MA 109: Calculus I

Tutorial Solutions

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Contents

§0. Notations

1. $\mathbb{N} = \{1,\ 2,\ \ldots\}$ denotes the set of natural numbers.
2. $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
3. $\mathbb{Q}$ denotes the set of rational numbers.
4. $\mathbb{R}$ denotes the set of real numbers.
5. $\subset$ is used for subset, not necessarily proper.

$$[0,1] \subset [0,1]$$

   is correct.
6. $\subsetneq$ is used for "proper subset."

§1. Tutorial 1

25th November, 2020

Sheet 1

2. (iv) $\lim_{n\to\infty} (n)^{1/n}$.

   Define $h_n := n^{1/n} - 1$.
   Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. (Why?)

   Now, for $n > 2$, we have

$$\begin{aligned} n &= (1 + h_n)^n \\ &= 1 + nh_n + \binom{n}{2}h_n^2 + \cdots + \binom{n}{n}h_n^n \\ &\geq 1 + nh_n + \binom{n}{2}h_n^2 \\ &> \binom{n}{2}h_n^2 \\ &= \frac{n(n-1)}{2}h_n^2. \end{aligned}$$

   Thus, $h_n < \sqrt{\dfrac{2}{n-1}}$ for all $n > 2$.

   Using Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$ which gives us that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

   (Where did we use that $h_n \geq 0$?)

3. (ii) We show that $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is *not* convergent.

Solution. Note that from the difference formula, we know that if $\{a_n\}$ converges, then

$$\lim_{n\to\infty}|a_{n+1}-a_n|=0.$$

(The limit *exists* and equals 0.)

We show that this is not true for the given sequence. We define

$$b_n := a_{n+1}-a_n,$$

where $\{a_n\}$ is the sequence given in the question.

Then, b_n is given as

$$\begin{aligned} b_n &= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)-(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\\ &= (-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n+1}\right)+(-1)^{n+1}\left(\frac{1}{2}-\frac{1}{n}\right)\\ &= (-1)^{n+1}+(-1)^n\left(\frac{1}{n+1}+\frac{1}{n}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} |b_n| &= \left|1-\left(\frac{1}{n+1}+\frac{1}{n}\right)\right|\\ &= \left|1-\frac{2n+1}{n(n+1)}\right| \end{aligned}$$

From the above, we conclude that

$$\lim_{n\to\infty}|b_n|=1.$$

This shows that a_n does not converge. $\square$

5. (iii) $a_1 = \sqrt{2},\ a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \geq 1.$

Solution. I first describe the general idea.

> The idea in these questions is to first prove a bound on a_n by induction. Then, using that bound we prove that the sequence is convergent.
>
> Once we do that, we then know that $\lim_{n\to\infty} a_n$ exists. Since that also equals $\lim_{n\to\infty} a_{n+1}$, we can take limit on both sides of the equation and solve for the limit L.

First, we prove that the sequence is bounded above.

> Claim 1. $a_n < 6$ for all $n \in \mathbb{N}$.
>
> *Proof.* We shall prove this via induction. The base case $n = 1$ is immediate as $2 < 6$.
> Assume that it holds for $n = k$. Then,
>
> $$a_{k+1} = 3 + \frac{a_n}{2} < 3 + \frac{6}{2} = 6.$$
>
> By principle of mathematical induction, we have proven the claim. □

> Claim 2. $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
>
> *Proof.* $a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n.$ □

Thus, we now know that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Then taking the limit on both sides of

$$a_{n+1} = 3 + \frac{a_n}{2}$$

gives us

$$L = 3 + \frac{L}{2},$$

which we can solve to get $L = 6$. □

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that
$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

Solution. Choose $\epsilon = \dfrac{|L|}{2}$. Note that this is indeed greater than 0.

By the $\epsilon - N$ definition, there exists $N \in \mathbb{N}$ such that
$$|a_n - L| < \epsilon = \frac{|L|}{2}$$
for all $n > N$. Using triangle inequality, we get
$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}.$$
Thus, we get
$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}.$$
Adding $|L|$ on both sides gives us
$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2}$$
for all $n > N$, as desired. $\square$

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   1. $\{a_nb_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
   2. $\{a_nb_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

   Solution. Both the statements are false. We give one counterexample for both.

$$
\begin{aligned}
a_n &:= 1 && \text{for all } n \in \mathbb{N}, \\
b_n &:= (-1)^n && \text{for all } n \in \mathbb{N}.
\end{aligned}
$$

   Clearly, $\{a_n\}_{n\geq 1}$ converges and $\{b_n\}_{n\geq 1}$ is bounded. However, the product is again the latter sequence which does not converge. $\square$

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for some $c \in [a, b]$. Prove or disprove the following statements.

    1. $\lim\limits_{x\to c}[f(x)g(x)] = 0.$
    2. $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if g is bounded.
    3. $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if $\lim\limits_{x\to c} g(x)$ exists.

    Solution.

    1. No. Consider $a = c = 0$ and $b = 1$. Let f, g be defined as
    $$f(x) = x, \quad g(x) = \frac{1}{x}.$$
    Verify that this works as a counterexample.

    2. We prove this statement. Since g is bounded, there exists $M > 0$ such that
    $$|g(x)| < M$$
    for all $x \in (a, b)$. Thus, we have
    $$|f(x)g(x)| \leq M|f(x)|$$
    for all $x \in (a, b)$. Since the LHS is clearly non-negative, using Sandwich theorem proves that
    $$\lim_{x\to c} |f(x)g(x)| = 0.$$
    This also gives us that
    $$\lim_{x\to c} f(x)g(x) = 0.$$
    (Why?)

    3. This is also true. We can simply use that limit of products is the product of limits if the individual limits exist.

$\square$

§2. Tutorial 2

2nd December, 2020

Sheet 1

13. (ii) Discuss the continuity of the following function:

$$f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Solution. For $x \neq 0$, the continuity of f at x follows from the fact that f is the product and composition of continuous functions.

For $x = 0$, we prove continuity using $\epsilon - \delta$. We show that

$$\lim_{x\to 0} f(x) = 0.$$

Since $f(0) = 0$, the continuity of f at 0 will follow.

To this end, let $\epsilon > 0$ be given. We show that $\delta := \epsilon$ works. Indeed, if $0 < |x - 0| < \delta$, then

$$\begin{aligned} |f(x) - 0| &= \left| x\sin\left(\frac{1}{x}\right)\right| \quad |\sin| \leq 1 \\ &\leq |x| \\ &= |x - 0| \\ &< \delta = \epsilon. \end{aligned}$$

Thus, we have shown that

$$0 < |x - 0| < \delta \implies |f(x) - 0| < \epsilon,$$

proving that

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

as desired. $\square$

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

Solution. As earlier, differentiability of f at $x \neq 0$ follows due to product/composition rules.

Now, for $h \neq 0$, note that

$$\frac{f(0+h) - f(0)}{h} = h \sin\left(\frac{1}{h}\right).$$

As saw earlier, the limit of the above as $h \to 0$ exists and is 0. Thus, we get that f is differentiable at 0 as well with $f'(0) = 0$.

Thus, f is differentiable on $\mathbb{R}$.

Now, for $x \neq 0$, we can compute the derivative using product/chain rule. Putting this together, we get

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

We now show that f' is not continuous at 0. We use the sequential criterion for this. Consider the sequence

$$x_n := \frac{1}{2n\pi}, \qquad n \in \mathbb{N}.$$

Clearly, we have that $x_n \to 0$ and $x_n \neq 0$. Thus, we get

$$f'(x_n) = -\cos(2n\pi) = -1.$$

Thus, we see that $f'(x_n) \to -1 \neq f'(0)$.
This shows that f' is not continuous. $\square$

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

Solution. Putting $x = y = 0$, we note that $f(0) = (f(0))^2$. If $f(0) = 0$, show that $f(x) = 0$ for all x and conclude that the given thing is indeed true.

Now, assume that $f(0) \neq 0$. Then, $f(0) = 1$.

Let $c \in \mathbb{R}$ be arbitrary. For $h \neq 0$, we note that

$$\begin{aligned}\frac{f(c+h) - f(c)}{h} &= \frac{f(c)f(h) - f(c)}{h} \\ &= f(c)\frac{f(h) - 1}{h} \\ &= f(c)\frac{f(h) - f(0)}{h}.\end{aligned}$$

Since f is given to be differentiable at 0, the above limit as $h \to 0$ exists and equals $f(c)f'(0)$. Thus, we see that $f'(c)$ exists and equals $f(c)f'(0)$. $\square$

Sheet 1 Optional

7. Let $f : (a, b) \to \mathbb{R}$ be differentiable and $c \in (a, b)$. Show that the following are equivalent:

   (i) f is differentiable at c.

   (ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$, and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and
   $$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

   (iii) There exists $\alpha \in \mathbb{R}$ such that
   $$\lim_{h\to 0} \frac{|f(c + h) - f(c) - \alpha h|}{|h|} = 0.$$

   Solution. We prove this by a usual technique in math by showing that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

   (i) $\implies$ (ii)
   First, we pick $\delta := \min \{c - a, b - c\}$. Note that $\delta > 0$ and $(c-\delta, c+\delta) \subset (a, b)$.

   Now, since f is differentiable at c, $f'(c)$ exists. We define $\alpha := f'(c) \in \mathbb{R}$.
   Now, we define $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ as
   $$\epsilon_1(h) := \begin{cases} \dfrac{f(c + h) - f(c)}{h} - \alpha & h \neq 0, \\ 0 & h = 0. \end{cases}$$

   (Note that $f(c + h)$ above makes sense because $(c - \delta, c + \delta) \subset (a, b)$.)

   Now, from the definition above, it is clear that
   $$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

   We only need to show that $\lim_{h\to 0} \epsilon_1(h) = 0$. However, note that, for $h \neq 0$, we have
   $$\epsilon_1(h) = \frac{f(c + h) - f(c)}{h} - \alpha.$$
   Since $f'(c) = \alpha$, we know that
   $$\lim_{h\to 0} \frac{f(c + h) - f(c)}{h} = \alpha$$

which gives us that $\epsilon_1(h) \to 0$ as $h \to 0$, as desired.

(ii) $\implies$ (iii)
Let α be as in (ii). Then, for $h \neq 0$, we note that

$$\begin{aligned}\frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \frac{|h\epsilon_1(h)|}{|h|} \\ &= |\epsilon_1(h)|.\end{aligned}$$

Since $\lim_{h\to 0} \epsilon_1(h) = 0$, we get that $\lim_{h\to 0} |\epsilon_1(h)| = 0$, which proves the desired limit.

(iii) $\implies$ (i)
We show that the α in (iii) is the derivative of f at c. Note that we are given

$$\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

or

$$\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0.$$

The above gives us that

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

or

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} \right) = \alpha.$$

Thus, $f'(c)$ exists and equals α. □

In the above, we used the following implicitly:

$$\lim_{x\to c} f(x) = 0 \iff \lim_{x\to c} |f(x)| = 0.$$

10. Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

    Solution. We need to show that there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$. Consider $g : [0, 1] \to \mathbb{R}$ defined by

    $$g(x) := f(x) - x.$$

    Then, showing that f has a fixed point is equivalent to showing that g has a zero.

    Note that

    $$g(0) = f(0) \geq 0$$

    and

    $$g(1) = f(1) - 1 \leq 0.$$

    If either of the equalities hold, then we are done. Otherwise, we have

    $$g(0) > 0 \quad \text{and} \quad g(1) < 0.$$

    By intermediate value property, $g(x_0) = 0$ for some $x_0 \in [0, 1]$, as desired. □

Sheet 2

2 Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Solution. The existence of x_0 is given by the intermediate value theorem since 0 lies between $f(a)$ and $f(b)$.

We now show uniqueness. Suppose that there exists $x_1 \in (a, b)$ such that $f(x_1) = 0$ and $x_1 \neq x_0$. We show that this leads to a contraction.
By LMVT, there exists c between x_0 and x_1 such that

$$\begin{aligned} f'(c) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= 0. \end{aligned}$$

A contraction since $c \in (a, b)$ and we were given that $f'(x) \neq 0$ for any $x \in (a, b)$. ☐

5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

   Solution. If $a = b$, then the inequality is clear. Suppose that $a \neq b$.

   Then, there exists c between a and b such that

$$\sin'(c) = \frac{\sin a - \sin b}{a - b}.$$

   Note that $\sin' = \cos$ and thus,

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos(c)| \leq 1.$$

   Cross-multiplying gives us the desired result. $\square$

§3. Tutorial 3

9th December, 2020

Sheet 2

8. In each case, find a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies all the given conditions, or else show that no such function exists.

   (ii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$.

   Solution. $f(x) := x + \dfrac{x^2}{2}$ is one such. Justify. □

   (iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$.

   Solution. Not possible.
   Assume not. As f'' is nonnegative, f' must be increasing everywhere. We are given that $f'(0) = 1$.

   Thus, given any $c > 0$, we know that

   $$f'(c) \geq 1. \qquad (*)$$

   Let $x \in (0, \infty)$. By MVT, we know that there exists $c \in (0, x)$ such that

   $$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$

   Thus, by $(*)$, we have it that $f(x) \geq x + f(0)$ for all positive x.

   This contradicts that $f(x) \leq 100$ for all positive x. (How?) □

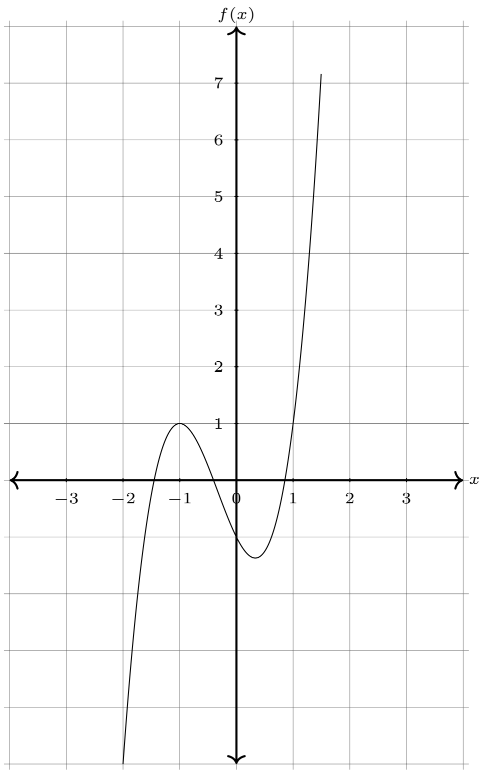
10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

    (i) $f(x) = 2x^3 + 2x^2 - 2x - 1$

    Solution. Note that this is a cubic and can have at most 3 roots. It is easy to locate that they're in $(-2, -1)$, $(-1, 0)$ and $(0, 1)$ since f changes signs consecutively at $-2, -1, 0, 1$.

    Moreover, f' has nice roots: -1 and $1/3$.

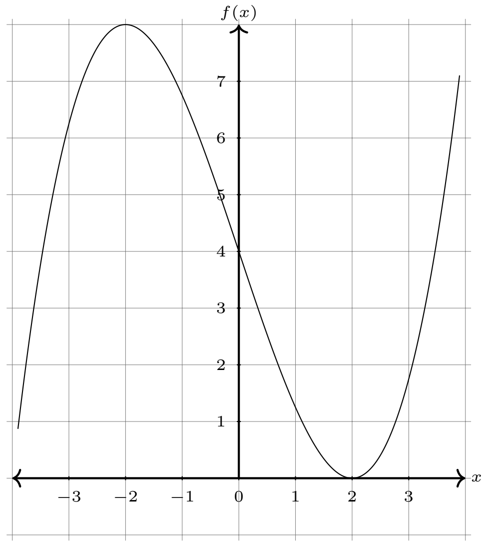
    Lastly, f'' has a root at $-1/3$. Using the above, we get pretty much all we want. Calculating $f(-1)$, $f(1/3)$ and $f(-1/3)$ also tells us the location of the roots with respect to minima/maxima and inflection point.



Above is the graph. □

11. Sketch a continuous curve $y = f(x)$ having all the following properties:
$f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ f'(-2) = f(2) = 0;$
$f'(x) > 0$ for $|x| > 2,\ f'(x) < 0$ for $|x| < 2;$
$f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0.$

Solution. Here is the graph:



I have actually graphed a polynomial that satisfies the given properties.
Can you come up with it?
Is there a unique such polynomial?
What's the minimum degree of such a polynomial?
Is there a unique polynomial with that degree?
Suppose you have two distinct polynomials f and g that satisfy the given conditions. Can you come up with a distinct third polynomial such that it satisfies the conditions as well? □

Sheet 3

1. Write down the Taylor series for $\arctan x$ about the point 0. Write down a precise remainder $R_n(x)$.

   Solution. For each of notation, let $f(x) := \arctan x$ and $g(x) := \dfrac{1}{1+x^2}$. Note that $f' = g$.

   Note that if $n \ge 1$, then $f^{(n)}(0) = g^{(n-1)}(0)$. For g, we have the easy Taylor expansion as
   $$g(x) = 1 - x^2 + x^4 - \cdots$$
   which is valid for $x \in (-1, 1)$.

   Thus, we easily see that
   $$g^{(n)}(0) = \begin{cases} 0 & n \text{ is odd,} \\ (-1)^{n/2} n! & n \text{ is even.} \end{cases}$$

   Thus,
   $$f^{(n)}(0) = \begin{cases} 0 & n \text{ is even,} \\ (-1)^{(n-1)/2}(n-1)! & n \text{ is odd.} \end{cases}$$
   (The above is for $n \ge 1$.) Using this, we get the $(2n+1)$-th Taylor polynomial as
   $$\begin{aligned} P_{2n+1}(x) &= \sum_{k=0}^{2n+1} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \sum_{k=1}^{2n+1} \frac{f^{(n)}(0)}{n!} x^n \\ &= 0 + x - \frac{2!}{3!} x^3 + \cdots + \frac{(-1)^n (2n)!}{(2n+1)!} x^{2n+1} \\ &= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n}{2n+1} x^{2n+1}. \end{aligned}$$
   Since $f^{(2n)} = 0$, we see that
   $$P_{2n}(x) = P_{2n-1}(x)$$
   for $n \ge 1$.

   This solves the problem for finding the Taylor polynomial. Now we solve for the remainder.

Once again, note that
$$g(t) = 1 - t^2 + t^4 - \cdots .$$
For $n \geq 1$, we note that
$$\begin{aligned} g(t) &= [1 - t^2 + \cdots + (-1)^n t^{2n}] + (-1)^{n+1} t^{2n+2} [1 - t^2 + \cdots] \\ &= [1 - t^2 + \cdots + (-1)^n t^{2n}] + (-1)^{n+1} \frac{t^{2n+2}}{1 + t^2} \end{aligned}$$
Integrating both sides from 0 to x gives
$$f(x) = P_{2n+1}(x) + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1 + t^2} \mathrm{d}t.$$
Thus, the term in red is the $(2n+1)$-th remainder $R_{2n+1}(x)$. Conclude as before, for $R_{2n}(x)$. $\square$

2. Write down the Taylor series of the polynomial $x^3 - 3x^2 + 3x - 1$ about the point 1.

   Solution. As one can easily calculate, we have

   $$f^{(n)}(1) = \begin{cases} 6 & n = 3 \\ 0 & n \neq 3, \end{cases}$$

   for $n \geq 0$. Thus, we get the Taylor "series" to actually be the following finite sum:

   $$\frac{f^{(3)}(1)}{3!}(x-1)^3.$$

   In other words, the Taylor series is simply $(x-1)^3$. □

4. Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. Prove that it converges as follows. Choose $N > 2|x|$. We see that for all $n > N$,
$$\frac{x^{n+1}}{(n+1)!} \leq \frac{1}{2}\frac{|x|^n}{n!}.$$
It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$), convergent.

Solution. If $N > 2|x|$ and $n > N$, then
$$\begin{aligned} \left|\frac{x^{n+1}}{(n+1)!}\right| &= \left|\frac{x^n}{n!}\right|\left|\frac{x}{n+1}\right| \\ &\leq \left|\frac{x^n}{n!}\right|\left|\frac{x}{N}\right| && n+1>n>N \\ &\leq \frac{1}{2}\left|\frac{x^n}{n!}\right|. && N > 2|x| \end{aligned}$$
Thus, we can repeatedly use the above to get:
$$\left|\frac{x^{n+1}}{(n+1)!}\right| \leq \frac{1}{2}\left|\frac{x^n}{n!}\right| \leq \cdots \leq \frac{1}{2^{n+1-N}}\left|\frac{x^N}{N!}\right|.$$
Let $s_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}$.

Now, given $m > n > N$, we have
$$\begin{aligned} |s_m(x) - s_n(x)| &= \left|\sum_{k=n+1}^{m} \frac{x^k}{k!}\right| \\ &\leq \sum_{k=n+1}^{m} \left|\frac{x^k}{k!}\right| \\ &= \left|\frac{x^{n+1}}{(n+1)!}\right| + \cdots + \left|\frac{x^m}{m!}\right| \\ &\leq \frac{|x|^N}{N!}\left(\frac{1}{2} + \cdots + \frac{1}{2^{m-n}}\right) \\ &\leq \frac{|x|^N}{N!}. \end{aligned}$$
Note that given any $\epsilon > 0$, we can pick $N \in \mathbb{N}$ such that $\dfrac{|x|^N}{N!} < \epsilon$. Conclude Cauchy-ness. $\square$

5. Using Taylor series, write down a series for

$$\int \frac{e^x}{x}\mathrm{d}x.$$

Solution. Note that

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}.$$

Dividing by x gives

$$\frac{e^x}{x} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}.$$

Integrating both sides gives us

$$\int \frac{e^x}{x}\mathrm{d}x = C + \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$$

□

§4. Tutorial 4

16th December, 2020

Sheet 4

2. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)\mathrm{d}x = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

Solution. For the first part, let

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

be an arbitrary partition of $[a, b]$. Note that

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x) \geq 0$$

for all $0 \leq i \leq n - 1$. (This is because 0 is a lower bound of f.)

Thus, we get that $L(f, P) \geq 0$.

In turn, we see that $L(f) \geq 0$, since $L(f)$ is the supremum of $L(f, P)$ over <u>all</u> partitions P of $[a, b]$. Since f is given to be Riemann integrable, we know that the integral is $L(f)$ and we are done.

For the second part, we prove by contrapositive. That is, if $f(x) \neq 0$ for some $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \neq 0$.

Suppose $c \in [a, b]$ is such that $f(c) \neq 0$. As $f(x) \geq 0$ for all $x \in [a, b]$, we have that $f(c) > 0$. Let $\epsilon := f(c)$.

As f is continuous, there is a $\delta > 0$ such that if $x \in [a, b]$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon/2$ which implies that $\epsilon/2 < f(x)$.

Note that even if $c = a$ or $c = b$, the above shows that we can find $c \in (a, b)$ with $f(c) > 0$. Thus, WLOG we may assume that $c \in (a, b)$. Moreover, we may also assume that $\delta > 0$ is small enough so that $(c - \delta, c + \delta) \subset (a, b)$.

Now, consider the partition of $[a, b]$ given as

$$P = \{a, c - \delta/2, c + \delta/2, b\}.$$

Now, note that
$$\inf_{x\in[c-\delta/2,c+\delta/2]} f(x) \geq \frac{\epsilon}{2}.$$

Thus, $L(P, f) > 0$. As $L(f)$ is the supremum over all such $L(P, f)$, we see that $L(f) > 0$. Since f is given to be Riemann integrable, we know that the integral is $L(f)$ and we are done. ☐

Here is an alternate easier solution for both the parts:

Solution. Consider the trivial partition $P_0 = \{a, b\}$ of $[a, b]$. Clearly,
$$\inf_{x\in[a,b]} f(x) \geq 0.$$

Thus,
$$L(f, P_0) = \left[\inf_{x\in[a,b]} f(x)\right][b-a] \geq 0$$

and hence,
$$L(f) \geq L(f, P_0) \geq 0.$$

Since f is given to be Riemann integrable, we know that the integral is $L(f)$ and we are done.

Second part:

Define $F : [a, b] \to \mathbb{R}$ as
$$F(x) := \int_a^x f(t)\mathrm{d}t.$$

Note that since f is continuous, F is differentiable with $F' = f$. (FTC Part I)

Thus, we get that $F' = f \geq 0$ and hence, F is increasing. Thus, we get
$$F(a) \leq F(x) \leq F(b)$$

for all $x \in [a, b]$. However, note that $F(a) = 0 = F(b)$ and hence, F is constant. Thus,
$$f(x) = F'(x) = 0,$$

for all $x \in [a, b]$, as desired. ☐

(b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\displaystyle\int_a^b f(x)\mathrm{d}x = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

Solution. Let $a = 0, b = 2$ and $f : [a, b] \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 0 & x \neq 1, \\ 1 & x = 1. \end{cases}$$

Show that f is actually Riemann integrable on $[0, 2]$ with the integral equal to 0. $\square$

3. Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an appropriate Riemann sum for a suitable function over a suitable interval.

   (ii) $S_n = \sum_{i=1}^{n} \dfrac{n}{i^2+n^2}$.

   (iv) $S_n = \dfrac{1}{n}\sum_{i=1}^{n} \cos\left(\dfrac{i\pi}{n}\right)$.

   Solution. For both the parts, we shall use the following theorem:

   > **Theorem 1**
   >
   > Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Suppose that (P_n, t_n) is a sequence of tagged partitions of $[a,b]$ such that $\|P_n\| \to 0$.
   > Then,
   > $$\lim_{n\to\infty} R(f, P_n, t_n) = \int_a^b f(x)\mathrm{d}x.$$

   Note very carefully in the above that we already need to know that f is Riemann integrable.

   (ii) Note that

   $$S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2} = \sum_{i=1}^{n} \frac{1}{\left(\frac{i}{n}\right)^2+1}\left(\frac{i}{n} - \frac{i-1}{n}\right).$$

   Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \tan^{-1} x$.
   Then, we have that $f'(x) = \dfrac{1}{x^2+1}$.

   As f' is continuous and bounded, it is (Riemann) integrable.
   For $n \in \mathbb{N}$, let
   $$P_n := \{0, 1/n, \ldots, n/n\}.$$

   The tags are given as follows: For the interval $\left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$, we pick the point $\dfrac{i}{n}$; for each $i = 1, \ldots, n$.

   This collection corresponding to P_n is denoted by t_n. Thus, we get a sequence (P_n, t_n) of tagged partitions.

   Then, $S_n = R(f', P_n, t_n)$. Since $\|P_n\| = 1/n \to 0$, it follows that

   $$\lim_{n\to\infty} R(f', P_n, t_n) = \int_0^1 \frac{1}{x^2+1}\mathrm{d}x = \int_0^1 f'(x)\mathrm{d}x.$$

By FTC Part II, we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\mathrm{d}x = f(1) - f(0) = \frac{\pi}{4}.$$

(iv) Note that

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right) = \sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \pi^{-1}\sin(\pi x)$.
Then, we have that $f'(x) = \cos(\pi x)$.

As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let

$$P_n := \{0, 1/n, \ldots, n/n\}.$$

The tags are given as follows: For the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, we pick the point $\frac{i}{n}$; for each $i = 1, \ldots, n$.

This collection corresponding to P_n is denoted by t_n. Thus, we get a sequence (P_n, t_n) of tagged partitions.

Then, $S_n = R(f', P_n, t_n)$. Since $\|P_n\| = 1/n \to 0$, it follows that

$$\lim_{n\to\infty} R(f', P_n, t_n) = \int_0^1 \cos(\pi x)\mathrm{d}x = \int_0^1 f'(x)\mathrm{d}x.$$

By FTC Part II, we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\mathrm{d}x = f(1) - f(0) = 0. \qquad \square$$

4. (b) Compute $F'(x)$, if for $x \in \mathbb{R}$

(i) $F(x) = \displaystyle\int_1^{2x} \cos(t^2)\mathrm{d}t.$

(ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)\mathrm{d}t.$

Solution. For both the parts, we shall use the following theorem:

Theorem 2

Let $v : \mathbb{R} \to \mathbb{R}$ be differentiable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Fix $a \in \mathbb{R}$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) := \int_a^{v(x)} g(t)\mathrm{d}t.$$

Then,

$$F'(x) = g(v(x))v'(x).$$

Note that using the above, we can state the more general result for when the lower limit is also a differentiable function.

Proof. First, define $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) := \int_a^{x} g(t)\mathrm{d}t.$$

By FTC Part I, we know that G is differentiable and

$$G'(x) = g(x).$$

On the other hand, note that

$$F(x) = G(v(x)).$$

An application of chain rule yields

$$F'(x) = G'(v(x))v'(x) = g(v(x))v'(x).$$ □

Both the parts are now solved easily.

(i) We have $a = 1$, $g(t) = \cos(t^2)$ and $v(x) = 2x$. Thus, $v'(x) = 2$ and

$$\boxed{F'(x) = 2\cos(4x^2).}$$

(ii) We have $a = 0,$ $g(t) = \cos(t)$ and $v(x) = x^2$. Thus, $v'(x) = 2x$ and

$$\boxed{F'(x) = 2x\cos(x^2).}$$

□

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin\left[\lambda(x-t)\right] \mathrm{d}t.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Solution. Just brute calculation. Note that

$$\begin{aligned}
g(x) &= \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \mathrm{d}t \\
&= \frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t\right) \mathrm{d}t \\
&= \frac{1}{\lambda} \sin \lambda x \int_0^x f(t) \cos \lambda t \mathrm{d}t - \frac{1}{\lambda} \cos \lambda x \int_0^x f(t) \sin \lambda t \mathrm{d}t.
\end{aligned}$$

Now, we can differentiate g using product rule and FTC Part I.

$$g'(x) = \cos \lambda x \int_0^x f(t) \cos \lambda t \mathrm{d}t + \sin \lambda x \int_0^x f(t) \sin \lambda t \mathrm{d}t$$

Since the limits of integrals appearing in the expressions for g and g' are both from 0 to x, we see that $g(0) = 0 = g'(0)$.

We can differentiate g' in a similar way and get,

$$\begin{aligned}
g''(x) &= -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t dt + f(x) \cos^2 \lambda x + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t dt \\
&\quad + f(x) \sin^2 \lambda x \\
&= f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t\right) dt \right) \\
&= f(x) - \lambda^2 g(x).
\end{aligned}$$

Rearranging the above gives

$$g''(x) + \lambda^2 g(x) = f(x)$$

□

§5. Tutorial 5

23rd December, 2020

Sheet 5

4. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions for $(x, y) \in \mathbb{R}^2$ are continuous:

   (i) $f(x) \pm g(x)$,

   (ii) $f(x)g(y)$,

   (iii) $\max\{f(x), g(y)\}$,

   (iv) $\min\{f(x), g(y)\}$.

   Solution. The idea in all is to use sequential criterion. To recap:

   > **Theorem 3: Sequential criterion**
   >
   > Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $(x_0, y_0) \in \mathbb{R}^2$. Then, h is continuous at (x_0, y_0) if and only if for every sequence $((x_n, y_n))$ converging to (x_0, y_0), we have that
   > $$\lim_{n\to\infty} h(x_n, y_n) = h(x_0, y_0).$$

   The proof of the above is identical to that for the case in one variable.

   We now prove the first two parts.
   Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary. Let $((x_n, y_n))$ be an arbitrary sequence converging to (x_0, y_0). Then we see that $x_n \to x_0$ and $y_n \to y_0$. Now, applying the (usual) sequential criterion of continuity to f and g, we see that

   $$\lim_{n\to\infty} f(x_n) = f(x_0) \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = g(y_0).$$

   Using the usual algebra of limits, we get that

   $$\begin{aligned} \lim_{n\to\infty} [f(x_n) \pm g(y_n)] &= \lim_{n\to\infty} f(x_n) \pm \lim_{n\to\infty} g(y_n) = f(x_0) \pm g(y_0), \\ \lim_{n\to\infty} [f(x_n)g(y_n)] &= \lim_{n\to\infty} f(x_n) \lim_{n\to\infty} g(y_n) = f(x_0)g(y_0). \end{aligned}$$

   Since the sequence was arbitrary, we have shown continuity at(x_0, y_0). Since (x_0, y_0) was arbitrary, we have shown that the desired functions are continuous on $\mathbb{R}^2$.

For the third and fourth parts, use the fact that

$$\min\{a, b\} = \frac{a + b - |a - b|}{2} \quad \text{and} \quad \max\{a, b\} = \frac{a + b + |a - b|}{2}.$$

A similar argument gives the answer, since $|\cdot|$ is continuous. $\square$

For an elaboration of the last argument, see https://aryamanmaithani.github.io/ma-109-tut/handwritten/5.pdf.

6. Examine the following function for the existence of partial derivatives at $(0,0)$.

   (ii) $f(x,y) := \begin{cases} \dfrac{\sin^2(x+y)}{|x|+|y|} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

   Solution. We shall show that neither partial derivative exists at $(0,0)$. First, we show this for the partial derivative in the first direction.

   For $h \neq 0$, we note that

$$\begin{aligned} \frac{f(0+h,0) - f(0,0)}{h} &= \frac{\frac{\sin^2(h)}{|h|} - 0}{h} \\ &= \frac{\sin^2 h}{h\,|h|} \end{aligned}$$

   It is easy to see that

$$\lim_{h \to 0} \frac{\sin^2 h}{h\,|h|}$$

   does not exist. (Consider the RHL and LHL.)

   Thus, we see that $\dfrac{\partial f}{\partial x_1}(0,0)$ does not exist. A similar computation shows the same for the second partial as well. □

8. Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y) & \text{if } x \neq 0,\ y \neq 0, \\ x\sin(1/x) & \text{if } x \neq 0,\ y = 0, \\ y\sin(1/y) & \text{if } x = 0,\ y \neq 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

Solution. To show continuity: First, for $(x,y) \neq (0,0)$, note that

$$|f(x,y)| \leq |x| + |y| \leq \sqrt{2}\sqrt{x^2 + y^2}.$$

The first inequality follows by taking the three cases and the second by simply squaring and verifying.

The above can be written as

$$|f(x,y) - f(0,0)| \leq \sqrt{2}\|(x,y) - (0,0)\|.$$

Thus, given any $\epsilon > 0$, $\delta = \epsilon/\sqrt{2}$ works in the definition of continuity.

Now, we show neither partial derivative exists. The calculations are similar and we show only the first. For $h \neq 0$, we note that

$$\frac{f(h,0) - f(0,0)}{h} = \frac{h\sin(1/h)}{h} = \sin\left(\frac{1}{h}\right).$$

The limit of the above expression as $h \to 0$ does not exist. Thus, we are done. $\square$

10. Let $f(x,y)=0$ if $y=0$ and

$$f(x,y)=\frac{y}{|y|}\sqrt{x^2+y^2}$$

otherwise. Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every unit vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Solution. For continuity: Let $(x,y)\neq(0,0)$. If $y=0$, then

$$|f(x,y)-f(0,0)|=0$$

and if $y\neq 0$, then

$$|f(x,y)-f(0,0)|=\sqrt{x^2+y^2}=\|(x,y)-(0,0)\|.$$

The cases put together give

$$|f(x,y)-f(0,0)|\leq\|(x,y)-(0,0)\|.$$

Thus, $\delta=\epsilon$ works as before.

Now to see the partial derivatives: We can write $\underline{u}=(u_1,u_2)$. Note that $u_1^2+u_2^2=1$.

If $u_2=0$, then for $t\neq 0$, note that

$$\begin{aligned}\frac{f(0+u_1t,0+u_2t)-f(0,0)}{t}&=\frac{f(u_1t,0)-0}{t}\\&=\frac{0-0}{t}=0.\end{aligned}$$

Clearly, the above limit exists as $t\to 0$ and is 0.

Now, for $u_2\neq 0$ and $t\neq 0$, note that

$$\begin{aligned}\frac{f(0+u_1t,0+u_2t)-f(0,0)}{t}&=\frac{f(u_1t,u_2t)-0}{t}\\&=\frac{1}{t}\frac{u_2t}{|u_2t|}\sqrt{(u_1^2+u_2^2)t^2}\\&=\frac{1}{t}\frac{u_2t}{|u_2t|}|t|\\&=\frac{u_2}{|u_2|}.\end{aligned}$$

Clearly, the above limit exists as $t\to 0$ and is $\frac{u_2}{|u_2|}$.

Thus, all directional derivatives exist and we have

$$D_{\underline{u}}f(0,0) = \begin{cases} 0 & u_2 = 0, \\ \frac{u_2}{|u_2|} & u_2 \neq 0, \end{cases}$$

Note that taking $\underline{u} = (1,0)$ and $(0,1)$ recovers the first and second partial derivatives, respectively. We now check for differentiability.

If f is differentiable at $(0,0)$, then the total derivative *must* be

$$A := \begin{bmatrix} \frac{\partial f}{\partial x_1}(0,0) & \frac{\partial f}{\partial x_2}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We now see whether that actually satisfies the limit condition. That is, we must check if

$$\lim_{(h,k)\to(0,0)} \frac{\left| f(0+h,0+k) - f(0,0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right|}{\|(h,k)\|} = 0.$$

We show that that is not case.

For $(h,k) \neq (0,0)$ and $k \neq 0$, we note that

$$\begin{aligned} \frac{\left| f(0+h,0+k) - f(0,0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right|}{\|(h,k)\|} &= \frac{|f(0+h,0+k) - f(0,0) - 0h - 1k|}{\sqrt{h^2+k^2}} \\ &= \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right| \end{aligned}$$

Note that along the curve $h = k$ with $(h,k) \neq (0,0)$, we see that the above expression equals

$$\left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right| = \left| \frac{k}{|k|} - \frac{k}{\sqrt{2k^2}} \right| = \left(1 - \frac{1}{\sqrt{2}} \right)$$

and the limit of *that* is not 0 as $k \to 0$.

Thus, we see that the original limit (which was supposed to be 0) also does not equal 0. Thus, f is not differentiable at $(0,0)$. $\square$

Note that we haven't actually shown that the limit equals $1 - \frac{1}{\sqrt{2}}$. (In fact, it doesn't exist.) All we have shown is that the limit is not 0.

§6. Tutorial 6

30th December, 2020

Sheet 6

2. Find the directions in which the directional derivative of $f(x, y) := x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

   Solution. Note that f is differentiable and thus, given a *unit* vector u, one has
   $$D_u f(1, 0) = (\nabla f(1, 0)) \cdot u.$$
   The gradient is easy to compute. Indeed, one notes that
   $$f_x(x_0, y_0) = 2x_0 + y_0 \cos(x_0 y_0) \quad \text{and} \quad f_y(x_0, y_0) = x_0 \cos(x_0 y_0).$$
   Thus, we get
   $$\nabla f(1, 0) = \begin{bmatrix} f_x(1, 0) & f_y(1, 0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$
   Now, we assume an arbitrary unit vector as $u = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$ for some $\theta \in [0, 2\pi)$.

   Taking the dot product and equating to it to 1 gives us
   $$2\cos\theta + \sin\theta = 1.$$
   Letting $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
   $$\begin{aligned} & 2\cos\theta + \sin\theta = 1 \\ \iff & \frac{2}{\sqrt{5}}\cos\theta + \frac{1}{\sqrt{5}}\sin\theta = \frac{1}{\sqrt{5}} \\ \iff & \sin(\theta + \alpha) = \cos\alpha \\ \iff & \sin(\theta + \alpha) = \sin\left(\frac{\pi}{2} - \alpha\right) \end{aligned}$$
   The above can be solved to give exactly two solutions in $[0, 2\pi)$ as:
   $$\theta = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} - 2\alpha.$$
   Note that
   $$\begin{aligned} \sin\left(\frac{5\pi}{2} - 2\alpha\right) &= \sin\left(\frac{\pi}{2} - 2\alpha\right) \\ &= \cos(2\alpha) \\ &= 2\cos^2\alpha - 1 \\ &= -\frac{3}{5} \end{aligned}$$

and similarly, its $\cos$ turns out to be $\frac{4}{5}$.

Thus, the two directions are

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \end{bmatrix}. \qquad \square$$

4. Find $D_{\underline{u}}F(2,2,1)$, where $F(x,y,z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

   Solution. The direction of the (outward) normal of the sphere is simply the same as that of the position vector. Thus, we have
$$\underline{u} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$
   Note that
$$\nabla f(2,2,1) = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix}.$$
   Since f is differentiable, we note that
$$D_{\underline{u}}f(2,2,1) = (\nabla f((2,2,1))) \cdot u.$$
   Thus, we get
$$D_{\underline{u}}f(2,2,1) = -\frac{2}{3}.$$
□

5. Given
$$\sin(x+y)+\sin(y+z)=1, \tag{$*$}$$
find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z)\neq 0$.

Solution. Differentiating ($*$) with respect to x and keeping y constant, we get
$$\cos(x+y)+\cos(y+z)\frac{\partial z}{\partial x}=0$$
or (since $\cos(y+z)\neq 0$),
$$\frac{\partial z}{\partial x}=-\frac{\cos(x+y)}{\cos(y+z)}. \tag{$\star$}$$
Differentiating ($*$) with respect to y and keeping x constant, we get
$$\cos(x+y)+\cos(y+z)\left(1+\frac{\partial z}{\partial y}\right)=0. \tag{$\star\star$}$$
Partially differentiating the above with respect to x gives
$$-\sin(x+y)-\sin(y+z)\left(\frac{\partial z}{\partial x}\right)\left(1+\frac{\partial z}{\partial y}\right)+\cos(y+z)\left(\frac{\partial^2 z}{\partial x \partial y}\right)=0.$$
Using ($\star$) and (a rearrangement of) ($\star\star$), the red term can be replaced to get
$$-\sin(x+y)-\sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)^2+\cos(y+z)\left(\frac{\partial^2 z}{\partial x \partial y}\right)=0.$$
Rearranging the above gives the answer as
$$\frac{\partial^2 z}{\partial x \partial y}=\frac{\sin(x+y)}{\cos(y+z)}+\tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}.$$
$\square$

8. Analyse the following functions for local minima, local maxima and saddle points:

    1. $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$.
    2. $f(x,y) = x^3 - 3xy^2$.

    Solution. The idea in both is to use the second derivative test. Note that both the functions are defined on all of $\mathbb{R}^2$ and all partials of order 2 exist and are continuous. Thus, the second derivative test is applicable.

    1. For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

        Note that $f_x(x,y) = xe^{1/2\left(-x^2-y^2\right)}\left(-x^2+y^2+2\right)$.

        Also, $f_y(x,y) = ye^{1/2\left(-x^2-y^2\right)}\left(-x^2+y^2-2\right)$.

        Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us precisely that

        $$(x_0, y_0) \in \{(0,0),\ (0,\sqrt{2}),\ (0,-\sqrt{2}),\ (-\sqrt{2},0),\ (\sqrt{2},0)\}.$$

        Recall the discriminant defined as

        $$D := f_{xx}\left(x_0, y_0\right) f_{yy}\left(x_0, y_0\right) - f_{xy}\left(x_0, y_0\right)^2.$$

        In our case, the right hand side becomes

        $$-e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4\right).$$

        Moreover, $f_{xx}(x,y) = e^{-\left(x^2+y^2\right)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$

        For $(x_0, y_0) = (0, 0)$, it is clear that it is a saddle point for f as discriminant is $-4 < 0$.

        Note that if $x = 0$, the discriminant reduces to $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$. Substituting $y = \pm\sqrt{2}$ gives us that the discriminant is positive with f_{xx} positive and hence, the points are points of local minima.

        Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima as they have discriminant positive and f_{xx} negative.

        Thus, we get

Point	Type
$(0,0)$	Saddle
$(\pm\sqrt{2},0)$	Maximum
$(0,\pm\sqrt{2})$	Minimum

2. The calculations here are thankfully easier. For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

   Note that $f_x(x, y) = 3x^2 - 3y^2$.

   Also, $f_y(x, y) = -6xy$.

   Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that $(x_0, y_0) = (0, 0)$.

   In our case, the discriminant at a general point is

$$\begin{aligned} D &= f_{xx}\left(x_0, y_0\right) f_{yy}\left(x_0, y_0\right) - f_{xy}\left(x_0, y_0\right)^2 \\ &= -36(x_0^2 + y_0^2). \end{aligned}$$

   Hence, we get that the discriminant test is inconclusive! This means that we must turn to some other analytic methods of determining the nature.

   Now, we note that $f(x, 0) = x^3$ for all $x \in \mathbb{R}$. This means that, given any $r > 0$, we have

$$f\left(\frac{r}{2}, 0\right) > f(0, 0) > f\left(-\frac{r}{2}, 0\right).$$

   However, note that

$$\left(-\frac{r}{2}, 0\right), \left(\frac{r}{2}, 0\right) \in D_r(0, 0).$$

   Thus, $(0, 0)$ can neither be a point of local minima nor of local maxima.

   Thus, $(0, 0)$ is a saddle point, by definition.

□

9. Find the absolute maximum and the absolute minimum of

$$f(x, y) = (x^2 - 4x)\cos y \quad \text{for } 1 \le x \le 3,\ -\pi/4 \le y \le \pi/4.$$

Solution. Note that $f_x(x, y) = (2x - 4)\cos y$ and $f_y(x, y) = -\left(x^2 - 4x\right)\sin y$ for interior points (x, y).
Thus, the only critical point is $p_1 = (2, 0)$.

Now we restrict ourselves to the boundaries to find the local extrema.

"Right boundary:" This is the line segment $x = 3, -\pi/4 \le y \le \pi/4$. The function now reduces to $-3\cos y$ on this segment. Using our theory from one-variable calculus, we get that we need to check the points $(3, 0),\ (3, \pi/4),\ (3, -\pi/4)$. (The first because it's an interior point at which the derivative is 0. The others because they are the boundary points now.)

Similar consideration of the "left boundary" gives us the points $(1, 0)$, $(1, \pi/4)$, $(1, -\pi/4)$.

Now, we look at the "top boundary." The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$. Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4),\ (2, \pi/4),\ (3, \pi/4)$. Similarly, checking the "bottom boundary" gives us the points $(1, -\pi/4),\ (2, -\pi/4),\ (3, -\pi/4)$.

(x_0, y_0)	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$
(x_0, y_0)	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\min} = -4$ at $(2, 0)$ and $f_{\max} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$. $\square$