$$f^{(n)}(x) \in \mathbb{R} \quad \forall x \in \mathbb{R} \cap D \quad \forall n \geq 0.$$

Thus, $f^{(n)}(0) \in \mathbb{R} \quad \forall n \geq 0$.

Thus, all coefficients of the pow. series are $\in \mathbb{R}$.

Q3.

28 August 2021 12:08

This is called Taylor series with integral remainder:

$$f(z) = f(0) + zf'(0) + \cdots + \frac{z^N}{N!} f^{(N)}(z)(0) + \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N f^{(N+1)}(tz)\mathrm{d}t$$

Use this to prove the following inequalities:

(a) $\left| e^z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| \le \frac{|z|^{N+1}}{(N+1)!}$; $\Re z \le 0$.

Useful bounds, in general:

(1) $|\exp(z)| = \exp(\mathrm{Re}(z)) \quad \forall z \in \mathbb{C}.$

Proof. Write $z = x + iy$. $(x, y \in \mathbb{R})$

Then,

$$\exp(z) = \exp(x)\exp(iy)$$

$$\Rightarrow |\exp(z)| = |\exp(x)| \cdot |\exp(iy)|$$
$$= \exp(x) \cdot 1$$
$$= \exp(\mathrm{Re}(z)).$$

(2) $|\cos(z)| \le \cosh(\mathrm{Im}(z)) \quad \forall z \in \mathbb{C}.$

Proof. Write $z = x + iy$ as earlier to get

$$|\cos(z)| = \frac{1}{2}\left| e^{iz} + e^{-iz} \right|$$
$$\le \frac{1}{2}\left\{ |e^{iz}| + |e^{-iz}| \right\}$$
$$= \frac{1}{2}\left\{ e^{\mathrm{Re}(iz)} + e^{\mathrm{Re}(-iz)} \right\}$$

using (1)

$$= \frac{1}{2}\left\{ e^{-y} + e^{y} \right\}$$

$$= \cosh(y) = \cosh(\mathrm{Im}(z)). \quad \blacksquare$$

Sol^n (1) (a) $\left| e^z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| \le \frac{|z|^{N+1}}{(N+1)!}$; $\Re z \le 0$.

$\Vert$ by Taylor formula $\qquad \left(\exp^{(N+1)} = \exp . \right)$

$$\left| \frac{z^{N+1}}{(N+1)!} \int_0^1 (1-t)^N \exp(tz)\, dt \right|$$

$$\le \frac{|z^{N+1}|}{(N+1)!} \int_0^1 (1-t)^N \underbrace{|\exp(tz)|}_{\text{Bound (1)}}\, dt$$

$$= \frac{|z^{N+1}|}{(N+1)!} \int_0^1 \underbrace{(1-t)^N}_{\le 1} \underbrace{\exp(t \cdot \mathrm{Re}(z))}_{\le 1}\, dt$$

$\mathrm{Re}(z) \le 0$
$\Downarrow$
$t\,\mathrm{Re}(z) \le 0$
$\Downarrow$
$\exp(t\,\mathrm{Re}(z)) \le 1$

$$\le \frac{|z^{N+1}|}{(N+1)!} \int_0^1 1\, dt = \frac{|z|^{N+1}}{(N+1)!}. \quad \blacksquare$$

(b) $\left| \cos(z) - \sum_{n=0}^{N} (-1)^n \frac{z^{2n}}{(2n)!} + 0 z^{2n+1} \right| \le \frac{|z|^{2N+2} \cosh R}{(2N+2)!}$; $|\Im z| \le R$.

Taylor $\Vert$ $\cos(z) = 1 + 0z - \frac{z^2}{2!} - 0z^3 + \frac{z^4}{4!} + \cdots$

$$\left| \frac{z^{2N+2}}{(2N+2)!} \int_0^1 (1-t)^{2N+1} \cos^{(2N+2)}(tz)\, dt \right|$$

$$\le \frac{|z|^{2N+2}}{} \int_0^1 (1-t)^{2N+1} \left| \cos^{(2N+2)}(tz) \right| dt$$

$$\leq \frac{|z|^{2N+2}}{(2N+2)!} \int_0^1 (1-t)^{2N+1} \left|\cos^{(2N+2)}(tz)\right| dt$$

$\cos^{(2N+2)} = \pm \cos$

$\Downarrow$

$\left|\cos^{(2N+2)}\right| = |\cos|$

$$= \frac{|z|^{2N+2}}{(2N+2)!} \int_0^1 (1-t)^{2N+1} |\cos(tz)| \, dt$$

Bound (2)

$$= \frac{|z|^{2N+2}}{(2N+2)!} \int_0^1 (1-t)^{2N+1} \cosh(t \, \mathrm{Im}(z)) \, dt$$

We are given: $|\mathrm{Im}(z)| \leq R$

Since $t \in [0,1]$, we have

$$|t \, \mathrm{Im}(z)| \leq R.$$

Thus, $\cosh(t \, \mathrm{Im}(z)) \leq \cosh(R)$. (WHY?!)

($\cosh(y)$ is increasing in $|y|$, i.e., if $|y_1| \leq |y_2|$, then $\cosh(y_1) \leq \cosh(y_2)$ $\forall y_1, y_2 \in \mathbb{R}$.)

$$\leq \frac{|z|^{2N+2}}{(2N+2)!} \int_0^1 \underbrace{(1-t)^{2N+1}}_{\leq 1} \underline{\cosh(R)} \, dt$$

$\hookrightarrow$ constant. Can take outside the integral

$$\leq \frac{|z|^{2N+2}}{(2N+2)!} \cosh(R).$$

$\blacksquare$

Q4.

28 August 2021 12:08

By computing $\boxed{n \geq 1}$

$$I := \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \mathrm{d}z,$$

in two different ways

show that

$$\int_0^{2\pi} \cos^{2n} \theta \mathrm{d}\theta = \frac{2\pi}{4^n} \cdot \frac{(2n)!}{(n!)^2}.$$

Way #1. Let us the "generalised" CIF.

$$I = \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \, dz$$

$$= \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} \, dz$$

Theorem 18 ("Generalised" CIF)

$$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0),$$

where f is a function which is holomorphic on an open disc $B_R(z_0)$ and $r < R$.

Here, we take $f(z) := (z^2+1)^{2n}$, which is entire.
$(z_0 = 0, \ r = 1, \ R = \infty)$

Thus,
$$\int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} \, dz = \frac{2\pi i}{(2n)!} f^{(2n)}(0). \qquad (1)$$

To calculate: $f^{(2n)}(0)$.

Note that $\dfrac{f^{(2n)}(0)}{(2n)!}$ is just the coefficient of z^{2n}

in the power series expansion of f around 0.

In our case, we can use the Binomial theorem to calculate the power series as

$$f(z) = \sum_{k=0}^{2n} \binom{2n}{k} z^{2k}.$$

Thus, $$\frac{f^{(2n)}(0)}{(2n)!} = \binom{2n}{n}.$$

Plug this back in (1) to get

$$I = 2\pi i \binom{2n}{n}. \qquad \text{—— (2)}$$

WAY # 2. Parameterise!

The contour is the circle. Parameterise it as

$$\gamma(t) = e^{it} = \cos(it) + i\sin(it).$$

$$I = \int_\gamma \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z}\, dz = \int_0^{2\pi} (2\cos t)^{2n} \frac{1}{e^{it}}\, i e^{it}\, dt$$

$$= \int_0^{2\pi} 2^{2n} \cos^{2n}(t)\, i\, dt$$

$$I = 4^n i \int_0^{2\pi} \cos^{2n}(t)\, dt. \qquad \text{—— (3)}$$

Equate (2) and (3) to get the desired equation. ∎

Q5.

28 August 2021 12:08

Let f be an entire function. Show that f is a polynomial of degree at most n if and only if there exists a positive real constant C such that $|f(z)| < C|z^n|$ for all z with $|z|$ sufficiently large.

($\Longrightarrow$) Assume that f is a polynomial of degree $\le n$.

Write $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ for $a_0, \ldots, a_n \in \mathbb{C}$.

(Note that a_n might be 0.)

Note that for $z \neq 0$, we have

$$\frac{f(z)}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n.$$

Thus, note that

$$\lim_{|z| \to \infty} \frac{f(z)}{z^n} = a_n. \quad \text{(Sum of limits.)}$$

By definition, this means that $\forall \varepsilon > 0, \exists R > 0$ s.t.

$$\left| \frac{f(z)}{z^n} - a_n \right| < \varepsilon \quad \forall z \in \mathbb{C} \text{ with } |z| > R.$$

Taking $\varepsilon = 1$, we get that $\exists R > 0$ s.t.

$$\left| \frac{f(z)}{z^n} - a_n \right| < 1 \quad \forall z \in \mathbb{C} \text{ with } |z| > R.$$

Thus, $$\left| \frac{f(z)}{z^n} \right| < |a_n| + 1 \quad \forall z \in \mathbb{C} \text{ with } |z| > R.$$

$\hookrightarrow$ holds for all z with $|z|$ sufficiently large.

Taking $C = |a_n| + 1$ does the job.

($\Leftarrow$) f is entire and $\exists R_0 > 0$, $\exists C > 0$ such that

$$|f(z)| < C|z|^n \quad \forall z \in \mathbb{C} \text{ with } |z| > R_0.$$

To show: f is a polynomial of degree $\leq n$.

→ WAY #1: Show that $f^{(n+1)}(z) = 0 \quad \forall z \in \mathbb{C}$.

→ WAY #2: Show that $f^{(m)}(0) = 0 \quad \forall m \geq n+1$.

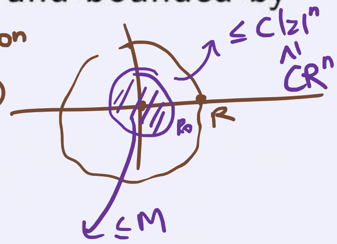
↳ We will use this.

Theorem 19 (Cauchy's estimate)

Suppose that f is holomorphic on $|z - \overset{0}{z_0}| < R$ and bounded by $M > 0$ on this disc. Then,

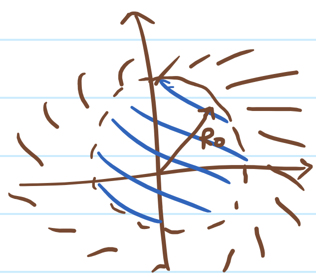
$$\left|f^{(n)}(\overset{0}{z_0})\right| \leq \frac{n!M}{R^n}.$$

bound on $B_R(0)$



Q: Is $|f|$ bounded on $D := \{z \in \mathbb{C} : |z| \leq R_0\}$?

↳ closed and bounded



Ans. Yes.

Since f is entire, f is continuous and hence, bounded on D.

Let $M > 0$ be an upper bound of $|f|$ on D.

$|f| \leq M + CR^n$

Now, let $R > 0$ be given.

Then, for $z \in B_R(0)$, we have

$$|f(z)| \leq \max\{M, C|z|^n\}$$
$$\leq M + C|z|^n$$

$$|f(z)| \le \max\{M, \ C|z|^n\}$$
$$\le M + C|z|^n$$
$$\le \underline{M + CR^n}.$$

Fix a natural $m > n$, note that

$$\underbrace{|f^{(m)}(0)|} \le \underline{\frac{m!\,(M + CR^n)}{R^m}}$$ Cauchy's estimate

for all $R > 0$.

$$= m! \left\{ \frac{M}{R^m} + \frac{C}{R^{m-n}} \right\} \longrightarrow 0 \text{ as } R \to \infty.$$

(Since $m > n \geq 0$.)

Thus, $\underline{f^{(m)}(0) = 0}$ for any $m > n$.

Thus, f is a polynomial of degree $\le n$. ∎

(WHY?!
Because f is entire and thus a power series expansion valid on all of $\mathbb{C}$.)

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \cdots + \frac{f^{(n)}(0)}{n!} z^n$$
$$\underbrace{+ \frac{f^{(n+1)}(0)}{(n+1)!} z^n + \cdots}_{=0}$$

Q6.

28 August 2021 12:08

Let f and g be entire nonvanishing functions such that

defined on $\mathbb{C}$ and is holomorphic on $\mathbb{C}$

$$\left(\frac{f'}{f}\right)\left(\frac{1}{n}\right) = \left(\frac{g'}{g}\right)\left(\frac{1}{n}\right)$$

for all $n \in \mathbb{N}$. Show that g is a nonzero scalar multiple of f.

Since f is entire and nonvanishing, f'/f is again entire.

Similarly, so is g'/g.

We are given that the entire functions f'/f and g'/g agree on $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

The above set has a limit point in $\mathbb{C}$. (Namely, 0.)

Thus, $\left(\frac{f'}{f}\right) = \left(\frac{g'}{g}\right)$ on all of $\mathbb{C}$. (WHY?!) (Use Tut 3 Q7)

Thus, $g(z)f'(z) - f(z)g'(z) = 0 \quad \forall z \in \mathbb{C}$. (1)

To show: g is a scalar multiple of f. That is, $\frac{g}{f}$ is constant.

Let us calculate $\left(\frac{g}{f}\right)'(z)$.

We have,

$$\left(\frac{g}{f}\right)'(z) = \frac{f(z)g'(z) - g(z)f'(z)}{(f(z))^2}$$

by (1)

$=$ 0

$= \quad 0.$

Since $\mathbb{C}$ is path-connected and $(g/f)' \equiv 0$, we see that g/f is constant, say λ.

Since $g \neq 0$, it follows that $\lambda \neq 0$.

$\therefore \quad g = \lambda f$ with $\lambda \neq 0$, as desired. ∎