Q1.

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1. Show that Cauchy Riemann equation take the form:

$$u_r = \frac{1}{r} v_\theta \ and \ v_r = -\frac{1}{r} u_\theta$$

in polar coordinates. As usual, $f = u + iv$.

Usually: $z = x + iy$

Here: $z = re^{i\theta}$.

$$\tilde{f}(r, \theta) := f(re^{i\theta})$$

Similarly, $\tilde{u}(r, \theta) := u(re^{i\theta})$ and

$\tilde{v}(r, \theta) := v(re^{i\theta})$.

Fix $z_0 = r_0 e^{i\theta_0} \neq 0$.

We are given that f is diff. at z_0.

Thus, we know this exists

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

As in slides, we calculate the limit in two ways.

<u>Way #1.</u> Fix $r = r_0$ and let $\theta \to \theta_0$.

Thus,

$$f'(z_0) = \lim_{\theta \to \theta_0} \frac{f(r_0 e^{i\theta}) - f(r_0 e^{i\theta_0})}{r_0 (e^{i\theta} - e^{i\theta_0})}$$

$$= \frac{1}{r_0} \lim_{\theta \to \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} + i \frac{v(r_0, \theta) - v(r_0, \theta_0)}{e^{i\theta} - e^{i\theta_0}} \right\} \quad (*)$$

For overall limit to exist, individual limits must exist. (WHY?!)

Let us concentrate on first term:

$$\lim_{\theta \to \theta_0} \frac{u(r_0,\theta) - u(r_0,\theta_0)}{e^{i\theta} - e^{i\theta_0}}$$

$$= \lim_{\theta \to \theta_0} \left(\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} \right) \cdot \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}$$

Note that $\lim_{\theta \to \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}$ exists and equals $\frac{1}{i e^{i\theta_0}}$.

Since this limit is non zero, even $\lim_{\theta \to \theta_0} \left(\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} \right)$ must exist.

This, by definition, is $u_\theta(r_0,\theta_0)$.

Putting this back in (*) gives:

$$f'(z_0) = \frac{1}{r_0} \left\{ \frac{u_\theta(r_0,\theta_0)}{i e^{i\theta}} + i \frac{v_\theta(r_0,\theta_0)}{i e^{i\theta}} \right\}. \quad (\dagger)$$

WAY#2. Fix $\theta = \theta_0$ and let $r \to r_0$.

$$f'(z_0) = \lim_{r \to r_0} \left\{ \frac{u(r,\theta_0) - u(r_0,\theta_0)}{(r - r_0) e^{i\theta_0}} + i \frac{v(r,\theta_0) - v(r_0,\theta_0)}{(r - r_0) e^{i\theta_0}} \right\}$$

$$= \frac{1}{e^{i\theta_0}} \lim_{r \to r_0} \left\{ \frac{u(r,\theta_0) - u(r_0,\theta_0)}{(r - r_0)} + i \frac{v(r,\theta_0) - v(r_0,\theta_0)}{(r - r_0)} \right\}$$

$$= \frac{1}{e^{i\theta_0}} \left\{ u_r(r_0, \theta_0) + i\, v_r(r_0, \theta_0) \right\} \qquad (\dagger\dagger)$$

Equate $(\dagger)$ and $(\dagger\dagger)$ and cancel $e^{-i\theta_0}$. Then, equate real and imaginary parts to conclude. $\boxtimes$

Q2.

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2. Prove Cauchy's theorem assuming Cauchy integral formula.

Theorem 14 (Cauchy's Theorem)

Let γ be a simple, closed contour and let f be a holomorphic function defined on an open set Ω containing γ as well as its interior. Then,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

now $\Uparrow$ $\Downarrow$ class

Theorem 16 (Cauchy Integral Formula)

Let f be holomorphic everywhere on an open set Ω. Let γ be a simple closed curve in Ω, oriented positively. If z_0 is interior to γ and Ω contains the interior of γ, then

$$\underline{f(z_0)} = \frac{1}{2\pi\iota}\underline{\int_\gamma \frac{f(z)}{z-z_0}\mathrm{d}z}.$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Sol^n Let γ be a simple, closed contour.
Let Ω be open, containing γ and int(γ). interior
Suppose $f : \Omega \to \mathbb{C}$ is holomorphic.

TS: $$\int_\gamma f(z)\,dz = 0.$$

Proof. Fix some $z_0 \in \text{int}(\gamma)$.
Consider $g : \Omega \to \mathbb{C}$ as $g(z) = f(z)(z-z_0)$.
Since f is holo. on Ω, so is g.
Thus, by CIF,

$$g(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-z_0}\,dz.$$

$\|$

0

note that z_0 does not lie on γ

$$\underset{0}{\|} \qquad 2\pi i \int_\gamma z - z_0$$

$$\| \qquad \text{note that } z_0 \text{ does not lie on } \gamma$$

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \cdot 0 = 0.$$

Q3.

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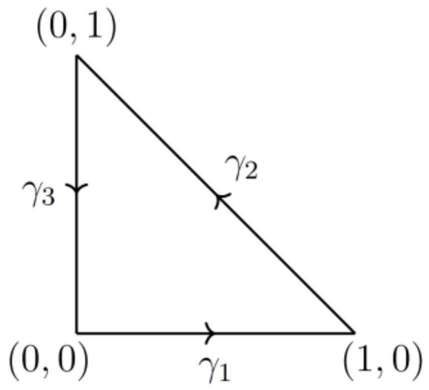
3. Let γ be the boundary of the triangle $\{0 < y < 1 - x;\ 0 \le x \le 1\}$ taken with the anticlockwise orientation. Evaluate:

a) $\int_\gamma Re(z)dz$ b) $\int_\gamma z^2 dz$ → This is 0. (WHY?!)

↳ It has a primitive on $\mathbb{C}$. → "FTC"

Aliter: It is holomorphic on $\mathbb{C}$ & Cauchy's Theorem



(a) WAY #1. By force. Parameterise each of γ_1, γ_2, and γ_3.

Calculate
$$\int_0^1 f(\gamma_i(t))\,\gamma_i'(t)\,dt.$$
Add.

WAY #2.
$$\int_\gamma Re(z)dz = \int_\gamma \frac{z+\bar{z}}{2}\,dz$$

$$= \frac{1}{2}\int_\gamma z\,dz + \frac{1}{2}\int_\gamma \bar{z}\,dz$$

has a primitive OR holomorphic on $\mathbb{C}$

By Q5

$$= 0 + \frac{1}{2}(2i\ \mathrm{Area}(\gamma))$$

$$= i\ \mathrm{Area}(\gamma) = \frac{i}{2}.$$

∎

https://aryamanmaithani.github.io/ma-205-tut/tut-solutions.pdf

(on bit.ly/ca-205)

For WAY #1, can check if you get stuck.

Q4.

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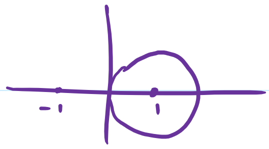

4. Compute $\int_{|z-1|=1} \frac{2z-1}{z^2-1} dz$

↳ Should scream, "CIF"

↳ Cauchy Integral Formula

[$z^2-1 = (z-1)(z+1)$.

Only $z=1$ is within the region]

$$\int_{|z-1|=1} \frac{2z-1}{z^2-1}\,dz = \int_{|z-1|=1} \frac{\frac{2z-1}{z+1}}{z-1}\,dz.$$

Define $f(z) := \dfrac{2z-1}{z+1}$ on $\underbrace{\mathbb{C} - \{-1\}}_{\Omega}$.

Then, Ω contains the curve of integration as well as its interior.

Thus, CIF tells us that

$(z_0 = 1)$

$$\int_{|z-1|=1} \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1).$$

$$\therefore \int_{|z-1|=1} \frac{2z-1}{z^2-1}\,dz = 2\pi i \left(\frac{2\cdot 1 - 1}{1+1} \right)$$

$$= \underline{\underline{\pi i}}. \qquad \blacksquare$$

Q5.

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5. ① Show that if γ is a simple closed curve traced counter clockwise, the integral $\int_\gamma \bar{z} dz$ equals $2\imath Area(\gamma)$. ② Evaluate $\int_\gamma \bar{z}^m dz$ over a circle γ centered at the origin.

① Idea is to use Green's theorem.

In going from the single integral to the double integral, we used Green's theorem which said that

$$\int_\gamma (M\mathrm{d}x + N\mathrm{d}y) = \iint_{\mathrm{Int}(\gamma)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}(x,y)$$

if γ is a (nice-enough) closed curve oriented counterclockwise. (Here is where we have used orientation.)

Want: $\int_\gamma \bar{z}\, dz$.

Step 1. Parameterise γ. Let $\gamma(t) = x(t) + iy(t)$, for $t \in [0, 1]$.

$$\int_\gamma \bar{z}\, dz = \int_0^1 \overline{x(t) + iy(t)} \cdot (x'(t) + iy'(t))\, dt$$

$$= \int_0^1 (x(t) - iy(t))(x'(t) + iy'(t))\, dt$$

$$= \int_0^1 [x'(t)\, x'(t) + y(t)\, y'(t)]\, dt$$

$$+ i \int_0^1 [x(t)\, y'(t) - y(t)\, x'(t)]\, dt$$

loose notation

$x(t) = x$
$x'(t)dt = dx$

$$= \oint (x\, dx + y\, dy) + i \oint (x\, dy - y\, dx)$$

$x(t) = x$
$x'(t)dt = dx$

$$= \oint_\gamma (x\,dx + y\,dy) + i\oint_\gamma (x\,dy - y\,dx)$$

$$= \iint_{\mathrm{int}(\gamma)} (0-0)\,d(x,y) + i\iint_{\mathrm{int}(\gamma)} [1-(-1)]\,d(x,y)$$

$$= 2i\iint_{\mathrm{int}(\gamma)} d(x,y) = 2i\,\mathrm{Area}(\gamma). \quad \blacksquare$$

② $\int_\gamma \bar{z}^m\,dz$ where γ is a circle, centered at 0. ($m \in \mathbb{Z}$)

Way #1. Parameterise and do it by "brute force".

That is, $\gamma(t) := Re^{it}$ for $t \in [0, 2\pi]$.

Way #2. Let $R > 0$ be the radius.

↳ If $m = 1$, then we are done by ①.

Thus,
$$\int_\gamma \bar{z}^m\,dz = \int_\gamma \bar{z}\,dz = 2i\,\mathrm{Area}(\gamma) = 2\pi i R^2.$$

Now, assume $m \neq 1$.

since $|z|^2 = R^2$ on γ

$$\int_\gamma \bar{z}^m\,dz = \int_\gamma \left(\frac{R^2}{z}\right)^m dz$$

$$= R^{2m}\int \frac{1}{z^m}\,dz$$

"FTC" since $z \mapsto \frac{1}{...}$ has

$$= R^{\cdots} \int_\gamma \frac{1}{z^m}\, dz$$

$$= 0.$$

"FTC" since $z \mapsto \frac{1}{z^m}$ has a primitive on $\mathbb{C} - \{0\}$. $(m \neq 1)$

Exercise: Confirm using way #1.

Q6.

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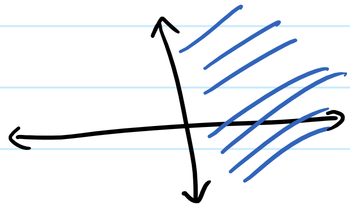
6. Let $\mathbb{H} = \{z \in \mathbb{C} | Re(z) > 0\}$ be the (strict) open right half plane. Construct a function non constant f which is holomorphic on $\mathbb{H}$ and such that $f(\frac{1}{n}) = 0$ for $n \in \mathbb{N}$.



Verify that $f(z) := \sin\left(\frac{\pi}{z}\right)$ works.

To check:

1. f is well-defined.
   True since $0 \notin \mathbb{H}$ and sin is entire.
2. f is holomorphic.
3. $f(\frac{1}{n}) = 0 \quad \forall n \in \mathbb{N}$.
   True since $\sin(n\pi) = 0 \quad \forall n \in \mathbb{N}$.
4. f is non constant.
   $f(1) = 0$ but $f(2) = 1$.

Q7.

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$\{\frac{1}{n} : n \in \mathbb{N}\}$ is discrete.

7. Let f be a holomorphic function on $\mathbb{C}$ such that $f(\frac{1}{n}) = 0$ for $n \in \mathbb{N}$. Show that f is a constant. (And necessarily, the constant is 0.)

Pf. **Claim 1.** $f(0) = 0$.

(Note that $f(0)$ makes sense earlier since domain is $\mathbb{C} \ni 0$.)

Proof. Since f is holomorphic, f is continuous.

Also, $\lim_{n\to\infty} \frac{1}{n} = 0$.

By sequential criterion of continuity, we have,

$$f(0) = f\left(\lim_{n\to\infty} \frac{1}{n}\right) = \lim_{n\to\infty} f\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 0 = 0.$$

Thus, $f(0) = 0$, as desired. ∎

Note: 0 is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$.

Thus, $Z = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is not discrete.

Since f vanishes on Z, it must vanish everywhere. (Identity theorem.)

In particular, f is constant. ∎

Defn. A subset $D \subseteq \mathbb{C}$ is said to be discrete if for every $z \in D$, there exists $\varepsilon > 0$ s.t.

$$B_\varepsilon(z) \cap D = \{z\}.$$

ball of radius ε centered at z

(That is, the ε-ball at z contains no other point of D.)

$D_1 = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is discrete

for $\frac{1}{n}$, take $\varepsilon = \frac{1}{n} - \frac{1}{n+1}$.

$Z = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$,

for 0 you cannot find such an ε.

Q8.

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8. Expand $\frac{1+z}{1+2z^2}$ into a power series around 0. Find the radius of convergence.

Recall that the power series expansion is unique. Thus, if we know *some* power series expansion with *some* positive radius of convergence, that must be *the* power series expansion with *that* radius of convergence.
Basically, if two different people compute two different sequence of coefficients with whatever methods, the coefficients and the radius of convergence will be equal.[2]

$$\frac{1+z}{1+2z^2}.$$

Note that $(1+2z^2)^{-1} = 1 - 2z^2 + (2z^2)^2 + \cdots$

for $|2z^2|<1$
$\Updownarrow$
$|z| < \frac{1}{\sqrt{2}}$

Thus, $\frac{1}{1+2z^2}$ has pow. series $= 1 - 2z^2 + 4z^4 - \cdots$
with radius of convergence $= \frac{1}{\sqrt{2}}$.

Since $1+z$ is a non zero polynomial, the radius of convergence does not change. Thus, we are done.

$$\frac{1+z}{1+2z^2} = (1+z)\sum_{n=0}^{\infty}(-2z^2)^n$$

$$= (1+z)(1 - 2z^2 + 4z^4 - 8z^6 + \cdots)$$

$$= 1 + z - 2z^2 - 2z^3 + 4z^4 + 4z^5 - \cdots$$

$$= \sum_{n=0}^{\infty} a_n z^n, \quad \text{where} \quad a_n = (-2)^{\lfloor n/2 \rfloor}.$$

Can check using root test that R.C = $\frac{1}{\sqrt{2}}$.