Sheet 6 2

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2. Find the directions in which the directional derivative of $f(x,y) := x^2 + \sin xy$ at the point $(1,0)$ has the value 1.

If f is differentiable at a point, ~~then f is continuous at that point and all directional derivatives at that point exist.~~
Moreover,

$$D_u f(x_0, y_0) = (\nabla f(x_0, y_0)) \cdot u$$

for every unit vector u.

Since f here is diff., we see that

$$D_u f(1,0) = [\nabla f(1,0)] \cdot u$$

(dot product)

for every unit vector u.

$$\nabla f(1,0) = [f_x(1,0) \quad f_y(1,0)]$$

$f(x,y) = x^2 + \sin xy$

Note

$$f_x(x_0,y_0) = 2x_0 + y_0 \cos(x_0 y_0)$$
$$f_y(x_0,y_0) = x_0 \cos(x_0 y_0)$$

Thus, $\nabla f(1,0) = [2 \quad 1]$.

Now assume $u = [\cos\theta \quad \sin\theta]$ for $\theta \in [0, 2\pi)$.

Thus, $D_u f(1,0) = 2\cos\theta + 1\sin\theta$.

For $D_u f(1,0) = 1$, we get

$$2\cos\theta + \sin\theta = 1$$

$$\Leftrightarrow \quad \underbrace{\frac{2}{\sqrt{5}}}_{=\sin\alpha} \cos\theta + \underbrace{\frac{1}{\sqrt{5}}}_{=\cos\alpha} \sin\theta = \frac{1}{\sqrt{5}}$$

$\alpha := \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \in [0, \frac{\pi}{2}]$

in fact, $\alpha > \frac{\pi}{4}$

$$\Leftrightarrow \quad \sin(\theta + \alpha) = \cos\alpha$$

$$\Leftrightarrow \quad \sin(\theta + \alpha) = \sin\left(\frac{\pi}{2} - \alpha\right)$$

exactly two solutions in $[0, 2\pi)$.

$\theta + \alpha = \frac{\pi}{2} - \alpha + 2\pi$

$\Rightarrow \theta = \frac{\pi}{2} - 2\alpha + 2\pi$

$$\theta = \frac{\pi}{2} \quad \text{or} \quad \frac{5\pi}{2} - 2\alpha$$

$\sin\theta = 1$, $\cos\theta$

$\sin\theta = \cos 2\alpha$

$= 2\cos^2\alpha - 1$

$= -\frac{3}{5}$

$\cos\theta = \frac{4}{5}$

Thus, $u = \begin{bmatrix} 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$.

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$\frac{\partial F}{\partial x} = 3, \ldots \leftarrow$ continuous $\therefore F$ is diff

4. Find $D_{\underline{u}}F(2,2,1)$, where $F(x,y,z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

$$\underline{u} = \frac{(2,2,1)}{\|(2,2,1)\|} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

Also, $\nabla f(2,2,1) = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix}$.

Since f is diff,

$$D_{\underline{u}} f(2,2,1) = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= -\frac{2}{3}.$$

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5. Given

$$\sin(x+y) + \sin(y+z) = 1, \qquad (*)$$

find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z) \neq 0$.

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

(Implicit function theorem)

$z = f(x, y)$

$\dfrac{\partial^2 f}{\partial x \partial y}$

$\sin(x+y) + \sin(y+z) = 1.$

$\dfrac{\partial}{\partial x}$: $\cos(x+y) + \cos(y+z)\left(\dfrac{\partial z}{\partial x}\right) = 0 \quad (1)$

$\partial/\partial y$: $\cos(x+y) + \cos(y+z)\left(1 + \dfrac{\partial z}{\partial y}\right) = 0 \quad (2)$

$\partial/\partial x$:

$$-\sin(x+y) - \sin(y+z)\underbrace{\left(\frac{\partial z}{\partial x}\right)}_{\text{from (1)}}\underbrace{\left(1+\frac{\partial z}{\partial y}\right)}_{\text{from (2)}} + \cos(y+z)\left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0$$

$$\Downarrow$$

$$-\sin(x+y) - \sin(y+z)\left[-\frac{\cos(x+y)}{\cos(y+z)}\right]^2 + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\Downarrow$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\,\frac{\cos^2(x+y)}{\cos^2(y+z)} \;!!$$

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8. Analyse the following functions for local minima, local maxima and saddle points:
   1. $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$.
   2. $f(x, y) = x^3 - 3xy^2$.

1. Double derivative test.

$f_x(x_0, y_0) = 0 = f_y(x_0, y_0) \iff (x_0, y_0) \in \{(0,0), (\pm\sqrt{2}, 0), (0, \pm\sqrt{2})\}$

Calculate D

$$D = -e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4\right).$$

$$f_{xx}(x, y) = e^{-\left(x^2+y^2\right)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$$

Note $D < 0$ at $(0,0)$. Thus, $(0,0)$ is a saddle point.

$D > 0$ at other four points.

$f_{xx}(\pm\sqrt{2}, 0) < 0 \longrightarrow$ maximum

$f_{xx}(0, \pm\sqrt{2}) > 0 \longrightarrow$ minimum

2. $f(x,y) = x^3 - 3xy^2$

$f_x(x_0, y_0) = 0 = f_y(x_0, y_0) \iff (x_0, y_0) = (0,0)$

That is, $(0,0)$ is the only critical point.

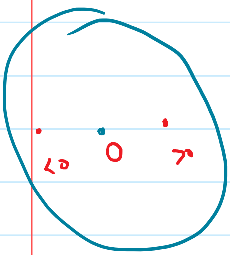
Here, $D = -36(0^2 + 0^2) = 0$

Thus, second derivative test fails.

$f(x, 0) = x^3$; $f(0, 0) = 0$

Now, given any $r > 0$, note that

$$\left(\frac{r}{2}, 0\right) \in D_r(0, 0)$$



and $$f\left(\frac{r}{2}, 0\right) = \left(\frac{r}{2}\right)^3 > 0 = f(0,0).$$

Thus, $(0,0)$ cannot be a local maximum.

Similarly, $$\left(-\frac{r}{2}, 0\right) \in D_r(0,0)$$

and $$f\left(-\frac{r}{2}, 0\right) = \left(-\frac{r}{2}\right)^3 < 0 = f(0,0).$$

Thus, $(0,0)$ cannot be a local minimum either.

Since $(0,0)$ is a critical point which is not a local extremum, it is a saddle point.

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given as $f(x,y) = x^4 + y^4$.

Then $(0,0)$ is the only crit. point.

Moreover, $D = 0$.

However, basic order properties tell us that
$f(x,y) = x^4 + y^4 \geq 0 = f(0,0)$.

Thus, $(0,0)$ is a local minimum.
(Take $r=1$)

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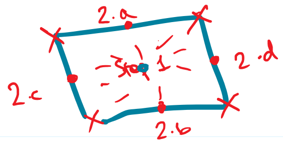

$$R = (1,3) \times \left(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \cup \{1\} \times \left(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \cup \{3\} \times \left(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \cup (1,3) \times \left\{\tfrac{\pi}{4}\right\} \cup (1,3) \times \left\{-\tfrac{\pi}{4}\right\} \cup \{4 \text{ corner points}\}$$

9. Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x^2 - 4x)\cos y \quad \text{for } 1 \le x \le 3,\ -\pi/4 \le y \le \pi/4.$$

Step 1. Locate critical points ~~and extrema~~ (in interior).

Note $f_x(x,y) = (2x-4)\cos y$

$f_y(x,y) = -(x^2 - 4x)\sin y$

Thus, the only critical point is $(2,0)$.

Step 2. Boundary.

2.a. "Top boundary"

$$g_1(x) = f(x, \pi/4) = \frac{x^2 - 4x}{\sqrt{2}} \quad \text{for } x \in [1,3]$$

$$= \frac{(x-2)^2 - 4}{\sqrt{2}}$$

$g_1'(x) = 0 \iff x = 2.$ interior of $[1,3]$

Thus, the points to be checked: $\left(2, \frac{\pi}{4}\right), \left(1, \frac{\pi}{4}\right), \left(3, \frac{\pi}{4}\right)$ boundary points

2.b. "Bottom boundary"

$$g_2(x) = f(x, -\pi/4) = \frac{x^2 - 4x}{\sqrt{2}}$$

Again, same thing gives:

$\left(1, -\frac{\pi}{4}\right), \left(2, -\frac{\pi}{4}\right), \left(3, -\frac{\pi}{4}\right).$

2.c. "Left boundary"

$g_3(y) = f(1, y) = -3\cos y$

$g_3'(y) = 0 \iff y = 0$

Thus, the points are: $(1,0)$, $(1, \frac{\pi}{4})$, $(1, -\frac{\pi}{4})$

2.d. "Right boundary"

The points here are: $(3,0)$, $(3, \pm\frac{\pi}{4})$.

Thus, we know the minima/maxima must occur at one of those nine points.

Writing it in the following table:

(x_0, y_0)	$(2,0)$	$(3,0)$	$(3,\pi/4)$	$(2,\pi/4)$	$(1,\pi/4)$
$f(x_0,y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$
(x_0, y_0)	$(1,0)$	$(1,-\pi/4)$	$(2,-\pi/4)$	$(3,-\pi/4)$	
$f(x_0,y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

$f_{min} = f(2,0) = -4$; $f_{max} = -\frac{3}{\sqrt{2}}$