Tutorial 0

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Complex Analysis

bit.ly/ca-205

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↳ B.S. in Mathematics

TA for
MA 205 → Complex Analysis

Tuts → Saturday 3 - 4:30PM
→ I will write live (Just like this.)
↳ Should attend all tuts.

bit.ly/ca-205 → Handwritten solutions (Live)
→ last year's tut solutions
↳ TSC slides are also there
+ writing

- 90 min tutorials → 30 mins. recap
  ↳ 60 mins tut. solving

- Interrupt me midway during tuts.

- Expectations : (i) You should've caught up with the lecs. for that tutorial.
  (ii) You should've attempted the tut. sheet.

- Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. (everywhere)

Is f continuous? (everywhere)

True

- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
  Is f differentiable?

Not necessarily. Define $f(x) := |x|$.
Then, f is not diff, since it is not diff. at 0.

- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
  Is f differentiable somewhere?

False. Weierstrass function. ← one counterexample

$$W(x) := \sum_{n=0}^{\infty} \frac{1}{4^n} \cos\left(\frac{x}{4^n}\right)$$

(Tough chore to do!)

- Let $f : \mathbb{C} \to \mathbb{C}$ be continuous
  Is f differentiable somewhere? (Complex differentiable.)

[Recall: f is differentiable at $z \in \mathbb{C}$ if

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{exists.}$$
]

False.

$$f(z) = W(Re(z))$$
$$f(x+iy) = W(x)$$
$$f(z) = W(|z|)$$

$\boxed{f(z) := \bar{z}.}$ → serves as a counterexample.

$f(z) := \bar{z}$. $\perp$ $\longrightarrow$ serves as a counterexample.

- Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable.
  Is f' continuous?

False. $$f(x) = \begin{cases} x^2 \sin(1/x^2) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Note: f' is not even bounded on $[-1, 1]$.
In particular, it is not (Riemann) integrable
(in the proper sense).

- Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be differentiable.
  Is f' continuous?

True. (Proof will be later.)

More is true: f' is differentiable.

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be infinitely differentiable.

  Let $g(x) := \sum_{n=0}^{\infty} \frac{f'(0)}{n!} x^n$.

  (i) Does g converge for <u>any</u> $x \neq 0$?

  (ii) Assume g converges on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.
  Is it true that
  $$f(x) = g(x) \quad \forall x \in (-\varepsilon, \varepsilon).$$

No, not necessarily.

No, not necessarily.

$$f(x) := \begin{cases} e^{-1/x^2} ; & x \neq 0, \\ 0 ; & x = 0. \end{cases}$$

Can check: f is inf. diff and

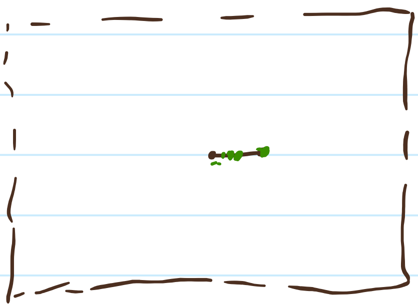
$$f^{(n)}(0) = 0 \quad \forall\, n \geq 0.$$

Thus, $g(x) = 0 \quad \forall\, x \in \mathbb{R}.$

But $g(x) \neq f(x) \quad \forall\, x \in \mathbb{R} \setminus \{0\}.$

(f is not "analytic".)

- If $f : \mathbb{C} \to \mathbb{C}$ is differentiable, then it is analytic.
- "Maximum modulus theorem", "identity theorem"



$f, g : \mathbb{C} \to \mathbb{C}$ diff.

and

$f(z) = g(z) \quad \forall\, z \in [0,1] \subseteq \mathbb{R},$

then $f(z) = g(z) \quad \forall\, z \in \mathbb{C}.$

$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

- What do you think (heuristic) is the cause for this difference between $\mathbb{R}$ and $\mathbb{C}$?

→ One reason is: in $\mathbb{C}$, the limit is "stronger". In $\mathbb{R}$, you only have "two directions" and LHL, RHL suffice.

→ Counter: Real differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ or

$f : \mathbb{R}^n \to \mathbb{R}$

also have this property but they are not as nice.
(Total derivative.)

Polynomials. $\mathbb{K}[x] \to$ polynomials with variable x and co-efficients in $\mathbb{K}$.
($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.)

(1) Let $f(x) \in \mathbb{K}[x]$ be a polynomial and $a \in \mathbb{K}$.
Then, there exists $g(x) \in \mathbb{K}[x]$ such that

$$f(x) - f(a) = (x-a)g(x).$$

Proof. Write $f(x) := a_n x^n + \cdots + a_1 x + a_0$ for $a_0, \dots, a_n \in \mathbb{K}$.

$$f(x) - f(a) = a_n(x^n - a^n) + \cdots + a_1(x-a). \cancel{+ a_0(1-1)}$$

$$\left(x^k - a^k = (x-a)\underbrace{(x^{k-1} + ax^{k-2} + \cdots + a^{k-2}x + a^{k-1})}_{\in \mathbb{K}[x]}\right)$$

Thus, take $x-a$ common from everything on RHS.

$$f(x) - f(a) = (x-a)\left[a_n(x^{n-1} + \cdots + a^{n-1}) + a_{n-1}(x^{n-2} + \cdots + a^{n-2}) + \cdots a_1\right]$$

$$f(x) - f(a) = (x-a)\,g(x). \quad \blacksquare$$

(2) If $f(x) \in \mathbb{K}[x]$ and $a \in \mathbb{K}$ is a root of $f(x)$, i.e., $f(a) = 0$. Then,

$$f(x) = (x-a)g(x) \quad \text{for some } g(x) \in \mathbb{K}[x].$$

(3) How do you think of a polynomial in two variables?

$\mathbb{K}[x]$: $\underline{a_0} + \underline{a_1} x + \cdots + \underline{a_n} x^n$ for $a_i \in \mathbb{K}$.

Two variables. $\mathbb{K}[x, y]$?

(Think of $\mathbb{K}[x,y]$ as $(\mathbb{K}[x])[y]$.)

"not a field"

$$\mathbb{K}[x,y] = \underline{p_0(x)} + \underline{p_1(x)}\, y + \cdots + \underline{p_n(x)}\, y^n$$

for polynomials $p_i(x) \in \mathbb{K}[x]$.

$$\mathbb{C}[x,y] \longrightarrow \underbrace{(i x^2)}_{p_0(x)} + \underbrace{(-3x)}_{p_1(x)} y + \underbrace{(2+i)}_{p_2(x)} y^2$$

$$x^2 y^3 + xy + 5x^{10} y^3 - 43 y^2 + 30 x^{10}$$

$$= \underbrace{(30x^{10})}_{p_0(x)} + \underbrace{(x)}_{p_1(x)} y + \underbrace{(-43)}_{p_2(x)} y^2 + \underbrace{(x^2 + 5x^{10})}_{p_3(x)} y^3$$

- Let $f(x) \in \mathbb{K}[x]$ be a polynomial. Can f have infinitely many roots?

Yes. $f(x) = 0$

Now, assume $f(x) \neq 0$. Can f have infinitely many roots?

No.

Proof. Suppose $a \in \mathbb{K}$ is a root of f. Then, write

$$f(x) = (x - a)\, g(x)$$

for some $g(x) \in \mathbb{K}[x]$

with $\deg(g(x)) = \deg(f(x)) - 1$.

If $b \neq a$ and $f(b) = 0$, then
$(b-a)g(b) = 0$ and hence,
$g(b) = 0$.

$\{\text{roots of } f(x)\} \subseteq \{a\} \cup \{\text{roots of } g(x)\}$.

But $\deg(g(x)) = \deg(f(x)) - 1$.

Use induction to complete the proof. $\blacksquare$

Takeaways for polynomials:

① "Root theorem"
$f(a) = 0 \Rightarrow f(x) = (x-a)g(x)$.

② Polynomial in two variable:
$p_0(x) + p_1(x)y + \cdots + p_n(x)y^n$.

③ Polynomials have finitely many roots.