Q1.

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1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions.

Assume: Domain is a domain (What does this mean?)

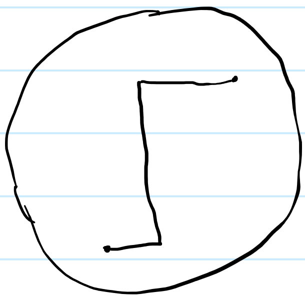
→ That u and v are defined on open & path-connected subsets of $\mathbb{R}^2$.

First, we have : $u_x = v_y$, — (1)
$u_y = -v_x$. — (2)

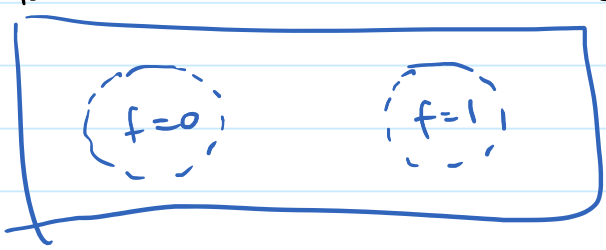
But we also have : $v_x = u_y$, — (2')
$v_y = -u_x$. — (1')

(1), (1') give $u_x = -u_x$ or $u_x \equiv 0$.
Similarly, $v_x \equiv 0$.
In turn, $u_y \equiv 0 \equiv v_y$.

Since the domain is connected, u and v are constants. ∎



On a connected domain:
total derivative $= 0 \Rightarrow$ constant.



Q2.

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2. Show that $u = XY + 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

Method #1. (Brute force)

Compute: $u_{xx}(x,y) = 6y,$

$u_{yy}(x,y) = -6y.$

$\therefore\ u_{xx} + u_{yy} \equiv 0.$

Want: v s.t. $v_y = u_x$

$= y + 6xy.$

integrate w.r.t. y

$$v = \frac{y^2}{2} + 3xy^2 + \phi(x). \quad \text{—①}$$

Now, we want $v_x = -u_y.$

$\Downarrow$ Plug it in

$$3y^2 + \phi'(x) = -x - 3x^2 + 3y^2$$

$$\Rightarrow \phi'(x) = -x - 3x^2.$$

Up to a constant $\phi(x) = -\dfrac{x^2}{2} - x^3.$

Use ①

Then, $v(x,y) = \frac{1}{2}y^2 + 3xy^2 - \frac{1}{2}x^2 - x^3.$

Method #2. (Smart)

$$u(x,y) = xy - 3x^2y - y^3$$

($z = x + iy$, the above resembles imaginary parts of z^2 and z^3.)

More precisely,

$$u(x, y) = \operatorname{Im}\left(\frac{1}{2} z^2 + z^3 \right).$$

Thus, $v = -\operatorname{Re}\left(\frac{1}{2} z^2 + z^3 \right)$ works. ∎

Q3.

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3. Find the radius of convergence of the following power series :

a) $\sum_{n=0}^{\infty} nz^n$

b) $\sum_{p \text{ prime}} z^p$

c) $\sum \frac{n!z^n}{n^n}$

(a) Ratio test. Note $L = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists.

Indeed $\lim_{n\to\infty} \frac{n}{n+1} = 1.$

$\therefore L = 1 \leftarrow$ Radius of convergence.

(c) Ratio test: $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{(n+1)^n}{n^n} \cdot (n+1)$$

$$= \lim_{n\to\infty} 1 \cdot \left(1+\frac{1}{n}\right)^n$$

$$= e.$$

$\therefore$ RoC $= \underline{\underline{e}}.$

(b) $\sum_{p:\text{prime}} z^p = \sum_{n\geq 1} a_n z^n,$

where

$$a_n := \begin{cases} 1, & n \text{ is prime,} \\ 0, & n \text{ is not prime.} \end{cases}$$

0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, ...

1

0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, ... $\begin{cases} 0, & n \text{ is not prime.} \\ 1 & \end{cases}$

<u>Root test:</u> $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$.

<u>Claim.</u> For every n, the supremum:

$$\sup\{a_m : m \geq n\} \quad \text{is equal to } 1.$$

<u>Proof.</u> There are infinitely many primes, i.e., given any $n \in \mathbb{N}$, $\exists\, m \geq n$ s.t. m is prime. ∎

Thus, $\alpha = \lim_{n\to\infty} 1 = 1$.

$$\therefore \text{RoC} = \frac{1}{1} = \underline{\underline{1}}.$$

Q4.

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4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not neccessarily imply that the series is divergent.

Lecture 3 slides:

Theorem (Ratio Test)

For a series $\sum_{i=1}^{\infty} a_i$, *let* $L = \limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. *Then, if* $L < 1$, *the series converges absolutely.*

<u>Remark</u> $L > 1$ in the above test doesn't imply that the series diverges. (Exercise!)

<u>Sol^n</u> Take a_n to be:

$$\underset{a_1}{\frac{1}{1^2}}, \underset{a_2}{\frac{1}{1^3}}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{4^2}, \frac{1}{4^3}, \dots$$

. . .

That is,

$$a_{2n} = \frac{1}{n^3}, \quad \text{and}$$

$$a_{2n-1} = \frac{1}{n^2}.$$

<u>Claim 1.</u> $\sum_{n=1}^{\infty} a_n$ converges

<u>Proof.</u> Because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converge. ∎

How? <u>Way #1.</u> Integral test.

<u>Way #2.</u> Comparison: $0 < \frac{1}{n^3} \leq \frac{1}{n^2} \leq \frac{1}{n(n-1)}$

$$\frac{\;}{n^3} \le \frac{\;}{n^2} \le \frac{\;}{n(n-1)}$$

$\sum \frac{1}{n(n-1)}$ telescopes. for $n \geq 2$.

Claim 2. $L = \infty$.

Proof. $$\left|\frac{a_{2n+1}}{a_{2n}}\right| = \frac{1/(n+1)^2}{1/n^3} = \frac{n^3}{(n+1)^2} \to \infty.$$

Thus, $$\limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| \geq \limsup_{n\to\infty} \left|\frac{a_{2n+1}}{a_{2n}}\right| = \infty.$$

$\therefore$ $L = \infty$. $\blacksquare$

To elaborate, consider $b_n := \left|\frac{a_{i+1}}{a_i}\right|$.

We wanted to compute $\limsup_{n\to\infty} b_n$.

I said that: $\limsup_{n\to\infty} b_n \geq \limsup_{n\to\infty} b_{2n}$.

$b_1, \underbrace{b_2, b_3, b_4, \ldots}$ $\longrightarrow \sup\{b_2, b_3, b_4, b_5, \ldots\}$

$\underbrace{b_2, b_4, b_6, \ldots}$ $\longrightarrow$ $\geq$ (VI) $\sup\{b_2, b_4, b_6, \ldots\}$.

Q5.

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5. (1) Construct a infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. (2) Show that there are no holomorphic functions which satisfy this property.

(1) Thought #1: f cannot be analytic.

#2: There is basically only one (or maybe two) example(s) that we know of an inf. diff. function which is not analytic.

$$g(x) := \begin{cases} e^{-1/x} & ; \quad x > 0, \\ 0 & ; \quad x \le 0. \end{cases}$$

g is inf. diff. and vanishes on $(-\infty, 0]$.

Define $f(x) := g(1-x^2)$.

Claim 1. f is inf. differentiable.
Proof. It is a composition of such functions.

Claim 2. f vanishes outside a bounded set.
More precisely, f vanishes outside $(-1, 1)$.

Proof. If $x \notin (-1, 1)$, then $|x| \geq 1$ or $x^2 \geq 1$.

Then, $1 - x^2 \le 0$.
Thus, $f(x) = g(1-x^2) = 0$, since g vanishes on $(-\infty, 0]$.

Claim 3. f is not the zero function.

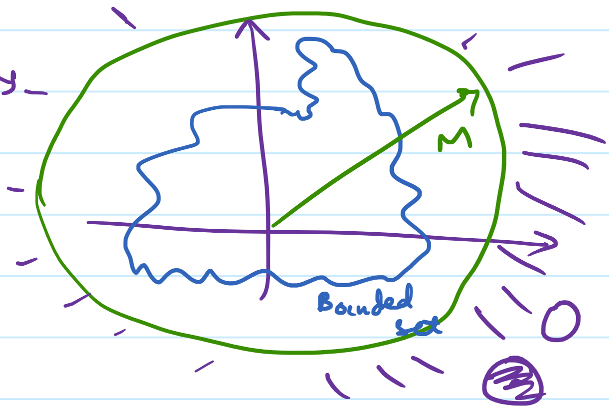
Claim 3. f is not the zero function.

Proof. $f(0) = g(1) = \exp(-1) \neq 0$. ∎

Thus, f fits the bill.

2 We use that holomorphic functions are analytic.
If $f : \mathbb{C} \longrightarrow \mathbb{C}$ vanishes outside a bounded, then its zeroes are non-discrete (WHY?!)
Thus, $f \equiv 0$. ∎

identity theorem



Recall: $S \subseteq \mathbb{C}$ is bounded if $\exists M > 0$ s.t.
$|z| < M \quad \forall\ z \in S$.
In other words, S is a subset of the ball of radius M centered at 0.

Q6.

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$$\exp(z) = w$$
$$\exp(x+iy) = \underbrace{\exp(x)}_{r} \boxed{\underbrace{[\cos(y) + i\sin(y)]}_{\text{unit}}}$$

6. Show that $\exp : \mathbb{C} \to \mathbb{C}^\times$ is onto. ($\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$)

Translation $\longrightarrow$ $\forall w \in \mathbb{C}^\times : \exists z \in \mathbb{C}$ such that $\exp(z) = w$.

So, let $w \in \mathbb{C}^\times$ be arbitrary.

Define $r := |w|$. Note: $r \neq 0$. (Why?! Since $w \in \mathbb{C}^\times$.)

Thus, $\exists x \in \mathbb{R}$ s.t. $\boxed{\exp(x) = r.}$ — (1)

Define $w_0 := \dfrac{w}{r}$.

Note: $|w_0| = \dfrac{|w|}{|r|} = \dfrac{r}{r} = 1$.

Thus, $w_0 = a + ib$ where $a, b \in \mathbb{R}$ s.t. $a^2 + b^2 = 1$.

Thus, $\exists y \in [0, 2\pi)$ s.t. $a = \cos(y)$, $b = \sin(y)$.

In other words, $\boxed{w_0 = \exp(iy).}$ — (2)

From (1) and (2):

$$\exp(x+iy) = \exp(x)\exp(iy)$$
$$= r w_0$$
$$= r \frac{w}{r} = w.$$

(Take $z = x+iy$.) ∎

Q7.

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7. Show that sin, cos : $\mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and cosine which were bounded by 1).

Sol: Will do this for sin. (cos is your homework.)

TS: $\forall w \in \mathbb{C} : \exists z \in \mathbb{C}$ s.t. $\sin(z) = w$.

Let $w \in \mathbb{C}$ be arbitrary.
To solve:

$$e^{iz} - e^{-iz} = 2iw \quad \text{for } z \in \mathbb{C}.$$

$$\Updownarrow$$

$$e^{2iz} - 2iw\, e^{iz} - 1 = 0.$$

For the moment, define $t = e^{iz}$ to see that the above equation is

$$t^2 - 2iw\,t - 1 = 0. \quad (*)$$

By FTA, the above has a root t_0.
Moreover, $t_0 \neq 0$ since 0 does not satisfy $(*)$.

Thus, we can find $z_0 \in \mathbb{C}$ such that $e^{iz_0} = t_0$.
(WHY?!)
(Just use the previous question appropriately. $\exp : \mathbb{C} \to \mathbb{C}^\times$ onto)

Thus, $\sin(z_0) = w$. ∎

Q8.

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8. Show that for any complex number z, $sin^2(z) + cos^2(z) = 1$.

Sol^n. **Method # 1**.

Write $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ and

$\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$.

Compute $\sin^2(z)$ and $\cos^2(z)$. You'll get

$$\sin^2(z) = \frac{e^{2iz} + e^{-2iz} - 2}{-4},$$

$$\cos^2(z) = \frac{e^{2iz} + e^{-2iz} + 2}{4}.$$

Add and complete. ∎

Method # 2. Note: sin and cos are holomorphic.

(Sum / diff / composition of holo. fn.)

Define $h(z) := \sin^2(z) + \cos^2(z) - 1$.

$h(0) = 0$ and $h'(z) = 0 \quad \forall z \in \mathbb{C}$.

Since $\mathbb{C}$ is connected, this implies $h \equiv 0$. ∎

Method #3. Let h be as earlier.

Then, h vanishes on $\mathbb{R}$.

Since h is analytic, it must be identically 0.

(Since otherwise, zeroes of h should be isolated.) ∎

Zeroes are isolated

Identity theorem
Zeroes are discrete