Sheet 1 2. (iv)

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(iv) $\lim_{n\to\infty} (n)^{1/n}$

Take $n \geq 3$. Consider the following n numbers:

$$\underbrace{1, \dots, 1}_{n-2 \text{ times}}, \sqrt{n}, \sqrt{n}$$

By $AM \geq GM$, we have:

$$\left(1^{n-2} \cdot \sqrt{n}\sqrt{n}\right)^{1/n} \leq \frac{n-2+2\sqrt{n}}{n}$$

$$\Rightarrow \quad n^{1/n} \leq 1 - \frac{2}{n} + \frac{2}{\sqrt{n}} \quad \text{— } (*)$$

Also, since $n \geq 1$, we have $n^{1/n} \geq 1$. — $(\star)$

By $(\star)$ and $(*)$, we have

$$1 \leq n^{1/n} \leq 1 - \frac{2}{n} + \frac{2}{\sqrt{n}} \quad \text{for all } n \geq 3.$$

Note that $\lim_{n\to\infty}\left(1 - \frac{2}{n} + \frac{2}{\sqrt{n}}\right) = 1$

individually exist

$\left(\begin{array}{ll} \frac{1}{n} \to 0 & \text{in class} \\ \frac{1}{\sqrt{n}} \to 0 & \text{similar argument} \end{array}\right)$

Thus, by Sandwich Theorem, $\lim_{n\to\infty} n^{1/n}$ exists and is equal to 1.

Define $\quad h_n := n^{1/n} - 1.$

$$\left\{\begin{array}{l} n \geq 1 \\ \Rightarrow n^{1/n} \geq 1^{1/n} = 1 \\ \Rightarrow n^{1/n} - 1 \geq 0 \\ \Rightarrow h_n \geq 0 \end{array}\right.$$

To show: $\quad \lim_{n\to\infty} h_n = 0$

$$h_n = n^{1/n} - 1 \quad \geq 0$$

$$\Leftrightarrow \quad n = (1+h_n)^n$$

Assume $n \geq 3$. Then,

$$n = (1+h_n)^n$$

Binom.

$$= \underbrace{1 + n h_n + \binom{n}{2} h_n^2}_{} + \underbrace{\cdots + \binom{n}{n} h_n^n}_{\geq 0}$$

$$n \geq \underbrace{1 + n h_n}_{\geq 0} + \binom{n}{2} h_n^2$$

(Why?)

$h_n \geq 0$. Why?

$$\Rightarrow \quad n > \binom{n}{2} h_n^2$$

$$\Rightarrow \quad h_n < \left\{\frac{n}{\binom{n}{2}}\right\}^{1/2} = \left(\frac{2}{n-1}\right)^{1/2}$$

$$\Rightarrow \quad 0 \leq h_n < \frac{\sqrt{2}}{(n-1)^{1/2}}$$

Use Sandwich again to get $h_n \to 0$.
Thus, concluden $n^{1/n} \to 1$.

Sheet 1 3. (ii)

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(ii) $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$

$$a_n = (-1)^n\left(\frac{1}{2}-\frac{1}{n}\right).$$

If a_n converges, then $\lim_{n\to\infty}(a_{n+1}-a_n) = 0$.

(Formulae for sum/ diff.)

(Converse is not true.) ($a_n := \sqrt{n}$ for example.)

$b_n := a_{n+1} - a_n$. We show that $\lim_{n\to\infty}|b_n| = 1$.

(Thus, $\lim_{n\to\infty} b_n$ is not zero.)

Does not mean it is ± 1.

{ If $\lim_{n\to\infty}|c_n|$ exists, does it mean $\lim_{n\to\infty} c_n$ exists? }

No.

{ If $\lim_{n\to\infty}|c_n| = 0$, then $\lim_{n\to\infty} c_n = 0$ }

$$b_n = (-1)^{n+1}\left\{\left(\frac{1}{2}\right) - \frac{1}{n+1}\right\} - (-1)^n\left\{\frac{1}{2} - \frac{1}{n}\right\}$$

$$= (-1)^{n+1}\left\{\left(\frac{1}{2}\right) - \frac{1}{n+1}\right\} + (-1)^{n+1}\left\{\frac{1}{2} - \frac{1}{n}\right\}$$

$$= (-1)^{n+1}\left[\left\{\left(\frac{1}{2}\right) - \frac{1}{n+1}\right\} + \left\{\frac{1}{2} - \frac{1}{n}\right\}\right]$$

$$b_n = (-1)^{n+1}\left[1 - \frac{1}{n+1} - \frac{1}{n}\right]$$

$$\Rightarrow |b_n| = \left|1 - \frac{1}{n+1} - \frac{1}{n}\right|$$

$$= 1 - \frac{1}{n+1} - \frac{1}{n} \quad \text{for } n \geq 2$$

$$\Rightarrow \lim_{n\to\infty} |b_n| = 1$$

Thus, $\lim_{n\to\infty} |a_{n+1} - a_n| = 1.$

Thus, $\lim_{n\to\infty} a_n$ does not exist!

Sheet 1 5. (iii)

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Show: monotone & bounded & find the limit

(iii) $a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2}\ \forall\, n \geq 1$

① To show: $a_n \leq 6 \quad \forall\, n \geq 1$

Proof. By induction.

$n=1$. $a_1 = 2 \leq 6$ is true.

Assume $a_n \leq 6$ for some $n \geq 1$.

Then, $a_{n+1} = 3 + \frac{a_n}{2} \overset{\text{by induc. hyp.}}{\leq} 3 + \frac{6}{2} = 6.$

Thus, $a_{n+1} \leq 6$.

By PMI, we are done. ∎

② To show: $a_{n+1} \geq a_n \quad \forall\, n \geq 1$

Proof. $a_{n+1} = 3 + \frac{a_n}{2}$

$$\Rightarrow a_{n+1} - a_n = 3 - \frac{a_n}{2}$$

$$= \frac{6 - a_n}{2}$$

by ①

$$\geq \frac{0}{2}$$

$$= 0$$

$\Rightarrow \quad a_{n+1} \geq a_n \quad \forall n \geq 1$ ☒

Thus, (a_n) is monotonic and bounded.
Therefore, it converges.

Now, recall $a_{n+1} = 3 + \frac{a_n}{2}$

Since the limits $\lim_{n \to \infty} a_n$ and $\lim_{n \to \infty} a_{n+1}$ exist and are equal, (call it L) we may lim on both sides.

Thus, $\lim_{n \to \infty} (a_{n+1}) = \lim_{n \to \infty} \left(3 + \frac{a_n}{2}\right)$

$$\Rightarrow \quad L = 3 + \frac{L}{2}$$

$$\Leftrightarrow \quad \boxed{L = 6}$$

Thus,

$$\lim_{n \to \infty} a_n = 6.$$

Note that you do need to show convergence first.

For ex: Consider $a_1 = 1$ & $a_{n+1} = 2a_n$ for $n \geq 1$.

Blindly taking limit: $\lim_{n \to \infty} a_n = 0$ ⇐ WRONG!

Suppose (a_n) is a sequence s.t. for every $k \in \mathbb{N}$, the following is true:

$$\lim_{n \to \infty} (a_{n+k} - a_n) = 0.$$

Is it true that (a_n) is Cauchy?

No.

Sheet 1 7

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7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \ \forall\, n \geq n_0.$$

Since $\lim_{n\to\infty} a_n = L$, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t.

$$|a_n - L| < \epsilon \quad \text{for all } n > N.$$

In particular, we can take $\epsilon = \frac{|L|}{2}$.

(Why is this a valid ϵ? Since $L \neq 0$, $\frac{|L|}{2} > 0$)

Thus, $\exists\, N \in \mathbb{N}$ s.t.

$$|a_n - L| < \frac{|L|}{2} \quad \text{for all } n > N.$$

Note that $\big||a_n| - |L|\big| < |a_n - L|$ and hence,

$$\big||a_n| - |L|\big| < \frac{|L|}{2}.$$

Hence,

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}$$

or

$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2} \quad \text{for all } n > N.$$

This proves the result.

Sheet 1 9

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9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

No (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

No (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

$a_n = 1$, $b_n = (-1)^n$ works for both

Sheet 1 11

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11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements. $(a, b) = (0, 1)$

(i) $\lim_{x\to c}[f(x)g(x)] = 0$. $\longrightarrow$ No. $\underline{c = 0,\ f(x)} = x,\ g(x) = 1/x$

(ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded. $\longrightarrow$ Sandwich

(iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists. $\longrightarrow$ Product rule

(ii) $\exists\, M > 0$ s.t. $|g(x)| < M \quad \forall x \in (a, b)$.

Then, $0 \le |f(x)g(x)| < M\,|f(x)|$

$\downarrow$

use sandwich.

If $\lim_{x\to c} f(x)$ exists (not nec. 0) and g is bounded, that does not mean $\lim_{x\to c} f(x)g(x)$ exists. $\sin\left(\frac{1}{x}\right)$

Ex. If $\lim_{x\to c} g(x)$ exists, then $\exists\, \delta > 0$ and $M > 0$ s.t. $|g(x)| < M$ for all $x \in (a, b)$ s.t. $0 < |x - c| < \delta$.

(Limit exists $\Rightarrow$ locally bounded)