Sheet 2 8. (ii) (iii)

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8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$

(ii) Rough Guess: $f = ax^2 + bx$ $(a > 0)$

$f'(0) = 1 \rightarrow b = 1$

$f'(1) = 2 \rightarrow 2a + b = 2 \rightarrow 2a = 1 \rightarrow a = 1/2$

Final answer: $f(x) = \dfrac{x^2}{2} + x$

Note $f''(x) = 1 > 0 \quad \forall x \in \mathbb{R}$

Moreover, $f'(x) = x + 1$

Thus, $f'(0) = 1$ and $f'(1) = 2$, as desired.

(iii) Since $f'' \geq 0$, f' is increasing.

Since $f'(0) = 1$, $f'(x) \geq 1$ for all $x \geq 0$.

Now, pick any $y > 0$. By MVT,

$\exists x \in (0, y)$ s.t.

$$f'(x) = \frac{f(y) - f(0)}{y - 0}$$

But $f'(x) \geq 1$ and thus,

$f(y) \geqslant f(0) + y \quad \forall y \in (0, \infty).$

$y = \max\{101 - f(0), 1\}$

Choose y large enough so that $f(0) + y \geqslant 101$

Then, we get $f(y) \geqslant 101 > 100$, a contradiction.

Sheet 2 10. (i)

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10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

(i) $y = 2x^3 + 2x^2 - 2x - 1$

$$y'(x) = 6x^2 + 4x - 2 = 6(x+1)(x - 1/3)$$

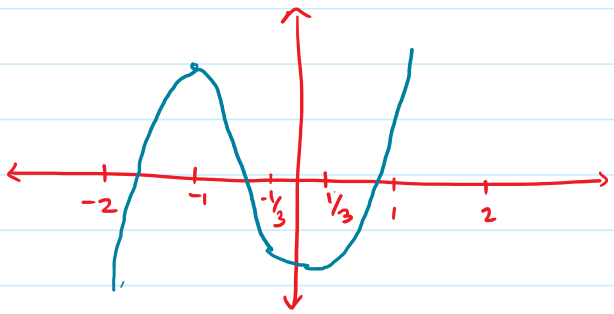
$$y''(x) = 12x + 4 = 12(x + 1/3)$$

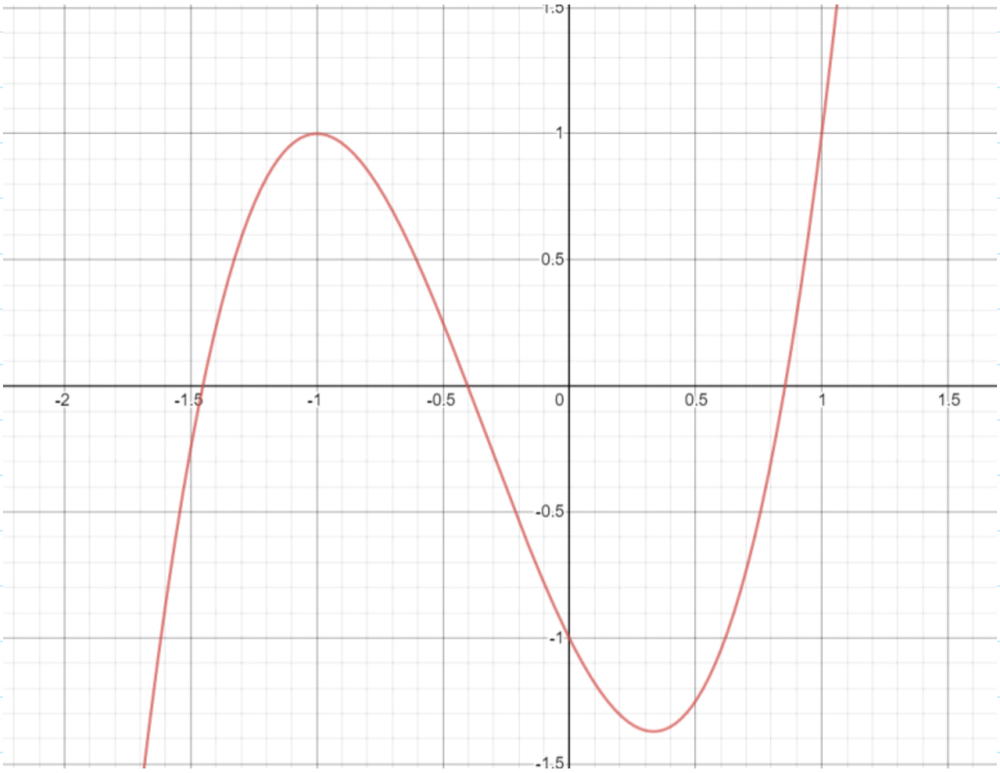
$$\lim_{x \to -\infty} y(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} y(x) = \infty.$$

$y(-1/3) < 0$

$y(-2) < 0, \quad y(-1) > 0, \quad y(0) < 0, \quad y(1) > 0$

one root in $(-2, -1)$, $(-1, 0)$, $(0, 1)$





Suppose $\frac{1}{x}$ was given

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Sheet 2 11

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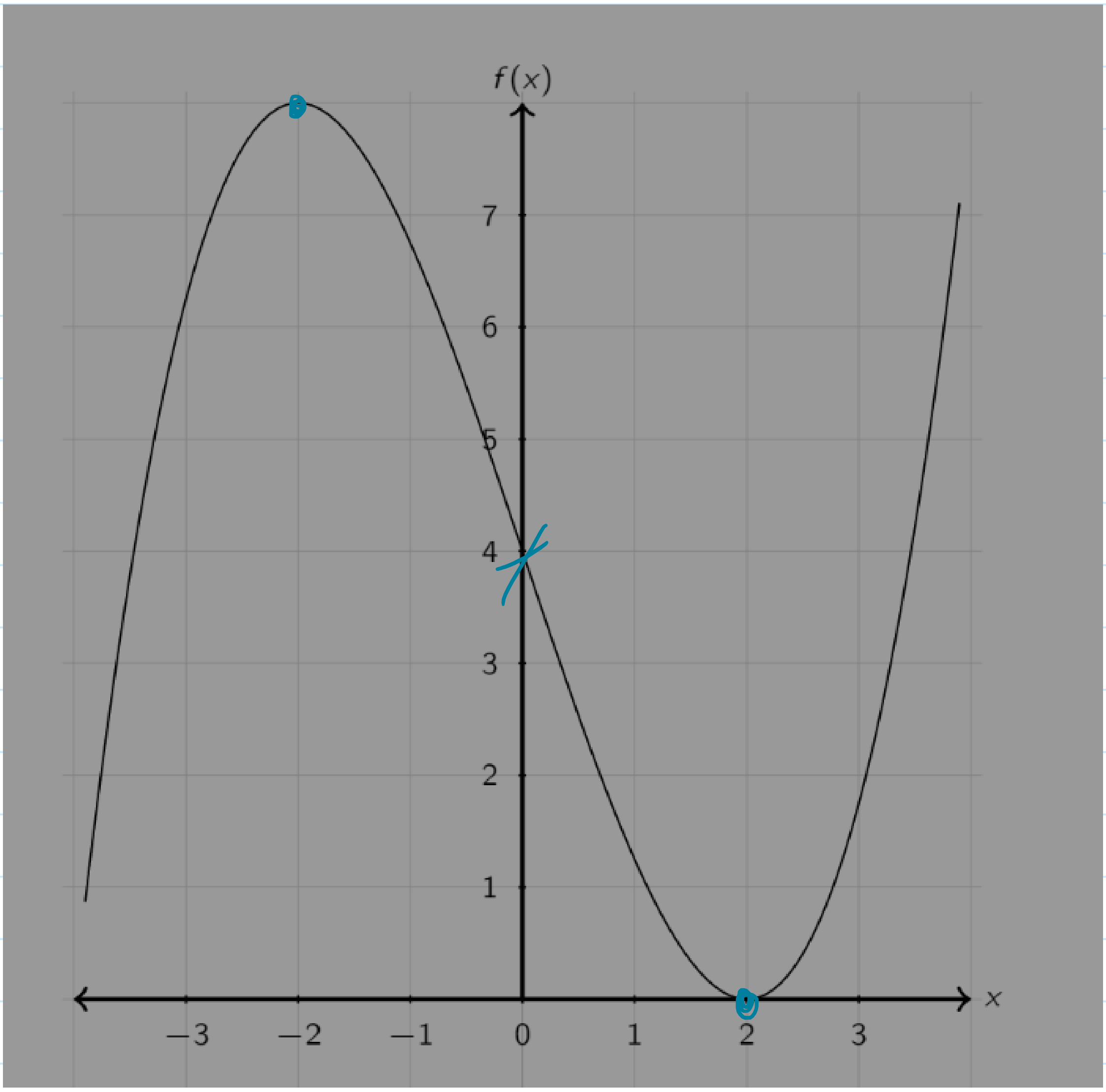
11. Sketch a continuous curve $y = f(x)$ having all the following properties:

$f(-2) = 8$, $f(0) = 4$, $f(2) = 0$; $f'(2) = f'(-2) = 0$;

$f'(x) > 0$ for $|x| > 2$, $f'(x) < 0$ for $|x| < 2$; → extrema ± 2

$f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.



Sheet 3 1 (ii)

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Exercise 1. *Write down the Taylor series for (i)* $\cos x$*, (ii)* $\arctan x$ *about the point* 0*. Write down a precise remainder term* $R_n(x)$ *in each case.*

① How to find $\arctan^{(n)}(0)$?

Answ: If $n=0$, $\arctan(0) = 0$.

Note $\arctan'(x) = \frac{1}{1+x^2} =: g(x)$.

Thus, $\arctan^{(n)}(0) = g^{(n-1)}(0)$.

Now, $g(x) = 1 - x^2 + x^4 - x^6 + \cdots \quad \forall\, x \in (-1,1)$

$\Rightarrow g(0) = 1$
$g'(0) = 0$
$g''(0) = -2$
$\vdots$

$$g^{(n)}(0) = \begin{cases} 0 & ; \ n \text{ is odd} \\ (-1)^{n/2}(n!) & ; \ n \text{ is even} \end{cases}$$

Thus, $P_{2n+1}(x) = \sum_{k=0}^{2n+1} \frac{\arctan^{(k)}(0)\, x^k}{k!} = 0 + \sum_{k=1}^{2n+1} \frac{\arctan^{(k)}(0)\, x^k}{k}$

$= \sum_{k=1}^{2n+1} \frac{g^{(k-1)}(0)\, x^k}{k!}$

$$\Rightarrow \left[P_{2n+1}(x) = x - \frac{x^3}{3} + \cdots + \frac{(-1)^n}{2n+1} x^{2n+1} \right]$$

$P_{2n+2}(x) = P_{2n+1}(x) + 0$

$R_{2n+1}(x) = f(x) - P_{2n+1}(x)$

$= \arctan(x) - \left\{ x - \frac{x^3}{3} + \cdots + \frac{(-1)^n}{2n+1} x^{2n+1} \right\}$

Now, we compute the remainder $R_{2n+1}(x)$ more explicitly.

$$\arctan'(x) = \left[1 - x^2 + x^4 + \cdots + (-1)^n x^{2n}\right] + (-1)^{n+1} x^{2n+2} + \cdots$$

$$= \left[1 - x^2 + \cdots + (-1)^n x^{2n}\right] + (-1)^{n+1} x^{2n+2} (1 - x^2 + x^4 \cdots)$$

$$\arctan'(x) = \left[1 - x^2 + \cdots + (-1)^n x^{2n}\right] + (-1)^{n+1} \frac{x^{2n+2}}{1+x^2}$$

(this is the $P_{2n+1}(x)$ from earlier)

$$\arctan(x) = \left[P_{2n+1}(x)\right] + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt$$

$$\boxed{\arctan(x) = P_{2n+1}(x) + \underbrace{(-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt}_{\text{This is the remainder } R_{2n+1}(x)}}$$

Exercise 2. *Our examples of Taylor's series have usually been series about the point 0. Write down the Taylor series of the polynomial $x^3 - 3x^2 + 3x - 1$ about the point 1.*

$$f(x) = x^3 - 3x^2 + 3x - 1$$

$P_0(x) = f(a)$

$a = 1$

$f(1) = 0 \rightarrow P_0(x) = 0$

$f^{(1)}(1) = 0 \rightarrow P_1(x) = 0 + \frac{0}{1!}(x-1) = 0$

$f^{(2)}(1) = 0 \rightarrow P_2(x) = P_1(x) + \frac{0}{2!}(x-1)^2 = 0$

$f^{(3)}(1) = 6 \rightarrow P_3(x) = P_2(x) + \frac{6}{3!}(x-1)^3 = (x-1)^3$

$f^{(n)}(1) = 0 \quad n \geq 4$

$P_3(x) = (x-1)^3$

$P_n(x) = (x-1)^3$ for all $n \geq 4$

Also, $f(x) - P_n(x) = 0 \quad \forall\, n \geq 3$

Thus $f(x) = (x-1)^3 \leftarrow$ Taylor "series" about 1.

In general, the above tells us that any polynomial

$$a_0 + a_1 x + \cdots + a_n x^n$$

can be written about any other point as

$$b_0 + b_1(x-a) + \cdots + b_n(x-a)^n.$$

$$a_0 + a_1 x + \cdots + a_n x^n = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \quad \forall\, x \in \mathbb{R}$$

$$\|$$

$$e^x = e + \frac{e^1}{1!}(x-1) + \frac{e^1}{2!}(x-1)^2 + \cdots \quad \forall\, x \in \mathbb{R}$$

$$\frac{1}{x^2+1} \xrightarrow{\text{about } 0} 1 - x^2 + x^4 - \cdots \quad \forall\, x \in (-1,1)$$

$$\xrightarrow{\text{about } 2} \quad \frac{1}{5 + (x^2 - 4)} = \frac{1}{5}\left(\frac{1}{1 + \left(\frac{x^2-4}{5}\right)}\right)$$

$$= \frac{1}{5}\left\{1 - \frac{(x^2-4)}{5} + \left(\frac{x^2-4}{5}\right)^2 + \cdots\right\}$$

Sheet 3 4

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Exercise 4. *Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. Prove that it converges as follows. Choose $N > 2|x|$. We see that for all $n > N$,*

$$\left|\frac{x^{n+1}}{(n+1)!}\right| \leq \frac{1}{2} \cdot \left|\frac{x^n}{n!}\right|. \qquad (*)$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$), convergent.

Proof of $(*)$: If $N > 2|x|$ and $n > N$, then

$$\left|\frac{x^{n+1}}{(n+1)!}\right| = \left|\frac{x^n}{n!}\right| \cdot \left|\frac{x}{n+1}\right| \overset{n+1>n>N}{\leq} \left|\frac{x^n}{n!}\right| \cdot \left|\frac{x}{N}\right|$$

$$\overset{N>2|x|}{\leq} \frac{1}{2}\left|\frac{x^n}{n!}\right|$$

Thus, if $n > N$:

$$\left|\frac{x^{n+1}}{(n+1)!}\right| \leq \frac{1}{2}\left|\frac{x^n}{n!}\right| \leq \cdots \overset{\text{by induction}}{\leq} \frac{1}{2^{n+1-N}}\left|\frac{x^N}{N!}\right|$$

Let $S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Then, for $m > n > N$

$$|S_m(x) - S_n(x)| = \left|\sum_{k=n+1}^{m} \frac{x^k}{k!}\right|$$

$$\leq \sum_{k=n+1}^{m} \frac{|x|^k}{k!}$$

$$= \frac{|x|^{n+1}}{(n+1)!} + \cdots + \frac{|x|^m}{m!}$$

$$\leq \frac{|x|^N}{N!}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}}\right)$$

$$\leq \frac{|x|^N}{N!}$$

Now, for $\varepsilon > 0$, pick N large enough ($N \in \mathbb{N}$) so that

$$\frac{|x|^N}{N!} < \varepsilon.$$

(Can do so because $\lim_{N\to\infty} \frac{|x|^N}{N!} = 0$)

Then, for all $n, m > N$, we have

$|s_m(x) - s_n(x)| < \varepsilon$, as desired!

$a_n := \frac{|x|^n}{n!}$.

Show (a_n) is event. decr.
Conclude (a_n) is convergent.

Use $a_{n+1} = \frac{|x|}{n+1} a_n$ to show limit is 0.

Sheet 3 5

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Exercise 5. *Using Taylor series write down a series for the integral*

$$\int \frac{e^x}{x} dx.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \quad \frac{e^x}{x} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

$$\Rightarrow \quad \int \frac{e^x}{x} dx = C + \log x + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \cdots$$

$$= \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!} + C.$$