Sheet 4 2. (a)

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2. (a) ① Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\int_a^b f(x)\mathrm{d}x \geq 0$. ② Further, if f is continuous and $\int_a^b f(x)\mathrm{d}x = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

① Recap

$$L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i) \qquad P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

$$L(f) = \sup \{ L(f,P) \mid P \text{ is a partition} \}.$$

||^{ly} $U(f)$ is defined.

If $L(f) = U(f)$, then $\int_a^b f(x)\,dx := U(f)$

Consider the partition $P_0 = \{a, b\}$.

By hypo., $f(x) \geq 0 \quad \forall x \in [a,b]$

$$\Rightarrow \inf_{x \in [a,b]} f(x) \geq 0$$

$$\Rightarrow L(f,P_0) \geq 0\,(b-a) = 0$$

$$\Rightarrow L(f) = \sup_{\text{all partitions } P} L(f,P)$$

$$\geq L(f, P_0) \geq 0$$

$$\Rightarrow L(f) \geq 0$$

Since f is R. I., we see that

$$\int_a^b f(x)\,dx = L(f) \geq 0. \qquad \blacksquare$$

② Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)\,dt.$$

Since f is continuous,
F is differentiable and $F' = f$, by FTC Part I.

But $f \geq 0$. Thus, $F' \geq 0$ and hence, F is increasing.

$$\Rightarrow \quad F(a) \leq F(x) \leq F(b) \quad (*) \quad \forall x \in [a, b]$$

But $F(a) = \int_a^a f(t)\,dt = 0$ and

$$F(b) = \int_a^b f(t)\,dt = 0. \quad \text{(Given)}$$

Thus, by $(*)$, we get $F(x) = 0 \quad \forall x \in [a, b]$.

$$\Rightarrow \quad f(x) = F'(x) = 0 \qquad \forall x \in [a, b] \quad \blacksquare$$

Sheet 4 2. (b)

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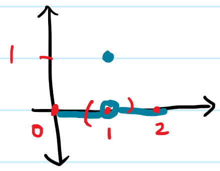
(b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\mathrm{d}x = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

$a = 0, \quad b = 2$

$f : [0, 2] \longrightarrow \mathbb{R}$ as



$$f(x) := \begin{cases} 0 ; & x \neq 1 \\ 1 ; & x = 1 \end{cases}$$

Show that f is Riemann integrable.

$P_n = \{ 0, 1 - \frac{1}{2n}, 1 + \frac{1}{2n}, 2 \}$

$U(P_n, f) = \frac{1}{n}$

Thus, given any $\varepsilon > 0$, $\exists\, n \in \mathbb{N}$ s.t. $U(P_n, f) < \varepsilon$.

$\Rightarrow$ there is no positive lower bound on $\{ U(P, f) \mid P$ is a part.$\}$

(inf = greatest lower bound) $\Rightarrow \inf U(P, f) \leq 0$

$\Rightarrow U(f) \leq 0$

But $L(f) = 0$, clearly.

Since $U(f) \geq L(f)$, in general, we see that $U(f) = L(f) = 0$.

① Clearly, $L(f) = 0$ since $L(f, P) = 0 \ \forall$ partitions P

② Clearly, 0 is a lower bound for every $U(f, P)$.

③ If $\varepsilon > 0$, then ε is not a lower bound for $\{U(f, P) : P \text{ is a partition of } [0,1]\}$.

Proof. Pick $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Consider $P_N = \left\{0, 1 - \frac{1}{2N}, 1 + \frac{1}{2N}, 2\right\}$

Thus,

$$U(f, P_N) = \sup_{[0, 1-\frac{1}{2N}]} f(x) \left(1 - \frac{1}{2N}\right) + (\sup)\left(\frac{1}{N}\right) + (\sup \quad)\left(1 - \frac{1}{2N}\right)$$

$$= 0 + \frac{1}{N} + 0$$

$$\Rightarrow U(f, P_N) = \frac{1}{N} < \varepsilon.$$

Thus, no $\varepsilon > 0$ is a lower bound.

$\therefore$ 0 is the greatest lower bound.

$$\Rightarrow 0 = U(f).$$

④ Thus, $L(f) = U(f)$ and hence, f is R.I.

with $\int_0^1 f(x)\,dx = 0$.

But $f(1) \neq 0$.

Sheet 4 3

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3 Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an appropriate Riemann sum for a suitable function over a suitable interval.

(ii) $S_n = \sum_{i=1}^n \frac{n}{i^2+n^2}$.

(iv) $S_n = \frac{1}{n}\sum_{i=1}^n \cos\left(\frac{i\pi}{n}\right)$.

Theorem 1

Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Suppose that (P_n, t_n) is a sequence of tagged partitions of $[a,b]$ such that $\|P_n\| \to 0$. Then,

$$\lim_{n\to\infty} R(f, P_n, t_n) = \int_a^b f(x)\mathrm{d}x.$$

Note very carefully in the above that we already need to know that f is Riemann integrable.

(ii) $$S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$$

$$= \sum_{i=1}^{n} \frac{1/n}{\left(\frac{i}{n}\right)^2+1}$$

$$= \sum_{i=1}^{n} \frac{1}{\left[\left(\frac{i}{n}\right)^2+1\right]} \left\{ \frac{i}{n} - \frac{i-1}{n} \right\}$$

Consider $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{x^2+1}$$

Note that f is continuous and hence, it is R.I.

Take the sequence of tagged partitions

$$P_n = \{0, 1/n, \dots, \tfrac{n-1}{n}, \tfrac{n}{n}\}$$ and tags t_n

$$\frac{i}{n} \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \quad \forall\ i = 1, \dots, n.$$

Then,
$$R(f, P_n, t_n) = \sum_{i=1}^{n} f(t_n^{(i)})(x_i - x_{i-1})$$

$$= \frac{1}{\left[\left(\frac{i}{n}\right)^2 + 1\right]}\left\{\frac{i}{n} - \frac{i-1}{n}\right\}$$

$$= S_n$$

Here $\|P_n\| = \frac{1}{n} \longrightarrow 0.$

Thus,
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} R(f, P_n, t_n)$$

recall that we knew that f is R.I. to begin with

$$= \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

FTC Part II since $\arctan'(x) = \frac{1}{1+x^2}$

$$= \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4}.$$

(iv) $S_n = \sum_{i=1}^{n} \frac{1}{n} \cos\left(\frac{i\pi}{n}\right)$

Here take $f : [0,1] \longrightarrow \mathbb{R}$ as $f(x) = \cos(\pi x)$.

Rest is same. Note that $F : [0, 1] \to \mathbb{R}$ defined as

$$F(x) = \frac{1}{\pi} \sin(\pi x) \text{ is}$$

an anti-der.

Thus, $$\int_0^1 f(x)\,dx = F(1) - F(0) = 0.$$

Sheet 4 **4. (b)**

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4. (b) Compute $F'(x)$, if for $x \in \mathbb{R}$

(i) $F(x) = \int_1^{2x} \cos(t^2)\mathrm{d}t.$

(ii) $F(x) = \int_0^{x^2} \cos(t)\mathrm{d}t.$

Theorem 2

Fix $a \in \mathbb{R}$.

Let $v : \mathbb{R} \to \mathbb{R}$ be differentiable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) := \int_a^{v(x)} g(t)\mathrm{d}t.$$

$$\int_{u(x)}^{v(x)} = \int_a^{v(x)} - \int_a^{u(x)}$$

Then,

$$F'(x) = g(v(x))v'(x).$$

Proof:

Define $G(x) = \int_a^x g(t)\,dt$

Since g is cont., G is diff &

$$G'(x) = g(x).$$

However, $F(x) = \int_a^{v(x)} g(t)\,dt$

$$\Rightarrow \quad F(x) = G(v(x))$$

chain rule

$$\Rightarrow \quad F'(x) = G'(v(x))\,v'(x) = g(v(x))\,v'(x)$$

(i) Here, $a = 1,$

$v(x) = x,$

$g(t) = \cos(t^2).$

Thus, $v'(x) = 2$ and

$$F'(x) = g(v(x))\,v'(x) = 2\cos(4x^2).$$

(i) Here, $a = 0,$

$v(x) = x^2,$

$g(t) = \cos(t).$

Thus, $v'(x) = 2x$ and

$$F'(x) = g(v(x))\,v'(x) = 2x\cos(x^2).$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin[\lambda(x-t)]\, dt. \qquad (1)$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

$$g(x) = \frac{1}{\lambda}\left\{ \int_0^x f(t) \sin\lambda x \cos\lambda t\, dt - \int_0^x f(t) \cos\lambda x \sin\lambda t\, dt \right\}$$

$$g(x) = \frac{1}{\lambda}\left\{ \sin\lambda x \int_0^x f(t)\cos\lambda t\, dt - \cos\lambda x \int_0^x f(t)\sin\lambda t\, dt \right\}$$

$$\frac{d}{dx}\int_0^x f(t)\cos\lambda t\, dt = f(x)\cos(\lambda x)$$

FTC Part I

$$\frac{d}{dx}\int_0^x f(t)\sin\lambda t\, dt = f(x)\sin(\lambda x)$$

FTC & Product Rule

$$g'(x) = \cos\lambda x \int_0^x f(t)\cos\lambda t\, dt + \sin\lambda x \int_0^x f(t)\sin\lambda t\, dt \qquad (2)$$

FTC & Product Rule

$$g''(x) = \lambda\left\{ -\int_0^x f(t)\left[\sin\lambda(x-t)\right] dt \right\} + f(x)$$

$$= \lambda\left\{-\lambda g(x)\right\} + f(x)$$

$$\Rightarrow \boxed{g''(x) + \lambda^2 g(x) = f(x)}$$

$$\Rightarrow \boxed{g''(x) + \lambda^2 g(x) = f(x)}$$

From ①: $g(0) = 0$

From ②: $g'(0) = 0$

Thus, we are done!