Calculus I Recap

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IIT Bombay

Autumn Semester 2020-21

Week 1

Start recording!

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Definition 1 (Sequences)

Definition 2 (Convergence)

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Proposition 4 (Convergence $\implies$ Cauchy)

If (a_n) is a convergent sequence in any space X, then (a_n) is Cauchy.

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Exercise: Show that $\mathbb{N}, \mathbb{Z}$ are complete. (What property do you really need? Can you generalise this?)

Now, we digress a bit to see what $\mathbb{R}$ and completeness really means.

It is okay if you don't understand every single thing. It is more or less for you to know "okay, whatever we say works" even if you don't know the exact details why.

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Note that **neither** of the above grey boxes is true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

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For us, all we need to know is that convergence of a series is just the convergence of the sequence of its *partial sums.* Thus, we are back in the case where we study sequences!

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If no such l exists, then we say that f does not have any limit at x_0.

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Similarly, we have the limit at $-\infty$.

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Take doubts.

Week 2

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Note carefully that the domain is an interval.

Now, we state another property, called the extreme value theorem.

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Once again, note that this only talks about "interior points."

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Stop recording. Start a new one.
Take doubts.

Week 3

Start recording!

Week 3

What did we see last week?

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If $f''(x_0) = 0$, then nothing can be concluded.

We now look at concavity and convexity.

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We now look at concavity and convexity. For what follows, I will always denote an interval. (It could be open/close/neither/unbounded.)

Definition 23 (Convex)

A function $f : I \to \mathbb{R}$ is said to be *convex* if for every $x_1, x_2 \in I$ and every $t \in [0, 1]$, we have

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

More graphically, given any two points on the graph of the function, the line segment joining the two points lies above the graph.

The definition of a *concave* function is obtained by replacing $\leq$ with $\geq$ and "above" with "below."

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However, if we do assume that it's differentiable, then we can say some things. If we assume twice differentiability, we can say some more things. I have put a summary of these on the next slide.

Read it some day.

Proposition 24

Suppose $f : I \to \mathbb{R}$ is differentiable. Then

1. f' is increasing on $I \iff f$ is convex on I.
2. f' is decreasing on $I \iff f$ is concave on I.
3. f' is strictly increasing on $I \iff f$ is strictly convex on I.
4. f' is strictly decreasing on $I \iff f$ is strictly concave on I.

Corollary 25

Suppose $f : I \to \mathbb{R}$ is twice differentiable. Then

1. $f'' \geq 0$ on $I \iff f$ is convex on I.
2. $f'' \leq 0$ on $I \iff f$ is concave on I.
3. $f'' > 0$ on $I \implies f$ is strictly convex on I.
4. $f'' < 0$ on $I \implies f$ is strictly concave on I.

Let's now talk about inflection points.

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$$f(x) := \begin{cases} \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

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Here's some more information being thrown at you.

Week 3

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Theorem 27 (Derivative tests)

1. (First derivative test) Suppose f is differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(x_0 - \delta, x_0)$ and f' is decreasing on $(x_0, x_0 + \delta)$, or vice-versa.
2. (Second derivative test) Suppose f is twice differentiable on $(x_0 - r, x_0) \cup (x_0, x_0 + r)$ for some $r > 0$. Then, x_0 is a point of inflection $\iff$ there is $\delta > 0$ with $\delta < r$ such that $f'' \geq 0$ on $(x_0 - \delta, x_0)$ and $f'' \leq 0$ on $(x_0, x_0 + \delta)$, or vice-versa.

Thus, if f is twice differentiable, then x_0 is inflection point iff f'' changes sign. (Note that $f''(x_0)$ is not required to exist. Recall the crazy example.)

The previous slide gives us a **necessary** condition for inflection point. We have the same notation as earlier.

Theorem 28 (Another second derivative test)

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P_2(x) &= f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 \\
&\vdots \\
P_n(x) &= f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
\end{aligned}
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Note that all the Taylor **polynomials** have only **finite**ly many terms, as a polynomial should have.

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$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where P_n is as in the previous slide.

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Week 3

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for all those x. For some nice functions, we get an $R > 0$ such that the above happens for all $x \in (a - R, a + R)$. If such an R exists for all $a \in I$, then f is said to be *analytic*.

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Week 3

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What was the example seen in class that illustrated this?

Stop recording. Start a new one.
Take doubts.

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$$M_i := \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i := \inf_{x \in [x_{i-1}, x_i]} f(x),$$

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for $i = 1, \ldots, n$.

Thus, m_i and M_i denote the infimum and supremum of f over the i-th interval, respectively.

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In the above, note that we have both f and P in the notation. This is crucial because the sums depend on the partition.

Using the earlier sums, we now define the upper and lower Darboux *integrals*. The notations are continuing from earlier.

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We now turn to the definition of Riemann integrals.

Some jargon.

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On the next slide, we state two equivalent definitions of Riemann integrability.

Week 4

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Week 4

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In both the cases above, the Darboux and Riemann integrals are the same.

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Now, we see how derivatives and integrals relate. These are the two parts of the Fundamental Theorem of Calculus.

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In particular, if f is continuous, then Riemann integrability of f is guaranteed and the above equation is true for *all* $c \in (a, b)$.

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Note that the if is crucial. It isn't necessary that the derivative of a function is Riemann integrable. It needn't even be bounded. (But even if it is bounded, it needn't be Riemann integrable. Although an example of this is harder.)

Some pathological remarks:

1. If a function is Riemann integrable, it doesn't mean that it is the derivative of a function. (That is, it needn't have an anti-derivative.)
2. If a function has an anti-derivative, it doesn't mean that it is Riemann integrable. (That is, derivatives needn't be Riemann integrable.)

For the first, take $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$. It cannot be the derivative of any function because it doesn't have IVP. (Recall Theorem 21.)

For the second, consider the derivative of $F : [-1, 1] \to \mathbb{R}$ defined by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$. F' here isn't bounded.

Start recording!

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Note that if $m = 1$, then $\|f(x) - L\|$ is just $|f(x) - L|$.

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Note that if $m = 1$, then $\|f(x) - L\|$ is just $|f(x) - L|$. In fact, for $n = m = 1$, the definition above coincides with the earlier one. (Definition 10.)

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Let $U \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$. c is said to be a *limit point* of U if for every $\delta > 0$,

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To see this, consider $n = 1$ and $U = [0, 1) \cup \{2\}$.

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To see this, consider $n = 1$ and $U = [0, 1) \cup \{2\}$. Then, 1 is a limit point of U while 2 is not.

As before, we can now define continuity easily.

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As before, the case $n = m = 1$ recovers the original one.

Now, let us assume $n = 2$ and $m = 1$.

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Note that the limit above is an ordinary one-variable limit of a real function, as we had seen earlier.

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Note that the limit above is an ordinary one-variable limit of a real function, as we had seen earlier. Also note that b is fixed in the numerator.

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The above says that not only is $c \in U$ but also that there is a "ball" around c contained in U.

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Note that the above says "there exists" and not "for every." Compare this with the definition of "limit point."

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provided it exists.

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As before, this is an ordinary limit. Taking $v = (1, 0)$ and $(0, 1)$ recovers the usual the partial derivatives with respect to x_1 and x_2, respectively.

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Suppose that f is differentiable is (x_0, y_0). Then, both the partial derivatives of f at (x_0, y_0) exist

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Suppose that f is differentiable is (x_0, y_0). Then, both the partial derivatives of f at (x_0, y_0) exist and

$$Df(x_0, y_0) = \left[\frac{\partial f}{\partial x_1}(x_0, y_0) \quad \frac{\partial f}{\partial x_2}(x_0, y_0)\right].$$

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$$Df(x_0, y_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x_0, y_0) & \frac{\partial f}{\partial x_2}(x_0, y_0) \end{bmatrix}.$$

The above matrix is also called the *gradient* and denoted by $\nabla f(x_0, y_0)$.

Week 5

Stop recording. Start a new one.
Take doubts.

Week 6

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If $n = m$, these vector valued functions are called vector fields.

We now look at the derivative of a vector valued function. As earlier, $U \subset \mathbb{R}^m$.

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Note in the above that h is a column matrix in the red space - $\mathbb{R}^m$, that is, the domain space. In the limit, note that the value in the numerator (inside the mod) is in $\mathbb{R}^n$ and denominator in $\mathbb{R}^m$.

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Recall that given a map $f : U \to \mathbb{R}^n$, we get n functions $f_1, \ldots, f_n : U \to \mathbb{R}$. All of these are real valued functions of m variables. Thus, it makes sense to talk about the m partial derivatives.

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Note that partial derivatives (as seen so far) only make sense for real valued functions.

Week 6

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Note that the matrix multiplication makes sense because $Dg(f(x))$ is a $p \times n$ matrix and $Df(x)$ an $n \times m$ matrix. Moreover, the product is a $p \times m$ matrix, as expected.

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A counterexample for the partials not being equal is given on the next slide. Of course, the function is not $\mathcal{C}^2$ in that case.

The promised counterexample:

Example 62 (Inequality of mixed partials)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Then,

$$\frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_1} f\right)(0,0) = -1 \neq 1 = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} f\right)(0,0).$$

Week 6

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Week 6

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And for the last time.

Stop recording. Start a new one.
Take doubts.