Q1.

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1. Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \ a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$. Add: $a_n \neq 0$.

(Assume FTA)

↳ any nonconstant polynomial has a root in $\mathbb{C}$.

Step 1. By FTA, $\exists\, z_0 \in \mathbb{C}$ s.t. $f(z_0) = 0$.

Case 1. $z_0 \in \mathbb{R}$

Then, use "factor theorem" to write

$$f(x) = (x - z_0)\, h(x) \quad \text{for } h(x) \in \mathbb{R}[x].$$

$g(x) = x - z_0$ and $h(x)$ as above fit the bill.

Case 2. $z_0 \notin \mathbb{R}$.

Claim. $f(\bar{z}_0) = 0$.

Proof.

$$\begin{aligned} f(\bar{z}_0) &= a_0 + a_1 \bar{z}_0 + \cdots + a_n \bar{z}_0^{\,n} \\ &= \overline{(a_0 + a_1 z_0 + \cdots + a_n z_0^n)} \\ &= \overline{f(z_0)} \\ &= \bar{0} = 0. \end{aligned}$$

$a_i \in \mathbb{R}$ so $a_i = \bar{a}_i$.

■

Since $z_0 \notin \mathbb{R}$, we have $z_0 \neq \bar{z}_0$.

But $f(z_0) = f(\bar{z}_0) = 0$.

Thus, applying the "factor theorem" twice, we get

$$\begin{aligned} f(x) &= (x - z_0)(x - \bar{z}_0)\, h(x) \\ &= \left(x^2 - (z_0 + \bar{z}_0)\, x + |z_0|^2\right) h(x) \end{aligned}$$

for some $h(x) \in \mathbb{C}[x]$.

Note: $g(x) := (x^2 - (z_0 + \bar{z}_0)x + |z_0|^2) \in \mathbb{R}[x]$.

Thus, we have
$$f(x) = g(x)\,h(x) \quad \text{where } f(x), g(x) \in \mathbb{R}[x] \text{ and } h(x) \in \mathbb{C}[x].$$

Claim. $h(x) \in \mathbb{R}[x]$.

Proof. Write $h(x) = b_0 + b_1 x + \cdots + b_{n-2}x^{n-2}$,
and $g(x) = c_0 + c_1 x + x^2$.

We know $c_0, c_1 \in \mathbb{R}$
Suppose there is some $b_i \in \mathbb{C} \setminus \mathbb{R}$.
Pick i largest with the above property.

$$f(x) = (\underbrace{b_{n-2}x^{n-2} + \cdots}_{\in \mathbb{R}[x]} + b_i x^i + \cdots + b_0)(x^2 + c_1 x + c_0)$$

look at co-eff of x^{i+2}: $a_{i+2} = \overbrace{b_i + \underbrace{b_{i+1}c_1 + b_{i+2}c_0}_{\in \mathbb{R}}}^{\in \mathbb{R}}$

$\therefore b_i \in \mathbb{R}$. A contradiction.
Thus, each b_i is real.

$\therefore h(x) \in \mathbb{R}[x]$ and we have written
$$f(x) = g(x)\,h(x)$$

Thus, we are done.

(Why is $h(x)$ non-constant?
Ans: $\deg(h(x)) \geq n-2 \geq 1$.)

Q2.

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2. Show that a non-constant polynomial $f(z_1, z_2)$ in complex variables z_1 and z_2 and with complex coefficients has infinitely many roots (in $\mathbb{C}^2$). (Assume FTA)

finitely many roots

$$\boxed{f(z_1, z_2) = p_0(z_1) + p_1(z_1) z_2 + \cdots + \underbrace{p_n(z_1)}_{\neq 0} z_2^n}$$

for some $n \geq 0$ and $p_0(z_1), \ldots, p_n(z_1) \in \mathbb{C}[z_1]$.

- Since f is non-constant, one of z_1 or z_2 must "appear". Wlog, assume z_2 "appears". That is, assume $n > 0$ and $p_n(z_1) \neq 0$.
  (otherwise swap z_1 and z_2)
  (That is: the polynomial is not in z_1 alone.)

- Recall the fact: A nonzero polynomial (in one variable) has finitely many roots.

Let $A :=$ roots of $p_n(z_1)$.
Thus, $A \subseteq \mathbb{C}$ is finite.

For $\alpha \in \mathbb{C} \setminus A$, define the new polynomial

$$\begin{aligned} f_\alpha(z_2) &:= f(\alpha, z_2) \\ &= \underset{\in \mathbb{C}}{p_0(\alpha)} + \underset{\in \mathbb{C}}{p_1(\alpha)} z_2 + \cdots + \underset{\in \mathbb{C}}{p_n(\alpha)} z_2^n. \end{aligned}$$

Thus, $f_\alpha(z_2)$ is a complex poly. in one-variable.
Moreover, since $\alpha \notin A$, $p_n(\alpha) \neq 0$.
Since $n > 0$, this means that $f_\alpha(z_2)$ has a root.
(FTA)

Let z_α denote any such root.

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Thus, for each $\alpha \in \mathbb{C} \setminus A$, we have found a root z_α of $f_\alpha(z_2)$.

In turn,

$$(\alpha, z_\alpha) \text{ is a root of } f(z_1, z_2).$$

But $\mathbb{C} \setminus A$ is infinite and thus, we are done.

WRONG ALTERNATIVE:

FTA does not let you factor polynomials of two variables.

So you ABSOLUTELY CANNOT write:

$$f(z_1, z_2) = (z_2 - p_1(z_1))(z_2 - p_2(z_1)) \cdots$$

Consider $f(z_1, z_2) = z_2^{100} - z_1$.

You cannot write it in the above form.

Q3.

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3. Show that the complex plane minus a countable set is path-connected.

FACTS : (1) Countable < uncountable

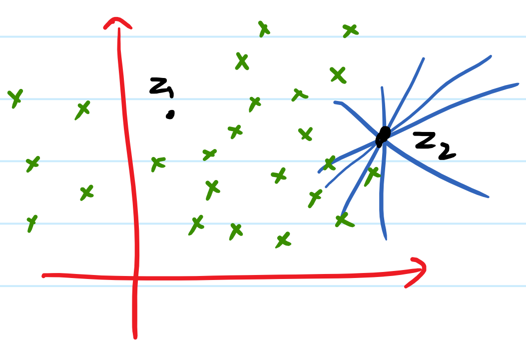
(2) any interval with at least point is uncountable.
$[0, \pi)$ is uncountable.

(Anti pigeonhole) Suppose I have 10 disjoint sets $A_1, \ldots, A_{10}$.
Suppose I remove 6 points (in total) from these sets.
Then, $\exists$ at least two A_i which are unchanged.

(Can replace "10" with "uncountable" and "6" with "countable".)

Sol^n. Let $A \subset \mathbb{C}$ be a countable subset.
TS : $\mathbb{C} \setminus A$ is path-connected.

Let $z_1, z_2 \in \mathbb{C} \setminus A$ be arbitrary. (Wlog $z_1 \neq z_2$.)



Consider $\mathcal{L}_2$ to be the set of all lines passing through z_2.

Claim 1. $\mathcal{L}_2$ is uncountable.

Proof. There is a bijection

$[0, \pi) \longrightarrow \mathcal{L}_2.$

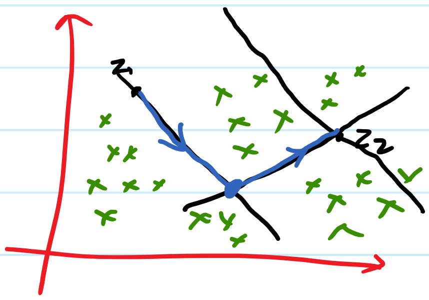
$\theta \longmapsto$ line passing through z_2 making an angle θ with +ve x-axis.

<u>Claim 2.</u> There are at least 2 lines in $\mathcal{L}_2$ which do not contain any point of A. Call them L_1 and L_2.

<u>Proof.</u> The antipigeon hole principle.

Similarly, consider $\mathcal{L}_1$ to be all lines passing through z_1. Then again, $\exists$ one line in $\mathcal{L}_1$ not intersecting A.

This line must intersect one of L_1 or L_2 (it can't be parallel to both).

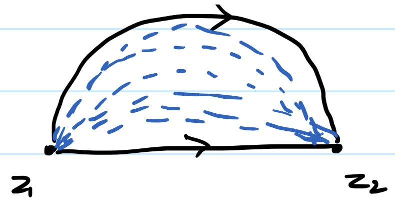


Thus, we have found a path in $\mathbb{C} \setminus A$ connecting z_1 to z_2.

WRONG ALTERNATIVE: By induction on no. of points.

(That is, doing induction on $|A|$ is incorrect.)

CORRECT ALTERNATIVE:



Q4.

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4. Check for real differentiability (RD) and holomorphicity:

1. $f(z) = c$ (Fix $c \in \mathbb{C}$.)
2. $f(z) = z$ → complex diff everywhere defined
3. $f(z) = z^n, n \in \mathbb{Z}$
4. $f(z) = \mathrm{Re}(z)$ → real diff every where, complex nowhere

1. f is RD with total derivative $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ at every point.

Easiest way to note that:

Claim. f is complex diff everywhere with $f'(z_0) = 0 \quad \forall z_0 \in \mathbb{C}$.

Proof. $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{c - c}{z - z_0} = 0. \qquad \blacksquare$$

Thus, f is RD everywhere and holomorphic on $\mathbb{C}$.

2. Again f is complex diff. everywhere.

$f(z) = z$

Proof. $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1. \qquad \blacksquare$$

Same as before.

3. $f(z) = z^n$ for $n \in \mathbb{Z}$.

- $n > 0$. In general, if $g, h : \Omega \to \mathbb{C}$ are (complex) diff. at $z_0 \in \Omega$, then

So is gh.

Thus, by induction, $z \mapsto z^n$ is (complex) diff. everywhere. (Use 2.)

- $n = 0$. Convention $0^0 = 1$.
  Then, f is constant and Q4.1. applies.
  (f is complex diff. every where.)

- $n < 0$. Thus, f is defined on $\mathbb{C} \setminus \{0\}$.

Fact. If $g : \Omega \to \mathbb{C}$ is complex diff. at $z_0 \in \Omega$ and $g(z_0) \neq 0$, then $1/g$ is diff. at z_0.

Thus, $f(z) = z^n = \dfrac{1}{z^{-n}}$ is complex diff. on $\mathbb{C} \setminus \{0\}$.

4. $f(z) = \mathrm{Re}(z)$.

$$\left(\underset{f(x+iy)}{\overset{\shortparallel}{f(z)}} = x \right)$$

$u(x, y) = x$
$v(x, y) = 0$

Thus, $u_x \equiv 1$, $u_y \equiv 0 \equiv v_x \equiv v_y$.

↳ ① All these are continuous everywhere.
Thus, f is RD every where.

② $u_x = 1 \neq 0 = v_y$.
Thus, CR equations hold <u>NOWHERE</u>.
Thus, f is complex diff. nowhere!

5. $f(z) = |z|$ → RD on $\mathbb{R}^2 - \{(0,0)\}$ and CD nowhere (holo nowhere)

6. $f(z) = |z|^2$ → RD everywhere, CD only at 0, holo. nowhere

7. $f(z) = \bar{z}$ → RD everywhere, CR equations hold nowhere, ∴ CD nowhere, holo nowhere

8. $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$ ↳ Check that not continuous at 0.
∴ not RD at 0
For $z_0 \neq 0$, use the properties

$\therefore$ not RD at 0

For $z_0 \neq 0$, use the properties about differentiability of g/f.

(Take $g(z) = z$.)

5. $f(z) = |z|$

$f(x+iy) = \sqrt{x^2+y^2}$.

$u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = 0$

Note: u_x and u_y do not exist at $(0,0)$.

$$u_x(0,0) \overset{def}{=} \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0} \frac{|h|}{h}. \quad \text{DNE!}$$

$\rightarrow$ f is not real differentiable at $(0,0)$.

For $(x_0, y_0) \in \mathbb{R}^2 - \{(0,0)\}$, u_x, u_y, v_x, v_y exist at (x_0,y_0) since u and v are compositions of "nice" functions.

(Recall $x \mapsto \sqrt{x}$ is diff. on $(0,\infty)$.)

For $(x_0, y_0) \neq (0,0)$, we have

$$u_x(x_0,y_0) = \frac{x_0}{\sqrt{x_0^2+y_0^2}}, \quad u_y(x_0,y_0) = \frac{y_0}{\sqrt{x_0^2+y_0^2}},$$

one of these is nonzero

but $v_x(x_0,y_0) = v_y(x_0,y_0) = 0$.

Thus, CR equations do not hold anywhere.

Thus, f is CD nowhere.

6. $f(z) = |z|^2$

$f(x+iy) = x^2+y^2$.

$u(x,y) = x^2+y^2$, $v(x,y) = 0$.

Clearly, f is RD everywhere since u and v

are polynomials

CR: $u_x(x_0, y_0) = 2x_0,$
$u_y(x_0, y_0) = 2y_0,$
$v_x(x_0, y_0) = 0 = v_y(x_0, y_0).$

Thus, CR equations hold only at $(0,0)$.

Since $(CR + RD) \Rightarrow CD$, it follows that f is CD at 0 but nowhere else.