Sheet 5 2

23 December 2020 13:36

2. Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x, y) = x - y$ (ii) $f(x, y) = x^2 + y^2$ (iii) $f(x, y) = xy$

One way is to study the level sets of the functions. These are the sets of the form $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$, where c is a constant. The level set "lives" in the xy-plane. $\rightarrow \subseteq \mathbb{R}^2$

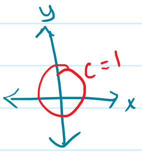
$\mathbb{R}^2 \not\subseteq \mathbb{R}^3$

When one plots the $f(x, y) = c$ for some constant c one gets a curve. Such a curve is usually called a contour line (the contour "lives" in the $z = c$ plane). $\rightarrow \subseteq \mathbb{R}^3$

(ii) $c = -3, -2, -1$

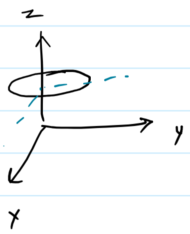
Level set $= \phi$
Contour line $= \phi$

$c = 0.$ Level set : $\{(0, 0)\}$
Contour line : $\{(0, 0, 0)\}$.



$C = 1$: Level set : unit circle in $\mathbb{R}^2$
$L = \{ (\cos\theta, \sin\theta) : \theta \in [0, 2\pi] \}$

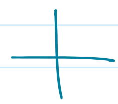
Contour set $= L \times \{1\}$



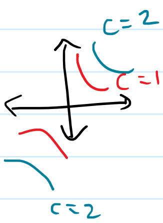
$= \{ (\cos\theta, \sin\theta, 1) : \theta \in [0, 2\pi] \}$

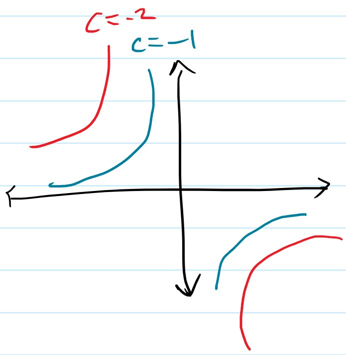
$c = 2, 3$ similar

(iii) $c = 0.$ Level set : $L = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$



Contour : $L \times \{0\}$





Sheet 5 4

23 December 2020 13:36

4. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions for $(x, y) \in \mathbb{R}^2$ are continuous:

   (i) $f(x) \pm g(x)$,

   (ii) $f(x)g(y)$,

   (iii) $\max\{f(x), g(y)\}$,

   (iv) $\min\{f(x), g(y)\}$.

Theorem 3: Sequential criterion

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $(x_0, y_0) \in \mathbb{R}^2$. Then, h is continuous at (x_0, y_0) if and only if for every sequence $((x_n, y_n))$ converging to (x_0, y_0), we have that

$$\lim_{n\to\infty} f(x_n, y_n) = f(x_0, y_0).$$

Idea: Use above and algebraic formulae of sequences.

Let $(x_0, y_0) \in \mathbb{R}^2$.
Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$ which converges to (x_0, y_0).

$$\lim_{n\to\infty} x_n = x_0 \quad \& \quad \lim_{n\to\infty} y_n = y_0$$

Now, f & g are cont. at x_0 & y_0, resp.

seq. criterion

Thus, $\lim_{n\to\infty} f(x_n) = f(x_0)$ & $\lim_{n\to\infty} g(y_n) = g(y_0)$.

$\because$ individual limits exist

Thus, $\lim_{n\to\infty} \left[f(x_n) \pm g(y_n)\right] = \lim_{n\to\infty} f(x_n) \pm \lim_{n\to\infty} g(y_n)$

$= f(x_0) \pm g(y_0)$

Thus, $(x,y) \mapsto f(x) \pm g(y)$ is continuous at (x_0, y_0). [$\because$ (x_n, y_n) was arbitrary]

Since (x_0, y_0) was arbit, the f^n is continuous on $\mathbb{R}^2$.

Similarly, product is continuous.

Using $\min\{a,b\} = \frac{a+b-|a-b|}{2}$ &

$\left[\left|\lim x_n\right| = \lim |x_n| \right]$ assuming $\lim x_n$ exists.

$\max\{a,b\} = \frac{a+b+|a-b|}{2}$,

we see the same holds for $\max\{f(x), g(y)\}$ & $\min\{f(x), g(y)\}$

$$\min\{f(x), g(y)\} = \frac{f(x)+g(y)-|f(x)-g(y)|}{2}$$

Now if $(x_n, y_n) \longrightarrow (x_0, y_0)$,

then $f(x_n) \to f(x_0)$, $g(y_n) \to g(y_0)$

$\Downarrow$ $f(x_n)+g(y_n) \to f(x_0)+g(y_0)$

$f(x_n)-g(y_n) \to f(x_0)-g(y_0)$

(seq. in $\mathbb{R}$)

$\Downarrow$

$$|f(x_n)-g(y_n)| \longrightarrow |f(x_0)-g(y_0)|$$

$$\Rightarrow \underbrace{\frac{f(x_n)+g(y_n)-|f(x_n)-g(y_n)|}{2}}_{= \min\{f(x_n), g(y_n)\}} \longrightarrow \underbrace{\frac{f(x_0)+g(y_0)-|f(x_0)-g(y_0)|}{2}}_{= \min\{f(x_0), g(y_0)\}}$$

Sheet 5 6. (ii)

23 December 2020 13:36

6. Examine the following function for the existence of partial derivatives at $(0,0)$.

(ii) $f(x,y) := \begin{cases} \dfrac{\sin^2(x+y)}{|x|+|y|} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

By def^n, $\dfrac{\partial f}{\partial x_1}(0,0) = \lim_{x_1 \to 0} \dfrac{f(x_1, 0) - f(0,0)}{x_1 - 0}$ (if it exists)

we show ↑ DNE

Let $x_1 \neq 0$. Then, note that

$$\frac{f(x_1, 0) - f(0,0)}{x_1} = \frac{\frac{\sin^2(x_1+0)}{|x_1|+|0|} - 0}{x_1}$$

$$= \frac{\sin^2(x_1)}{x_1|x_1|}$$

The limit of the above expression as $x_1 \to 0$ does not exist since RHL $= \lim_{x_1 \to 0^+} \frac{\sin^2 x_1}{x_1^2} = 1$

& LHL $= \lim_{x_1 \to 0^-} -\frac{\sin^2 x_1}{x_1^2} = -1$

Thus, $\frac{\partial f}{\partial x_1}(0,0)$ DNE. Similarly, neither does $\frac{\partial f}{\partial x_2}(0,0)$.

Sheet 5 8

23 December 2020 13:36

8. Let $\boxed{f(0,0) = 0}$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y) & \text{if } x \neq 0,\ y \neq 0, \\ x\sin(1/x) & \text{if } x \neq 0,\ y = 0, \\ y\sin(1/y) & \text{if } x = 0,\ y \neq 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

CONTINUITY: For $(x,y) \in \mathbb{R}^2$, note that

$$|f(x,y) - f(0,0)| = |f(x,y)|$$

$$\leq |x| + |y|$$

take 4 cases

Case 1. $(x,y) = (0,0)$, Obvious.

Case 2. $x \neq 0$ & $y \neq 0$.

$$|f(x,y)| = |x\sin(1/x) + y\sin(1/y)|$$
$$\leq |x\sin(1/x)| + |y\sin(1/y)|$$
$$\leq |x| + |y|$$

Case 3. $x = 0$, $y \neq 0$

$$|f(x,y)| = |y\sin(1/y)| \leq |y| = |x| + |y|$$

Case 4. $x \neq 0$, $y = 0$. Similar

Thus,

$$|f(x,y) - f(0,0)| \leq |x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2}$$

$$\Rightarrow |f(x,y) - f(0,0)| \leq \sqrt{2}\,\|(x,y) - (0,0)\| \quad (*)$$

Thus, given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{\sqrt{2}}$.

Then,

$$\left.\begin{aligned} &\|(x,y) - (0,0)\| < \delta \\ \Rightarrow\ &|f(x,y) - f(0,0)| \le \sqrt{2}\,\delta < \epsilon \end{aligned}\right\} (*)$$

Thus, f is continuous at $(0,0)$.

$\frac{\partial f}{\partial x_1}$ DNE: For $x_1 \neq 0$, note

$$\frac{f(x_1,0) - f(0,0)}{x_1} = \frac{x_1 \sin(1/x_1) - 0}{x_1}$$

$$= \sin\left(\frac{1}{x_1}\right)$$

[The sequence $x_n = \frac{2}{(2n+1)\pi}$ shows this.]

Since $\lim_{x_1 \to 0} \sin\left(\frac{1}{x_1}\right)$ does not exist,

we see that $\frac{\partial f}{\partial x_1}(0,0)$ DNE.

Similarly, $\frac{\partial f}{\partial x_2}(0,0)$ DNE.

Sheet 5 10

23 December 2020 13:36

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}$$

otherwise. Show that f is continuous at $(0,0)$, $D_{\underline{u}} f(0,0)$ exists for every unit vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Again, for $(x,y) \in \mathbb{R}^2$, note that

$$|f(x,y) - f(0,0)| = |f(x,y)| = \begin{cases} 0 & ; \ y=0 \\ \underbrace{\left|\frac{y}{|y|}\right|}_{1} \sqrt{x^2+y^2} & ; \ y \neq 0 \end{cases}$$

$$\Rightarrow |f(x,y)| \leq \sqrt{x^2+y^2} = \|(x,y)-(0,0)\|$$

$$\Rightarrow |f(x,y) - f(0,0)| \leq \|(x,y) - (0,0)\|$$

Given $\epsilon > 0$, $\delta = \epsilon$ works.

Thus, f is continuous at $(0,0)$.

Let $u = (u_1, u_2) \in \mathbb{R}^2$ where $u_1, u_2 \in \mathbb{R}$ and $u_1^2 + u_2^2 = 1$.

$$\left\{ \lim_{t\to 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} \right\}$$

If $u_2 = 0$, then, for $t \neq 0$, we note

$$\frac{f(tu_1, tu_2) - f(0,0)}{t} = \frac{f(tu_1, 0) - 0}{t} = \frac{0}{t} = 0.$$

1

Thus, $\lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$ exists and is 0.

2

If $u_2 \neq 0$, then, for $t \neq 0$, we note

$$\frac{f(tu_1, tu_2) - f(0,0)}{t} = \frac{\frac{tu_2}{|tu_2|}\sqrt{t^2(u_1^2+u_2^2)} - 0}{t}$$

$$= \frac{tu_2}{|tu_2|} \frac{\sqrt{t^2}}{t}$$

$$= \frac{t\, u_2\, |t|}{|t|\,|u_2|\, t} = \frac{u_2}{|u_2|}$$

Thus, $\lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$ exists and is $\frac{u_2}{|u_2|}$.

Thus, $D_u f(0,0)$ exists for all $u \in \mathbb{R}^2$ with $\|u\| = 1$.

NOT DIFF

If f were diff at $(0,0)$, then

$$Df(0,0) = \left[\frac{\partial f}{\partial x_1}(0,0) \quad \frac{\partial f}{\partial x_2}(0,0)\right]$$

$$= \begin{bmatrix} D_{(1,0)} f(0,0) & D_{(0,1)} f(0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \quad =: A \quad \text{(say)}$$

and

$$\lim_{(h,k)\to(0,0)} \frac{\left| f(h,k) - f(0,0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right|}{\|(h,k)\|} = 0.$$

We now show that the above limit is not equal to 0. (In fact, it doesn't exist.)

For $(h,k) \neq (0,0)$ & $k \neq 0$,

$$\frac{\left| f(h,k) - f(0,0) - A \begin{bmatrix} h \\ k \end{bmatrix} \right|}{\|(h,k)\|} = \frac{\left| \frac{k}{|k|}\sqrt{h^2+k^2} - 0 - \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{pmatrix} h \\ k \end{pmatrix} \right|}{\sqrt{h^2+k^2}}$$

$$= \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right|$$

Along the line $h = k$, the above expression becomes

$$\left| \frac{k}{|k|} - \frac{k}{\sqrt{2k^2}} \right| = \left| \frac{k}{|k|}\left(1 - \frac{1}{\sqrt{2}}\right) \right| = 1 - \frac{1}{\sqrt{2}}$$

Along $h = 2k$, the expression becomes $1 - \frac{1}{\sqrt{5}}$

Thus, the limit does not exist.

12. Let (x_0, y_0) be an interior point of a subset D of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$. Suppose the following conditions hold:
    (a) Both partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.
    (b) The directional derivative $(\mathbf{D_u} f)(x_0, y_0)$ exists for every unit vector $\mathbf{u} \in \mathbb{R}^2$.
    (c) $(\mathbf{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \mathbf{u}$ for every unit vector $\mathbf{u} \in \mathbb{R}^2$.
    (d) f is continuous at (x_0, y_0).

    It is not necessary that f is differen tiable at (x_0, y_0).
    Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) := \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

5 points (a) — (d) & "f is diff at (x_0, y_0)"