Sheet 1 13. (ii)

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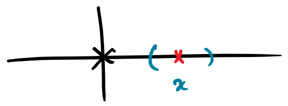
13. (ii) Discuss the continuity of the following function:



$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

- For $x \neq 0$, f is continuous at x because it product / composition of continuous functions.

- For 0, we show that

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

Let $\epsilon > 0$ be given.
If $0 < |x - 0| < \epsilon$, then:

$$|f(x) - f(0)| = \left| x \sin\left(\frac{1}{x}\right) - 0 \right|$$

$$= |x \sin(1/x)| \qquad \because \left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\leq |x|$$

$$= |x - 0| < \epsilon$$

Thus,

$$0 < |x - 0| < \epsilon \implies |f(x) - f(0)| < \epsilon.$$

Hence, $\delta = \epsilon$ works.

Thus, $\lim_{x \to 0} f(x) = f(0)$. $\therefore$ f is continuous at 0.

Hence, f is continuous on $\mathbb{R}$.

In general: $\lim_{x \to c} f(x) = L$ means

$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad$ s.t. $\quad 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

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15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

↳ nope!

Again, $x \neq 0$ argument is similar as earlier.

For 0: For $h \neq 0$, we note

$$\frac{f(0+h) - f(0)}{h} = \frac{h^2 \sin(1/h) - 0}{h}$$

$$= h \sin\left(\frac{1}{h}\right)$$

As seen earlier, $\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$.

Thus, $f'(0)$ exists and is equal to 0.

Hence, f' is diff. on $\mathbb{R}$.

$$f'(x) = \begin{cases} 0 & ; \quad x = 0 \\ 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & ; \quad x \neq 0 \end{cases}$$

We show that f' is not cont. at 0 using the seq. criterion.

Consider $x_n := \dfrac{1}{2n\pi}$ for $n \in \mathbb{N}$.

$\left[f'(x_n) = 2 \sin(2n\pi) - \cos(2n\pi) \right]$

Clearly, $x_n \to 0$. $\left[f'(x_n) = \frac{2}{2n\pi}\sin(2n\pi) - \cos(2n\pi) \right]$

However, $f'(x_n) = -1 \quad \forall n \in \mathbb{N}$.

Thus, $\lim_{n\to\infty} f'(x_n) = -1 \neq f'(0)$.

Thus, f' is not continuous at 0.

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g is discontinuous at a

$\Updownarrow$

There exists a sequence (x_n) s.t.

$x_n \to a$ and

$g(x_n) \not\to g(a)$.

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18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

$$\frac{f(c+h) - f(c)}{h} = f(c)\left\{\frac{f(h)-1}{h}\right\}$$

Put $x = y = 0$ to get $\left[f(0) = (f(0))^2. \right]$ $(*)$

Case 1 $f(0) = 0$.

In this case, $f(x+0) = f(x) f(0) = 0$

$\Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}$

Clearly, f is diff. at any $c \in \mathbb{R}$ and

$$f'(c) = 0 = \underbrace{f'(0)}_{\text{"}0\text{"}} f(c).$$

Case 2. $f(0) \neq 0$.

Thus, $f(0) = 1$. [By $(*)$]

Now, for $c \in \mathbb{R}$, note that

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \overset{\text{using the eq. given}}{=} \lim_{h \to 0} \frac{f(c) f(h) - f(c)}{h}$$

$$= \lim_{h \to 0} f(c) \left\{ \frac{f(h) - 1}{h} \right\}$$

$\because f(0) = 1$

$$= \lim_{h \to 0} f(c) \left\{ \underbrace{\frac{f(h) - f(0)}{h}} \right\}$$

$$\lim_{h \to 0} \ldots) \underbrace{\frac{\ldots}{h}}_{\text{deriv. limit at } 0}]$$

Since $f'(0)$ exists, the above limit exists and equals $f(c)\, f'(0)$.

Thus, $f'(c)$ exists & $f'(c) = f'(0) f(c)$, as desired. $\blacksquare$

Sheet 1 Optional 7

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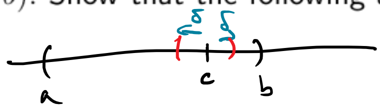
7. Let $f : (a, b) \to \mathbb{R}$ be differentiable and $c \in (a, b)$. Show that the following are equivalent:



(i) $\Leftrightarrow$ (ii)

(i) $\Leftrightarrow$ (iii)

(i) f is differentiable at c.

(ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$, and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and

$$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \quad \text{for } h \in (-\delta, \delta).$$

$$\epsilon_1(h) = \frac{f(c+h) - f(c)}{h} - \alpha$$

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|f(c + h) - f(c) - \alpha h|}{|h|} = 0.$$

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)

Let $\alpha := f'(c)$. [Exists since f is given to be diff at c, by (i).]

Let $\delta := \min\left\{ \dfrac{c-a}{2}, \dfrac{b-c}{2} \right\}$.

Note $\delta > 0$. Moreover, $(c-\delta, c+\delta) \subset (a, b)$.

Let $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ be defined as

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c)}{h} - \alpha & ; \quad h \neq 0 \\ 0 & ; \quad h = 0 \end{cases}$$

Claim 1: $\lim_{h\to 0} \epsilon_1(h) = 0$.

Proof. By definition: $\lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h} = \alpha$

($\because h \neq 0$ when talking about limit)

$$\Rightarrow \lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

$$\Rightarrow \lim \epsilon_1(h) = 0 \qquad \square$$

$$\Rightarrow \lim_{h\to 0} \epsilon_1(h) = 0$$ ☒

Claim2:

$$f(c+h) = f(c) + \alpha h + h\,\epsilon_1(h) \quad \forall\, h \in (-\delta, \delta)$$

Proof. For $h=0$, clearly true.
For $h \neq 0$, plug in the definition of ϵ_1.

Thus, $\boxed{(i) \Rightarrow (ii)}$.

(ii) $\Rightarrow$ (iii)

Let α be as in (ii).

$$\lim_{h\to 0} \left| \frac{f(c+h) - f(c) - \alpha h}{h} \right|$$

since $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$

$$= \lim_{h\to 0} \left| \frac{h\,\epsilon_1(h)}{h} \right|$$

$$= \lim_{h\to 0} |\epsilon_1(h)| = 0$$

($\because \lim_{h\to 0} \epsilon_1(h) = 0$ is given)

(iii) $\Rightarrow$ (i)

Given: $$\lim_{h\to 0} \left| \frac{f(c+h) - f(c) - \alpha h}{h} \right| = 0$$

(Why?)

$$\Rightarrow \lim_{h\to 0} \frac{f(c+h) - f(c) - \alpha h}{h} = 0$$

h

$$\Rightarrow \lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

Thus, $f'(c)$ exists.

Sheet 1 Optional 10

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Q : Is this true if $[0,1]$ is replaced by $(0,1]$ or $(0,1)$?

x_0 is a fixed pt. if $f(x_0) = x_0$

10. Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

Define $g : [0,1] \longrightarrow \mathbb{R}$ as

$$g(x) := f(x) - x.$$

Note that

$$g(0) = f(0) - 0 = f(0) \geqslant 0,$$

(since $f(x) \in [0,1]$ $\forall x \in [0,1]$)

and

$$g(1) = f(1) - 1 \leqslant 0.$$

Thus, by IVP, $\exists x_0 \in [0,1]$ s.t.

$$g(x_0) = 0$$

$$\Leftrightarrow f(x_0) - x_0 = 0$$

$$\Rightarrow f(x_0) = x_0$$

Thus, x_0 is a fixed point of f.

Sheet 2 2

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2 Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Idea: Existence. IVP (since 0 lies b/w f(a) & f(b).)

Uniqueness. Suppose $x_0 < x_1$ are pts s.t.
$f(x_0) = f(x_1) = 0$.
Then, $\exists\, x_2 \in (x_0, x_1)$ s.t.

(By Rolle's.) $f'(x_2) = 0$

Contradiction!

Sheet 2 5

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5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

Proof. If $a = b$, then it is clearly true. $(\because 0 \leq 0)$

Else assume $a \neq b$.

By MVT, $\exists c$ between a and b s.t.

$$\sin'(c) = \frac{\sin a - \sin b}{a - b}$$

But $\sin'(c) = \cos(c)$ and thus,

$$\left|\frac{\sin a - \sin b}{a - b}\right| = |\cos(c)| \leq 1$$

$$\Rightarrow |\sin a - \sin b| \leq |a - b| \qquad \blacksquare$$